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A TWO PARAMETERS AMBROSETTI–PRODI PROBLEM*

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1 – Introduction

The study of the Ambrosetti–Prodi problem has started with the paper of A. Ambrosetti–G. Prodi [2] who consider the problem

$$\Delta u + f(u) = h(x), \quad \text{in } \Omega ,$$
$$u = 0, \quad \text{on } \partial\Omega ,$$

where f is a convex function which satisfies

$$0 < \lim_{u\to-\infty} f'(u) < \lambda_1 < \lim_{u\to+\infty} f'(u) < \lambda_2 .$$

They prove that in the space $H = \mathrm{Im}(\Delta u + f(u))$, there is a manifold M which separates the space in two regions O_0 and O_2 such that the above problem has zero, exactly one or exactly two solutions according to $h \in O_0$, $h \in M$ or $h \in O_2$. In 1975, M.S. Berger–E. Podolak [4] use the decomposition of h in terms of the first eigenfunction φ of $-\Delta$ with Dirichlet condition i.e. $h(x) = \nu\,\varphi(x) + \widetilde{h}(x)$, where $\int_\Omega \widetilde{h}(x)\,\varphi(x)\,dx = 0$, and characterize the manifold M in terms of the parameter ν. In the same year, J.L. Kazdan–F.W. Warner [10] have used the lower and upper solution method to study this problem but were only able to prove the existence of one solution if $h \in O_2$. It is only in 1978 that E.N. Dancer [6] and H. Amann–P. Hess [1] have obtained independently the multiplicity result by combining lower and upper solutions technique with degree theory. An exact count of solutions was then obtained by H. Berestycki [3] in case f is convex.

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See also the survey paper of D.G. de Figueiredo [7]. Later, R. Chiappinelli–J. Mawhin–R. Nugari [5] have considered the problem

$$
\begin{aligned}
&u'' + u + f(t,u) = \nu\,\varphi(t)\ , \\
&u(0) = 0\,, \qquad u(\pi) = 0\ ,
\end{aligned} \tag{1}
$$

where $\varphi(t) = \sqrt{\frac{2}{\pi}}\,\sin t$. They have proved that, if $\lim_{|u|\to\infty} f(t,u) = +\infty$, uniformly in t, then there exist ν_0 and $\nu_1 \geq \nu_0$ such that

i) If $\nu < \nu_0$, the problem (1) has no solution;

ii) If $\nu_0 \leq \nu \leq \nu_1$, the problem (1) has at least one solution;

iii) If $\nu_1 < \nu$, the problem (1) has at least two solutions.

The question is then to know whether $\nu_0 = \nu_1$. Under the additional condition that $f(t,u) + Ku$ is nondecreasing on $[-u_0, u_0]$, they prove that $\nu_0 = \nu_1$. C. Fabry [8], using another argument, proves $\nu_0 = \nu_1$ with the help of a Hölder condition. R. Chiappinelli–J. Mawhin–R. Nugari [5] have also considered a generalized case where instead of the coercivity condition they assume some growth restriction related to the classical Landesman–Lazer condition.

As already observed by J.L. Kazdan–F.W. Warner [10], the important point is not that φ is the first eigenfunction but that φ is of one sign. If we consider a function $\varphi > 0$ on $[0,\pi]$, for example $\varphi(t) \equiv 1$ then, using the same argument as R. Chiappinelli–J. Mawhin–R. Nugari [5], we obtain easily the same kind of results with $\nu_0 = \nu_1$. In order better to understand the characterization of the manifold M together with the existence of the interval $[\nu_0, \nu_1]$, we consider in this work the two parameters problem

$$
\begin{aligned}
&u'' + u + f(t,u) = \mu + \nu\,\varphi(t)\ , \\
&u(0) = 0\,, \qquad u(\pi) = 0\ .
\end{aligned} \tag{2}
$$

An other aspect of our work concerns existence of $W^{2,p}$ solutions i.e. we consider the case where f is a L^p-Carathéodory function. With the above coercivity assumption on f, we prove the existence of a nonincreasing Lipschitz function $\mu_0(\nu)$ such that

i) If $\mu < \mu_0(\nu)$, the problem (2) has no solution;

ii) If $\mu = \mu_0(\nu)$, the problem (2) has at least one solution;

iii) If $\mu_0(\nu) < \mu$, the problem (2) has at least two solutions.

This result will be deduced from a more general one where we consider a Landesman–Lazer type condition similar to the one used by R. Chiappinelli–J. Mawhin–R. Nugari [5]. Moreover, under some regularity assumption related to those used by R. Chiappinelli–J. Mawhin–R. Nugari [5] and C. Fabry [8] we prove that μ_0 is decreasing. In the particular case $\mu = 0$ we generalize the results of [5] and [8]. The main techniques are lower and upper solutions together with degree theory.

To end this introduction, let us introduce some notations. For any function $u \in L^1(0,\pi)$, we write

$$\overline{u} := \int_0^\pi u(t)\,\varphi(t)\,dt \ , \tag{3}$$

where $\varphi(t) = \sqrt{\frac{2}{\pi}}\,\sin t$. We abbreviate "almost everywhere" as a.e. We use the space

$$C_0^1([0,\pi]) := \Big\{x \in C^1([0,\pi])\colon\ x(0) = 0,\ x(\pi) = 0\Big\} \ .$$

2 – Preliminary results

Consider the boundary value problem

$$\begin{gathered} x'' + f(t,x) = 0 \ , \\ x(0) = 0\,, \qquad x(\pi) = 0 \ , \end{gathered} \tag{4}$$

where f verifies L^p-Carathéodory conditions, i.e.

i) For a.e. $t \in [0,\pi]$, the function $f(t,\cdot)$ is continuous;

ii) For all $x \in \mathbb{R}$, the function $f(\cdot,x)$ is measurable;

iii) For all $r > 0$, there exists $m \in L^p(0,\pi)$ such that for a.e. $t \in [0,\pi]$ and all $x \in [-r,r]$, $|f(t,x)| \le m(t)$.

A basic notion to prove existence results for the problem (4) is the notion of upper and lower solution. In this paper, however, we will use stronger concepts. In order to use degree theory, we want to define curves

$$x = \beta(t) \quad \text{and} \quad x = \alpha(t)$$

so that solution curves of (4) cannot be tangent to them, respectively from below or from above. This property is basic in defining a degree with respect to sets

$$\Omega = \Big\{x \in C([0,\pi]) \mid \alpha(t) < x(t) < \beta(t)\Big\} \subset C([0,\pi])$$

or

$$\Omega = \left\{x \in C^1([0,\pi]) \mid \alpha(t) < x(t) < \beta(t),\ |x'(t)| < N\right\} \subset C^1([0,\pi]) \ .$$

Definition 1. A function $\beta \in C([0,\pi])$ is a strict upper solution of (4) if it is not a solution of (4), $\beta(0) \geq 0$, $\beta(\pi) \geq 0$ and for any $t_0 \in [0,\pi]$, one of the following is satisfied:

i) $t_0 \in]0,\pi[$ and $D^-\beta(t_0) > D_+\beta(t_0)$;

ii) $t_0 \in \{0,\pi\}$ and $\beta(t_0) > 0$;

iii) There exist an interval $I_0 \subset [0,\pi]$ and $\delta > 0$ such that $t_0 \in \operatorname{int} I_0$ or $t_0 \in I_0 \cap \{0,\pi\}$, $\beta \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $x \in [\beta(t) - \delta \sin t, \beta(t)]$ we have

$$\beta''(t) + f(t,x) \leq 0 \ .$$

Notice that this definition allows the curve $x = \beta(t)$ to have angles, provided they are downward. Also, $\beta(0)$ and $\beta(\pi)$ can be zero. In this case, condition iii) imposes some second order condition near $t = 0$ or $t = \pi$ but restricted to some angular region below the curve $x = \beta(t)$.

The notion of strict lower solutions is defined in a dual way.

Definition 2. A function $\alpha \in C([0,\pi])$ is a strict lower solution of (4) if it is not a solution of (4), $\alpha(0) \leq 0$, $\alpha(\pi) \leq 0$ and for any $t_0 \in [0,\pi]$, one of the following is satisfied:

i) $t_0 \in]0,\pi[$ and $D_-\alpha(t_0) < D^+\alpha(t_0)$;

ii) $t_0 \in \{0,\pi\}$ and $\alpha(t_0) < 0$;

iii) There exist an interval $I_0 \subset [0,\pi]$ and $\delta > 0$ such that $t_0 \in \operatorname{int} I_0$ or $t_0 \in I_0 \cap \{0,\pi\}$, $\alpha \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $x \in [\alpha(t), \alpha(t) + \delta \sin t]$ we have

$$\alpha''(t) + f(t,x) \geq 0 \ .$$

Notice that if f is continuous, a function $\beta \in C^2([0,\pi])$ such that $\beta(0) > 0$, $\beta(\pi) > 0$ and

$$\forall\, t \in]0,\pi[\quad \beta''(t) + f(t,\beta(t)) < 0$$

is a strict upper solution. However, if f is not continuous but L^p-Carathéodory, this is not the case anymore. In fact, even the stronger condition

$$\text{for a.e. } t \in [0,\pi] \quad \beta''(t) + f(t,\beta(t)) \leq -1$$

does not prevent solutions $x(t)$ of (4) to be tangent to the curve $x = \beta(t)$ from below. This is for example the case for the bounded function

$$
\begin{array}{lll}
f(t,x) & := -1 & x \geq 1 \ , \\
 & := \frac{\sin t - x^2}{1 - \sin t} & 1 > x \geq \sin t \ , \\
 & := \sin t & \sin t > x \ ,
\end{array}
$$

$\beta(t) \equiv 1$ and $x(t) \equiv \sin t$.

To obtain such a result, one has to impose additional assumptions on f such as:

(**H-1**) For any $t_0 \in [0,\pi]$ and any bounded set $E \subset \mathbb{R}$, there exists an interval I_0 with $t_0 \in I_0$ and

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \text{ for a.e. } t \in I_0,\ \forall\, y \in E,\ \forall\, x \in [y - \delta \sin t, y] \quad f(t,x) - f(t,y) \leq \varepsilon \ .$$

Typical examples of L^p-Carathéodory functions that satisfy (H-1) are $f(t,x) = g(x) + h(t)$, with g continuous and $h \in L^p$.

Lower solutions must be smaller than upper ones. To make this precise we introduce the following notation.

Definition 3. Let $x, y \in C([0,\pi])$. We write $x \prec y$ if there exists $\varepsilon > 0$ such that for any $t \in [0,\pi]$

$$y(t) - x(t) \geq \varepsilon \sin t \ .$$

The main tool in this paper is the following result, the proof of which is given for the sake of completeness.

Theorem 1. *Assume f satisfies L^p-Carathéodory conditions. Let α, β be strict lower and upper solutions of (4) such that $\alpha \prec \beta$. Then, the problem (4) has at least one solution $x_1 \in W^{2,p}(0,\pi)$ such that*

$$\forall\, t \in [0,\pi] \quad \alpha(t) \leq x_1(t) \leq \beta(t) \ .$$

If, moreover, there exists $r > 0$, such that for all $s < 0$ and all solutions x of

$$x'' + f(t,x) = s \ ,$$
$$x(0) = 0 \ , \quad x(\pi) = 0 \ ,$$

we have

$$\|x\|_\infty < r \ ,$$

then the problem (4) has at least two solutions $x_1, x_2 \in W^{2,p}(0,\pi)$.

Proof: Observe first that the problem (4) is equivalent to the fixed point equation

$$x = Tx := \int_0^\pi G(t,s)\, f(s, x(s))\, ds \ ,$$

where $G(t,s)$ is the Green function of the boundary value problem

$$-x'' = h(t) \ ,$$
$$x(0) = 0\,, \qquad x(\pi) = 0 \ .$$

The operator

$$T\colon\ C_0^1([0,\pi]) \to C_0^1([0,\pi])$$

is completely continuous. Moreover, for every $r > 0$, there exists $R > 0$ such that every solutions x of (4) with $\|x\|_\infty < r$ satisfies $\|x'\|_\infty < R$.

Let us consider the modified problem

$$\text{(5)} \qquad \begin{aligned} & x'' + f(t, \gamma(t,x)) = 0 \ , \\ & x(0) = 0\,, \qquad x(\pi) = 0 \ , \end{aligned}$$

where

$$\begin{aligned} \gamma(t,x) \ &= \alpha(t), && \text{if } \ x < \alpha(t) \ , \\ &= x, && \text{if } \ \alpha(t) \le x \le \beta(t) \ , \\ &= \beta(t), && \text{if } \ x > \beta(t) \ . \end{aligned}$$

We will prove that, if x is a solution of (5) then $\alpha \prec x$. In a similar way, we prove that also $x \prec \beta$. Assume by contradiction that $\alpha \not\prec x$, i.e.

$$\inf_t \frac{x(t) - \alpha(t)}{\sin t} \le 0 \ .$$

This implies that either there exists $\hat t \in]0,\pi[$ such that $\min_t (x-\alpha) = x(\hat t) - \alpha(\hat t) \le 0$ or for any $t \in]0,\pi[$, $x(t) - \alpha(t) > 0$ and for $\hat t = 0$ or π,

$$\liminf_{t \to \hat t} \frac{x(t) - \alpha(t)}{\sin t} = 0 \ .$$

In both cases, one deduces that if $x - \alpha$ has a derivative at $t = \hat t$, then $(x - \alpha)'(\hat t) = 0$. Observe next that we cannot have $x(t) - \alpha(t) = x(\hat t) - \alpha(\hat t) \le 0$ for all $t \in [0,\pi]$. This would imply $x(t) = \alpha(t)$ and α would be a solution of (4). Hence there exists $\bar t$ such that $x(\bar t) - \alpha(\bar t) > x(\hat t) - \alpha(\hat t)$. Assume $\bar t > \hat t$, the other case being similar, and define

$$t_0 = \inf\Big\{ t > \bar t \mid x(t) - \alpha(t) = x(\hat t) - \alpha(\hat t) \Big\} \in]0,\pi] \ .$$

From the definition of strict lower solutions and the contradiction assumption, we can choose I_0, δ and $t_1 \in I_0$ with $t_1 < t_0$ such that

i) for every $t \in]t_1, t_0[$, $x(t) \leq \alpha(t) + \delta \sin t$,

ii) $(x-\alpha)'(t_1) < 0$, $(x-\alpha)'(t_0) = 0$ and

iii) for a.e. $t \in]t_1, t_0[$, $\alpha''(t) + f(t, \gamma(t, x(t))) \geq 0$.

Hence we have the contradiction

$$0 < (x-\alpha)'(t_0) - (x-\alpha)'(t_1) = \int_{t_1}^{t_0} \Big[-f(t, \gamma(t, x(t))) - \alpha''(t)\Big]\, dt \leq 0 \ .$$

Now, let us define the operator

$$\overline{T}\colon\ C_0^1([0,\pi]) \to C_0^1([0,\pi])\colon\ x \to \overline{T}x := \int_0^\pi G(t,s)\, f(s, \gamma(s, x(s)))\, ds \ .$$

Observe that there exists $\overline{R} > 0$ such that $\overline{T}(C_0^1([0,\pi])) \subset B(0, \overline{R})$. So by the properties of the degree,

$$\deg(I - \overline{T}, B(0, \overline{R})) = 1 \ .$$

Moreover, if R is large enough, every fixed point of $\overline{T}$ is in

$$\Omega := \Big\{ x \in C_0^1([0,\pi])\colon\ \alpha \prec x \prec \beta,\ \|x'\| < R \Big\} \ ,$$

whence

$$\deg(I - \overline{T}, \Omega) = \deg(I - \overline{T}, B(0, \overline{R})) = 1 \ .$$

Finally, as T and $\overline{T}$ coincide on $\overline{\Omega}$ we obtain

$$\deg(I - T, \Omega) = \deg(I - \overline{T}, \Omega) = 1 \ .$$

In this way, we have the existence of a solution x_1 of (4) such that, for all $t \in [0,\pi]$, $\alpha(t) \leq x_1(t) \leq \beta(t)$.

Let us prove the second part of the result. Without loss of generality, we can assume $r > \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. Let R be such that, every solution x of (4) with $\|x\|_\infty < r$ satisfies $\|x'\|_\infty < R$. We will prove

$$\deg(I - T, \Omega_1) = 0 \ ,$$

where

$$\Omega_1 := \Big\{ x \in C_0^1([0,\pi])\colon\ \|x\|_\infty < r,\ \|x'\|_\infty < R \Big\} \ .$$

Hence, as $\Omega \subset \Omega_1$ we have, by excision, the existence of a second solution.

We have assumed that, for all $s < 0$, the solutions x of

$$x'' + f(t,x) = s \ ,$$
$$x(0) = 0 \,, \qquad x(\pi) = 0 \ ,$$

i.e. the solutions of

$$x = Tx - s\,\frac{t(\pi - t)}{2} \tag{6}$$

are such that

$$\|x\|_\infty < r \ .$$

We can assume this is true also for $s = 0$. Otherwise the result is proved.

As T maps bounded sets into bounded sets, there exists $r_1 > 0$ such that, for all $x \in \overline{\Omega}_1$,

$$\|x - Tx\|_\infty < r_1 \ .$$

Hence, if s is large enough, the problem (6) has no solution and

$$\deg(I - T, \Omega_1) = \deg\Big(I - T + s\,\frac{t(\pi - t)}{2}, \Omega_1\Big) = 0 \ .$$

We obtain the existence of the second solution by the excision property of the degree. ∎

Remark. For the first part of this theorem to hold, it is not necessary to assume the upper and lower solutions to be strict.

3 – Existence of strict upper and lower solutions

In this section, we consider the boundary value problem

$$\begin{aligned} &x'' + x + f(t,x) = \mu + \nu\,\varphi(t) \ , \\ &x(0) = 0 \,, \qquad x(\pi) = 0 \ , \end{aligned} \tag{7}$$

where $\varphi(t) = \sqrt{\frac{2}{\pi}}\sin t$ and f verifies L^p-Carathéodory conditions.

Proposition 2. *Let β be a solution of problem (7) with $(\mu,\nu) = (\mu_0,\nu_0)$. Assume (H-1) is satisfied.*

Then β is a strict upper solution of (7) for any (μ,ν) such that

$$\mu > \mu_0 \quad \text{ and } \quad \mu + \sqrt{\frac{2}{\pi}}\,\nu > \mu_0 + \sqrt{\frac{2}{\pi}}\,\nu_0 \ .$$

Proof: Let $\varepsilon > 0$ be such that

$$\mu \geq \mu_0 + \varepsilon \quad \text{and} \quad \mu + \sqrt{\frac{2}{\pi}}\,\nu \geq \mu_0 + \sqrt{\frac{2}{\pi}}\,\nu_0 + \varepsilon \ .$$

It follows that, in case $\nu \leq \nu_0$,

$$\mu + \nu\,\varphi(t) \geq \mu_0 + \nu_0\,\varphi(t) + \left(\sqrt{\frac{2}{\pi}} - \varphi(t)\right)(\nu_0 - \nu) + \varepsilon \geq \mu_0 + \nu_0\,\varphi(t) + \varepsilon \ ,$$

and if $\nu > \nu_0$,

$$\mu + \nu\,\varphi(t) = \mu_0 + \nu_0\,\varphi(t) + (\mu - \mu_0) + (\nu - \nu_0)\,\varphi(t) \geq \mu_0 + \nu_0\,\varphi(t) + \varepsilon \ .$$

For any $t_0 \in [0,\pi]$, we can find from (H-1) an interval I_0 containing t_0 and $\delta > 0$, such that for a.e. $t \in I_0$ and all $x \in [\beta(t) - \delta \sin t, \beta(t)]$, we have $f(t,x) - f(t,\beta(t)) \leq \varepsilon$. Hence, for such (t,x),

$$\begin{aligned}\beta''(t) + x + f(t,x) - (\mu + \nu\,\varphi(t)) &\leq \beta''(t) + \beta(t) + f(t,\beta(t)) + (x - \beta(t)) \\ &\qquad + \Big(f(t,x) - f(t,\beta(t))\Big) - (\mu_0 + \nu_0\,\varphi(t)) - \varepsilon \\ &\leq 0\end{aligned}$$

and the thesis follows. ∎

It is easy to find strict upper solutions for large values of μ.

Proposition 3. *Let f satisfy L^p-Carathéodory conditions together with (H-1). Then, for any $\nu \in \mathbb{R}$, if μ is large enough, the problem (7) has a strict upper solution.*

Proof: Let $m \in L^1(0,\pi)$ be such that for a.e. $t \in [0,\pi]$ and all $x \in [-1,1]$, $|f(t,x)| \leq m(t)$ and let $K > 0$ be such that for any $r \in L^1(0,\pi)$ with $\overline{r} = 0$ (see (3)), the solution of

$$\text{(8)} \qquad \begin{gathered} u'' + u + r(t) = 0 \ , \\ u(0) = 0 \, , \quad u(\pi) = 0 \ , \end{gathered}$$

with $\overline{u} = 0$, verifies

$$\|u\|_\infty \leq K\,\|r\|_{L^1} \ .$$

Next, we choose $p \in C([0,\pi])$ such that $\|m - p\|_{L^1} < \frac{1}{3K}$ and define β to be the solution of (8) with $r = m - p - (\overline{m} - \overline{p})\,\varphi$. We have $\|\beta\|_\infty \leq 1$ and therefore

$$\begin{aligned}\beta''(t) + \beta(t) + f(t,\beta(t)) &\leq \beta''(t) + \beta(t) + m(t) \\ &= p(t) + (\overline{m} - \overline{p})\,\varphi(t) \leq \mu_0 + \nu\,\varphi(t)\end{aligned}$$

if μ_0 is large enough. As in the proof of Proposition 2, we deduce now that β is a strict upper solution for $\mu > \mu_0$. ∎

To build strict lower solutions, we need the nonlinearity to be large enough for large negative values of x.

Proposition 4. *Let f satisfy L^p-Carathéodory conditions. Assume*

(H-2) *There exist $\widehat{\nu} \in \mathbb{R}$ and a function $h \in L^1(0,\pi)$ such that*

$$\int_0^\pi h(t)\,\varphi(t)\,dt > \widehat{\nu}$$

and

$$\liminf_{x\to-\infty} f(t,x) \geq h(t) \tag{9}$$

uniformly in t.

Then, for any $z \in C_0^1([0,\pi])$ and each (μ,ν) such that

$$2\sqrt{\frac{2}{\pi}}\,\mu + \nu \leq \widehat{\nu}$$

the problem (7) has a strict lower solution α such that

$$\alpha \prec z\ .$$

Proof: Let us choose $\varepsilon > 0$ such that

$$2\sqrt{\frac{2}{\pi}}\,\varepsilon < \frac{1}{2}\left[\int_0^\pi h(t)\,\varphi(t)\,dt - 2\sqrt{\frac{2}{\pi}}\,\mu - \nu\right]\ .$$

From (9), we can pick $r > 0$ such that for a.e. t and all $x \leq -r$,

$$f(t,x) \geq h(t) - \varepsilon\ . \tag{10}$$

From the Carathéodory conditions and (10), there exists $m \in L^1(0,\pi)$ such that for a.e. t and all $x \in]-\infty, r]$

$$f(t,x) \geq -m(t)\ .$$

Let us choose next $\delta > 0$ small enough so that

$$\int_{F_\delta} \Big(m(t) + h(t)\Big)\,\varphi(t)\,dt < \frac{1}{2}\left[\int_0^\pi h(t)\,\varphi(t)\,dt - 2\sqrt{\frac{2}{\pi}}\,\mu - \nu\right]\ ,$$

where $F_\delta := [0,\delta] \cup [\pi-\delta,\pi]$. The function

$$\begin{aligned} g(t) \quad &:= -m(t) - \mu, &&\text{if } \ t \in F_\delta \ , \\ &:= h(t) - (\mu+\varepsilon), &&\text{if } \ t \in I_\delta = [\delta, \pi-\delta] \ , \end{aligned}$$

is such that

$$\begin{aligned} \overline{g} &= \int_0^\pi g(t)\,\varphi(t)\,dt \\ &= \left[\int_0^\pi h(t)\,\varphi(t)\,dt - 2\sqrt{\frac{2}{\pi}}\,\mu\right] - \int_{F_\delta} \Big(m(t)+h(t)\Big)\,\varphi(t)\,dt - \varepsilon \int_{I_\delta} \varphi(t)\,dt > \nu \ . \end{aligned}$$

Define now w to be the solution of

$$\begin{gathered} w'' + w + g(t) - \overline{g}\,\varphi(t) = 0 \ , \\ w(0) = 0\,, \qquad w(\pi) = 0 \ , \end{gathered}$$

and choose a negative enough so that

$$\begin{aligned} \alpha(t) \quad &:= a\,\varphi(t) + w(t) \le 0 &&\text{on } \ F_\delta \ , \\ &:= a\,\varphi(t) + w(t) \le -2r &&\text{on } \ I_\delta \ . \end{aligned}$$

The function α is a strict lower solution since

$$\alpha(0) = \alpha(\pi) = 0$$

and for a.e. $t \in [0,\pi]$ and all $x \in [\alpha(t), \alpha(t) + r\sin t]$,

$$\begin{aligned} &\alpha''(t) + x + f(t,x) - \mu - \nu\,\varphi(t) \ge \\ &\qquad\qquad \ge \alpha''(t) + \alpha(t) + g(t) - \nu\,\varphi(t) = (\overline{g} - \nu)\,\varphi(t) \ge 0 \ . \ \blacksquare \end{aligned}$$

4 – Existence results

Theorem 5. *Let f satisfy L^p-Carathéodory conditions together with (H-1) and (H-2).*

Then, there exists a nonincreasing, Lipschitz function $\mu_0 \colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with

$$2\sqrt{\frac{2}{\pi}}\,\mu_0(\nu) + \nu \le \widehat{\nu}$$

such that

i) *if* $\mu < \mu_0(\nu)$, *the problem (7) has no solution;*

ii) *if* $\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(\hat{\nu} - \nu)$, *the problem (7) has at least one solution.*

Proof: Step 1 – *Definition of* μ_0. Define $\mu_0(\nu) \in \mathbb{R} \cup \{-\infty\}$ by

$$\mu_0(\nu) := \inf\Big\{\mu \mid 2\sqrt{\frac{2}{\pi}}\,\mu + \nu = \hat{\nu} \text{ or (7) has a solution for } (\mu,\nu)\Big\} \ .$$

Let (μ,ν) be given such that

$$\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(\hat{\nu} - \nu) \ .$$

By definition, there exists $\mu_1 \in [\mu_0(\nu), \mu[$ and a solution x_1 of (7) for (μ_1,ν). From Proposition 2, x_1 is a strict upper solution of (7) for (μ,ν), from Proposition 4 there exists a strict lower solution $\alpha \prec x_1$ and at last we obtain from the first part of Theorem 1 the existence of a solution of (7) for the given (μ,ν).

Step 2 – *The function* $\mu_0(\nu)$ *is nonincreasing and Lipschitz.* Let ν_2 be such that $\mu_0(\nu_2) < \frac{1}{2}\sqrt{\frac{\pi}{2}}(\hat{\nu} - \nu_2)$. For any $\eta > 0$, small enough, there exist

$$\mu_2 \in \Big[\mu_0(\nu_2), \mu_0(\nu_2) + \eta\Big] \subset \Big[\mu_0(\nu_2), \frac{1}{2}\sqrt{\frac{\pi}{2}}(\hat{\nu} - \nu_2)\Big]$$

and a solution x_2 of (7) for (μ_2,ν_2). Let $\nu_1 < \nu_2$ be close enough to ν_2. From Propositions 2 and 4, x_2 is a strict upper solution of (7) for (μ_1,ν_1) such that $\mu_1 > \mu_2 + \sqrt{\frac{2}{\pi}}(\nu_2 - \nu_1)$ and there exists a strict lower solution $\alpha \prec x_2$ if $2\sqrt{\frac{2}{\pi}}\mu_1 + \nu_1 \le \hat{\nu}$. This together with the first part of Theorem 1 implies $\mu_0(\nu_1) \le \mu_1$. Further, we can choose μ_1 so that $\mu_1 \le \mu_2 + \sqrt{\frac{2}{\pi}}(\nu_2 - \nu_1) + \eta$. It follows that

$$\mu_0(\nu_1) \le \mu_1 \le \mu_0(\nu_2) + \sqrt{\frac{2}{\pi}}\,(\nu_2 - \nu_1) + 2\eta$$

and as η is arbitrary,

$$\mu_0(\nu_1) \le \mu_0(\nu_2) + \sqrt{\frac{2}{\pi}}\,(\nu_2 - \nu_1) \ .$$

On the other hand, there exists a strict upper solution β of (7) for (μ_2,ν_2) such that $\mu_2 > \mu_0(\nu_1)$. Using the same arguments, it follows that

$$\mu_0(\nu_2) \le \mu_0(\nu_1) \ .$$

Hence, the claim follows. ∎

Notice that this theorem allows two degenerate cases.

a) The function $f(t,x) := -x$ satisfies (H-1) and (H-2) and the corresponding problem (7) has a unique solution for any (μ,ν), i.e. $\mu_0(\nu) = -\infty$.

b) The function $f(t,x) := 0$ satisfies also (H-1) and (H-2) and (7) has a solution if and only if

$$0 = \int_0^\pi (\mu + \nu\,\varphi(t))\,\varphi(t)\,dt = 2\sqrt{\frac{2}{\pi}}\,\mu + \nu\ .$$

In this case, $h(t) \le 0$, we must choose $\widehat{\nu} < 0$ and there is no solution if

$$\mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(\widehat{\nu} - \nu) < -\frac{1}{2}\sqrt{\frac{\pi}{2}}\,\nu\ ,$$

i.e. $\mu_0(\nu) = \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu)$.

To rule out the case where $\mu_0(\nu) = -\infty$, we shall impose some lower bound on f.

Theorem 6. *Let f satisfy L^p-Carathéodory conditions together with (H-1), (H-2) and*

(**H-3**) *There exists $k \in L^1(0,\pi)$ such that for a.e. $t \in [0,\pi]$ and all $x \in \mathbb{R}$, $f(t,x) \ge k(t)$.*

Then, there exists a nonincreasing Lipschitz function $\mu_0 \colon \mathbb{R} \to \mathbb{R}$ with

$$2\sqrt{\frac{2}{\pi}}\,\mu_0(\nu) + \nu \le \widehat{\nu}$$

such that

i) *if $\mu < \mu_0(\nu)$, the problem (7) has no solution;*

ii) *if $\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu)$, the problem (7) has at least one solution.*

Proof: Claim 1 – *Given any $\nu \in \mathbb{R}$, problem (7) has no solution for μ negative enough.* For any solution x of (7), multiplying the equation by φ and integrating, one obtains a lower bound on μ

$$\int_0^\pi k(t)\,\varphi(t)\,dt \le \int_0^\pi f(t,x(t))\,\varphi(t)\,dt = 2\sqrt{\frac{2}{\pi}}\,\mu + \nu\ .$$

Step 2 – The remainder of the proof follows from Theorem 5. ∎

To make sure that there is a non empty region

$$\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(\widehat{\nu} - \nu) ,$$

where there is at least one solution, we reinforce the assumption on the nonlinearity for large negative values of x.

Theorem 7. *Let f satisfy L^p-Carathéodory conditions together with (H-1), (H-3) and*

(H-2*) $\liminf_{x \to -\infty} f(t,x) = +\infty$, *uniformly in t.*

Then, there exists a nonincreasing Lipschitz function $\mu_0\colon \mathbb{R} \to \mathbb{R}$ such that

i) *if $\mu < \mu_0(\nu)$, the problem (7) has no solution;*

ii) *if $\mu_0(\nu) < \mu$, the problem (7) has at least one solution.*

Proof: From Proposition 3, for any $\nu \in \mathbb{R}$, there exists μ large enough so that problem (7) has a strict upper solution β. Let us take $\widehat{\nu} > 0$ large enough so that

$$2\sqrt{\frac{2}{\pi}}\,\mu + \nu < \widehat{\nu} .$$

Notice that for such a $\widehat{\nu}$, assumption (H-2) is satisfied (choose $h(t) = \sqrt{\frac{\pi}{2}}\widehat{\nu}$). Hence from Proposition 4, the problem (7) has, for (μ,ν), a strict lower solution α such that $\alpha \prec \beta$. We deduce then from Theorem 1 the existence of a solution of (7). Also, from Theorem 6, the function $\mu_0(\nu)$ exists such that $\mu_0(\nu) \le \mu$ and

i) if $\mu < \mu_0(\nu)$, the problem (7) has no solution;

ii) if $\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu)$, the problem (7) has at least one solution.

As $\widehat{\nu}$ is arbitrary large, the conclusion holds for all $\mu > \mu_0(\nu)$. ∎

5 – Multiplicity results

Consider the problem (7) with

$$\begin{aligned} f(t,x) \; &:= -x \qquad \text{if } x \le 0 , \\ &:= 0 \qquad\;\; \text{if } x > 0 . \end{aligned}$$

It is easy to see that $\mu_0(0) = 0$. First, one proves there is no solution if $\nu = 0$ and $\mu < 0$. Next, considering a solution for $\nu = 0$ and $\mu > 0$, the distance between two consecutive zeros of a positive hump is larger than π. Hence, the solution has to be negative and is unique. This means that for $\nu = 0$, Theorem 7 gives an exact count of the number of solutions. To obtain a multiplicity result, we will have to reinforce (H-2) and (H-3), assuming a better control on the nonlinearity for large values of x.

Theorem 8. *Let f satisfy L^p-Carathéodory conditions together with (H-1) and*

(H-4) *There exist $\widehat{\nu} \in \mathbb{R}$ and a function $h \in L^1(0,\pi)$ such that*

$$\int_0^\pi h(t)\,\varphi(t)\,dt > \widehat{\nu}$$

and

$$\liminf_{|x|\to\infty} f(t,x) \geq h(t)$$

uniformly in t.

Then there exists a nonincreasing Lipschitz function $\mu_0\colon \mathbb{R} \to \mathbb{R}$ with

$$2\sqrt{\frac{2}{\pi}}\,\mu_0(\nu) + \nu \leq \widehat{\nu}$$

such that

i) *if $\mu < \mu_0(\nu)$, the problem (7) has no solution;*

ii) *if $\mu = \mu_0(\nu) < \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu)$, the problem (7) has at least one solution;*

iii) *if $\mu_0(\nu) < \mu \leq \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu)$, the problem (7) has at least two solutions.*

To prove this result, we will need the following lemma which provides the necessary a-priori bounds to apply Theorem 1.

Lemma 9. *Suppose the assumptions of Theorem 8 are verified. Then, for all ν, there exists $r > 0$, such that for all μ with $2\sqrt{\frac{2}{\pi}}\mu + \nu \leq \widehat{\nu}$ and all solution x of (7), we have*

$$\|x\|_\infty < r\ .$$

Proof: Let x be a solution of (7) for (μ,ν) such that

$$2\sqrt{\frac{2}{\pi}}\,\mu + \nu \leq \widehat{\nu}\ .$$

Let us write $x(t) = \overline{x}\,\varphi(t) + \widetilde{x}(t)$, with $\overline{x} \in \mathbb{R}$ and $\int_0^\pi \widetilde{x}(t)\,\varphi(t)\,dt = 0$.

It is known [11] that there exists $K > 0$ such that

$$\|\widetilde{x}\|_\infty \le K \int_0^\pi |x''(t) + x(t)|\,\varphi(t)\,dt\ .$$

Using (H-4) and the Carathéodory conditions, we can find $m \in L^1(0,\pi)$ such that

$$\text{for a.e. } t \in [0,\pi],\quad \forall\, x \in \mathbb{R} \qquad f(t,x) \ge m(t)\ . \tag{11}$$

Hence, we compute

$$\begin{aligned} |x''(t) + x(t)| &\le |f(t,x(t))| + |\mu| + |\nu|\,\varphi(t) \\ &\le f(t,x(t)) - m(t) + |m(t)| + |\mu| + |\nu|\,\varphi(t) \end{aligned}$$

and

$$\begin{aligned} &\int_0^\pi |x''(t) + x(t)|\,\varphi(t)\,dt \le \\ &\qquad \le \int_0^\pi f(t,x(t))\,\varphi(t)\,dt + 2\int_0^\pi |m(t)|\,\varphi(t)\,dt + 2\sqrt{\frac{2}{\pi}}\,|\mu| + |\nu| \\ &\qquad \le \int_0^\pi (\mu + \nu\,\varphi(t))\,\varphi(t)\,dt + 2\int_0^\pi |m(t)|\,\varphi(t)\,dt + 2\sqrt{\frac{2}{\pi}}\,|\mu| + |\nu|\ . \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|\widetilde{x}\|_\infty &\le K\Big[2\sqrt{\frac{2}{\pi}}\,(\mu + |\mu|) + (\nu + |\nu|) + 2\int_0^\pi |m(t)|\,\varphi(t)\,dt\Big] \\ &\le 2K\Big[|\widehat{\nu} - \nu| + |\nu| + \int_0^\pi |m(t)|\,\varphi(t)\,dt\Big] =: r\ . \end{aligned}$$

Assume now there exists $(\mu_k)_k$ with $2\sqrt{\frac{2}{\pi}}\mu_k + \nu \le \widehat{\nu}$ and $x_k = \overline{x}_k\varphi + \widetilde{x}_k$ solution of (7) for (μ_k, ν) such that $\|x_k\|_\infty \to \infty$. From the first part of the proof, $\|\widetilde{x}_k\|_\infty$ is bounded and $|\overline{x}_k| \to \infty$. Going to a subsequence, we have $x_k(t) \to \infty$ (or $x_k(t) \to -\infty$) for all $t \in]0,\pi[$. Hence, using (11), Fatou's theorem and (H-4), we obtain

$$\begin{aligned} \widehat{\nu} &\ge \liminf_{k\to\infty}\Big(2\sqrt{\frac{2}{\pi}}\,\mu_k + \nu\Big) = \liminf_{k\to\infty}\int_0^\pi f(t,x_k(t))\,\varphi(t)\,dt \\ &\ge \int_0^\pi h(t)\,\varphi(t)\,dt > \widehat{\nu} \end{aligned}$$

which is a contradiction. ∎

Proof of Theorem 8: Let μ_0 be defined from Theorem 6 and (μ,ν) be such that $\mu_0(\nu) < \mu \le \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu}-\nu)$. From the proof of Theorem 5, we can find strict lower and upper solutions $\alpha \prec \beta$. From Lemma 9, there exists $r > 0$, such that for all $s \le 0$ and all solutions x of

$$x'' + x + f(t,x) = \mu + \nu\,\varphi(t) + s\;,$$
$$x(0) = 0\,, \qquad x(\pi) = 0\;,$$

we have $\|x\|_\infty < r$. Hence by Theorem 1, the problem (7) has at least two solutions.

Let now (μ,ν) be such that $\mu = \mu_0(\nu) < \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu}-\nu)$. We can find $\mu_n > \mu_0(\nu)$ such that $\lim_{n\to\infty}\mu_n = \mu_0(\nu)$ and solutions x_n of (7) for (μ_n,ν). From Lemma 9, there exists $r > 0$ such that $\|x_n\|_\infty \le r$ and from Carathéodory condition, we have $m \in L^1(0,\pi)$ with $|x_n''(t)| \le m(t)$. Using Arzela–Ascoli Theorem, we can find a subsequence $(x_{n_k})_k$ that converges in C^1 to some function $x \in C^1$. Going to the limit in (7), it is easy to see that x solves (7) for $(\mu,\nu) = (\mu_0(\nu),\nu)$. ∎

Just as for Theorem 7, we can obtain a result for all values of (μ,ν).

Theorem 10. *Let f satisfy L^p-Carathéodory conditions together with (H-1) and*

(H-4*) $\liminf_{|x|\to\infty} f(t,x) = +\infty$, *uniformly in t.*

Then there exists a nonincreasing Lipschitz function $\mu_0\colon \mathbb{R} \to \mathbb{R}$ such that

i) *if $\mu < \mu_0(\nu)$, the problem (7) has no solution;*

ii) *if $\mu = \mu_0(\nu)$, the problem (7) has at least one solution;*

iii) *if $\mu_0(\nu) < \mu$, the problem (7) has at least two solutions.*

The proof of this result is similar to the proof of Theorem 7 and will be omitted.

In case $\mu = 0$, Theorem 10 reduces to the following result which extends [5] to L^p-Carathéodory functions.

Corollary 11. *Let f satisfy L^p-Carathéodory conditions together with (H-1) and (H-4*).*

Then there exist ν_0 and $\nu_1 \ge \nu_0$ such that

i) *if $\nu < \nu_0$, the problem*

$$
(12) \qquad \begin{aligned} & x'' + x + f(t,x) = \nu\,\varphi(t) \ , \\ & x(0) = 0\,, \quad x(\pi) = 0\ , \end{aligned}
$$

has no solution;

ii) *if $\nu_0 \le \nu \le \nu_1$, the problem (12) has at least one solution;*

iii) *if $\nu_1 < \nu$, the problem (12) has at least two solutions.*

Notice that ν_0 is different from ν_1 in case $\mu_0(\nu)$ is not a decreasing function.

6 – Decreasing of μ_0

In this section, we give regularity assumptions on f which imply that the function μ_0 given by Theorem 6 is decreasing. The first condition is a one-sided continuity for a norm $\|x\| := \sup_t \frac{|x(t)|}{\varphi(t)}$.

Theorem 12. *Let f satisfy L^p-Carathéodory conditions together with (H-1) and (H-4). Assume*

(**H-5**) *One of the functions $g(t,x) = f(t,x)$ or $g(t,x) = -f(t,x)$ is such that, given $R > 0$, we have*

$$
\begin{aligned} & \forall\,\varepsilon > 0,\ \exists\,\delta > 0, \quad \text{for a.e. } t \in [0,\pi],\ \forall\, x,y \in [-R\varphi(t), R\varphi(t)], \\ & \qquad 0 \le x - y \le \delta\,\varphi(t) \ \Rightarrow\ g(t,y) - g(t,x) \le \varepsilon\,\varphi(t)\ . \end{aligned}
$$

Then, the function $\mu_0(\nu)$ is decreasing.

Proof: Let $\nu_0 \in \mathbb{R}$ be such that

$$
\mu_0(\nu_0) < \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(\widehat{\nu} - \nu_0)\ .
$$

Claim 1 – *For all $\nu > \nu_0$, there exists $\mu < \mu_0(\nu_0)$ and a strict upper solution β of (7) with (μ,ν).*

Assume $g(t,x) = f(t,x)$ in (H-5). The proof is similar in case $g(t,x) = -f(t,x)$. From Theorem 8, there is a solution x_0 of

$$
\begin{aligned} & x_0'' + x_0 + f(t,x_0) = \mu_0(\nu_0) + \nu_0\,\varphi(t)\ , \\ & x_0(0) = 0\,, \quad x_0(\pi) = 0\ . \end{aligned}
$$

Also, there exists $\widetilde{K} > 0$ such that for every $m > 0$, the solution w of

$$w'' + w = -m\left(1 - 2\sqrt{\frac{2}{\pi}}\,\varphi(t)\right) ,$$
$$w(0) = 0 , \qquad w(\pi) = 0 ,$$

with $\overline{w} = 0$, is such that, for all $t \in [0, \pi]$,

$$|w(t)| \le m\,\widetilde{K}\,\varphi(t) \ .$$

Define

$$\beta(t) := x_0(t) + w(t) - m\,\widetilde{K}\,\varphi(t) \ ,$$

where $m \in]0, 1]$ will be chosen later. Observe that

$$0 \le x_0(t) - \beta(t) \le 2\,m\,\widetilde{K}\,\varphi(t) \ .$$

Let $R > 0$ be such that, for all $m \in]0, 1]$,

$$|x_0(t)| < R\,\varphi(t) , \qquad |\beta(t)| < R\,\varphi(t) \ .$$

Define $\varepsilon > 0$ such that

$$\nu_0 + \varepsilon < \nu$$

and pick $\delta > 0$ from assumption (H-5). Next, we choose $m \in]0, 1]$ small enough so that

$$2\,m\,\widetilde{K} \le \delta , \qquad \nu_0 + 2\sqrt{\frac{2}{\pi}}\,m + \varepsilon \le \nu$$

and set $\mu_1 := \mu_0(\nu_0) - m$. Notice that

$$x_0(t) - \beta(t) \le \delta\,\varphi(t) \ ,$$

whence we compute

$$\begin{aligned}
\beta''(t) + \beta(t) + f(t, \beta(t)) &= \\
&= x_0''(t) + x_0(t) + w''(t) + w(t) + f(t, \beta(t)) \\
&= \mu_0(\nu_0) + \nu_0\,\varphi(t) - m\Big(1 - 2\sqrt{\tfrac{2}{\pi}}\varphi(t)\Big) + f(t, \beta(t)) - f(t, x_0(t)) \\
&\le \Big(\mu_0(\nu_0) - m\Big) + \Big(\nu_0 + 2\sqrt{\tfrac{2}{\pi}}\,m + \varepsilon\Big)\,\varphi(t) \\
&\le \mu_1 + \nu\,\varphi(t) \ .
\end{aligned}$$

Using (H-1), we deduce now as in Proposition 2 that β is a strict upper solution for $\mu \in]\mu_1, \mu_0(\nu_0)]$ and ν.

Step 2 – If $2\sqrt{\frac{2}{\pi}}\,\mu + \nu \le \widehat{\nu}$, there exists, from Proposition 4, a strict lower solution $\alpha \prec \beta$ and we obtain, from the first part of Theorem 1, the existence of a solution of (7) for the given (μ, ν). This implies $\mu_0(\nu) \le \mu < \mu_0(\nu_0)$. If $2\sqrt{\frac{2}{\pi}}\mu + \nu > \widehat{\nu}$, we have

$$2\sqrt{\frac{2}{\pi}}\,\mu_0(\nu_0) + \nu > \widehat{\nu} \ge 2\sqrt{\frac{2}{\pi}}\,\mu_0(\nu) + \nu$$

and also $\mu_0(\nu) < \mu_0(\nu_0)$. ∎

The assumption (H-5) can be thought of as some one-sided uniform continuity assumption when $t \ne 0, \pi$, together with a Lipschitz condition near zero. More precisely, we can prove the following result.

Proposition 13. *Let $f(t,x)$ be a continuous function and*

(**H-5***) *There exist $L > 0$ and $r > 0$ such that*

$$f(t,x) + Lx \quad \text{or} \quad -f(t,x) + Lx$$

is nondecreasing for a.e. $t \in [0,\pi]$ and $x \in [-r, r]$.

Then assumption (H-5) holds.

Proof: Assume $f(t,x) + Lx$ is nondecreasing. Let $R > 0$ be given and choose $\tau \in]0, \frac{\pi}{2}]$ so that $R\varphi(\tau) < r$. For $t \in F_\tau := [0,\tau] \cup [\pi - \tau, \pi]$ and all x, y in $[-R\varphi(t), R\varphi(t)]$, we have

$$0 \le x - y \;\Rightarrow\; f(t,y) - f(t,x) \le L(x-y)\ .$$

If $t \in [\tau, \pi - \tau]$, we compute from the continuity of f

$$\text{(13)}\quad \forall\, \varepsilon' > 0,\ \exists\, \delta_2 > 0\,, \qquad |x-y| < \delta_2 \;\Rightarrow\; f(t,y) - f(t,x) \le \varepsilon' \le \frac{\varepsilon'}{\varphi(\tau)}\,\varphi(t)\ .$$

Now, given ε, we choose δ_1 and ε' small enough so that

$$L\,\delta_1 < \varepsilon\,, \qquad \frac{\varepsilon'}{\varphi(\tau)} < \varepsilon\ ,$$

we choose δ_2 from (13) and $\delta := \min(\delta_1, \delta_2\sqrt{\frac{\pi}{2}})$. Now, it is easy to see that if $|x|, |y| < R\varphi(t)$ are such that $0 \le x - y \le \delta\,\varphi(t)$, we have:

for $t \in F_\tau$,

$$f(t,y) - f(t,x) \le L\,\delta\,\varphi(t) < \varepsilon\,\varphi(t) \ ,$$

and for $t \in [\tau, \pi-\tau]$, $0 < x-y < \delta\,\varphi(t)$ implies $|x-y| < \delta_2$ and

$$f(t,y) - f(t,x) \le \frac{\varepsilon'}{\varphi(\tau)}\,\varphi(t) < \varepsilon\,\varphi(t) \ . \ \blacksquare$$

The following corollary is a direct consequence of Theorem 12. It improves a result of P. Habets and P. Omari [9] and also the corresponding one in R. Chiappinelli–J. Mawhin–R. Nugari [5].

Corollary 14. *Let f satisfy L^p-Carathéodory conditions together with (H-1), (H-4*) and (H-5). Then there exists $\nu_0 \in \mathbb{R}$ such that*

i) *for $\nu < \nu_0$, the problem (12) has no solution;*

ii) *for $\nu = \nu_0$, the problem (12) has at least one solution;*

iii) *for $\nu > \nu_0$, the problem (12) has at least two solutions.*

Our second result considers a Hölder condition on the function f.

Theorem 15. *Let f be a continuous function that satisfies (H-4) and*

(**H-6**) *Either $g(t,x) = f(t,x)$ or $g(t,x) = -f(t,x)$ is such that, for some $L \ge 0$, $\alpha \in]1/3, 1[$ and $r > 0$, for a.e. t and all $x, y \in [-r, r]$ with $0 \le x-y \le 1$, we have*

$$g(t,x) - g(t,y) \le L(x-y)^\alpha \ .$$

Then, the function $\mu_0(\nu)$ is decreasing.

Proof: Let ν_0 be such that $\mu_0(\nu_0) < \frac{1}{2}\sqrt{\frac{\pi}{2}}(\widehat{\nu} - \nu_0)$. We shall prove that, for all $\nu > \nu_0$, there exists $\mu < \mu_0(\nu_0)$ such that the problem (7) has a strict upper solution β. Then, by Proposition 4 there exists a strict lower solution $\alpha \prec \beta$ and we obtain, from the first part of Theorem 1, the existence of a solution of (7) for the given (μ, ν).

Case 1 – *Assume $g(t,x) = f(t,x)$.* Let $\nu > \nu_0$ and x_0 be a solution of

$$\begin{aligned} &x'' + x + f(t,x) = \mu_0(\nu_0) + \nu_0\,\varphi(t) \ , \\ &x(0) = 0\,, \quad x(\pi) = 0 \ . \end{aligned} \tag{14}$$

Let $\tau \le \pi/2$ be such that $|x_0(t)| \le r/2$ on $[0,\tau] \cup [\pi-\tau,\pi]$ and $R > \max|x_0(t)|$. By continuity, we have δ such that, if $|x|, |y| \le R+1$, $|x-y| \le \delta$ and $t \in [\tau, \pi-\tau]$,

$$|f(t,y) - f(t,x)| \le \frac{(\nu-\nu_0)}{2}\,\varphi(\tau)\ .$$

Let $\varepsilon > 0$ be a quantity to be chosen later but small enough so that

$$\varepsilon \le \frac{r}{2}\,, \qquad \varepsilon \le 1 \ \text{ and } \ \varepsilon \le \delta\ . \tag{15}$$

Let also $A := (L+1)\,\varepsilon^\alpha$. The problem

$$\begin{gathered} w'' + w = -A\ , \\ w(0) = 0\,, \qquad w(t_1) = \varepsilon\,, \qquad w'(t_1) = 0\ , \end{gathered}$$

has a solution

$$\overline{w}(t) := -A + A\cos t + (A+\varepsilon)\sin t_1 \sin t$$

with t_1 such that

$$\cos t_1 := \frac{A}{A+\varepsilon}, \qquad \text{i.e. } \ \varphi(t_1) = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\varepsilon^2 + 2\,\varepsilon\, A}}{A+\varepsilon} = O(\varepsilon^{\frac{1-\alpha}{2}})\ .$$

Let us choose ε small enough so that $t_1 \le \tau$. Next we define

$$\begin{aligned} w(t) \ &:= \overline{w}(t), && \text{if } \ t \in [0,t_1]\ , \\ &:= \varepsilon, && \text{if } \ t \in [t_1, \pi - t_1]\ , \\ &:= \overline{w}(\pi - t), && \text{if } \ t \in [\pi - t_1, \pi] \end{aligned}$$

and $\beta(t) := x_0(t) + w(t)$. Observe that $\beta \in \mathcal{C}^1([0,\pi]) \cap W^{2,1}(0,\pi)$ and for $t \in [0,t_1[\cup]\pi - t_1, \pi]$, we can write

$$\begin{aligned} \beta''(t) + \beta(t) + f(t,\beta(t)) &= x_0''(t) + x_0(t) + f(t,x_0(t)) + \overline{w}''(t) + \overline{w}(t) \\ &\quad + f(t,\beta(t)) - f(t,x_0(t)) \\ &\le \mu_0(\nu_0) + \nu_0\,\varphi(t) - A + L\,\overline{w}^\alpha \\ &\le \mu + \nu\,\varphi(t) + \Big(\mu_0(\nu_0) - \mu - \varepsilon^\alpha\Big) + (\nu_0 - \nu)\,\varphi(t)\ . \end{aligned}$$

On $]t_1,\tau[\cup]\pi-\tau, \pi-t_1[$, we have

$$\begin{aligned} \beta''(t) + \beta(t) + f(t,\beta(t)) &\le \mu_0(\nu_0) + \nu_0\,\varphi(t) + \varepsilon + L\,\varepsilon^\alpha \\ &\le \mu + \nu\,\varphi(t) + \Big(\mu_0(\nu_0) - \mu + \varepsilon - \varepsilon^\alpha\Big) \\ &\quad + \left(\nu_0 + \frac{(L+1)\,\varepsilon^\alpha}{\varphi(t_1)} - \nu\right)\varphi(t)\ . \end{aligned}$$

At last, for $t \in]\tau, \pi-\tau[$, we compute

$$\begin{aligned}\beta''(t)+\beta(t)+f(t,\beta(t)) &\le \mu_0(\nu_0)+\nu_0\,\varphi(t)+\varepsilon+\frac{(\nu-\nu_0)}{2}\,\varphi(\tau)\\ &\le \mu+\nu\,\varphi(t)+\Big(\mu_0(\nu_0)-\mu+\varepsilon-\frac{(\nu-\nu_0)}{2}\,\varphi(\tau)\Big)\\ &\quad +(\nu_0-\nu)\,(\varphi(t)-\varphi(\tau))\ .\end{aligned}$$

Now it is easy to see that, as $\alpha \in]1/3,1[$, we can choose $\varepsilon>0$ and next $\mu_0(\nu_0)-\mu>0$ small enough so that in all cases

$$\beta''(t)+\beta(t)+f(t,\beta(t)) \le \mu+\nu\,\varphi(t)\ .$$

Case 2 – Assume $g(t,x)=-f(t,x)$. Let us fix $\nu>\nu_0$ and x_0 solution of (14). Next we choose, as in Case 1, τ, R and δ. At last, $\varepsilon>0$ will be chosen later but is small enough to verify (15). Let $t_1 := \sqrt{6}\,(L+1)^{-1/2}\,\varepsilon^{\frac{1-\alpha}{2}}$, $A := (L+1)\,\varepsilon^\alpha$, $B := \sqrt{\frac{2}{3}}\,(L+1)^{\frac{3}{2}}\,\varepsilon^{\frac{3\alpha-1}{2}}$ and consider the problem

$$\begin{gathered} w''=-A+Bt\ ,\\ w(0)=0,\quad w(t_1)=-\varepsilon,\quad w'(t_1)=0\ .\end{gathered}$$

Its solution reads

$$\overline{w}(t) := -A\,\frac{t^2}{2}+B\,\frac{t^3}{6}\ \in\ [-\varepsilon,0]\ .$$

Next we define

$$\begin{aligned} w(t)\ &:= \overline{w}(t), && \text{if}\ \ t\in[0,t_1]\ ,\\ &:= -\varepsilon, && \text{if}\ \ t\in[t_1,\pi-t_1]\ ,\\ &:= \overline{w}(\pi-t), && \text{if}\ \ t\in[\pi-t_1,\pi]\end{aligned}$$

and $\beta(t) := x_0(t)+w(t)$. Again, we have $\beta\in\mathcal{C}^1([0,\pi])\cap W^{2,1}(0,\pi)$ and for $t\in[0,t_1[$, we can write

$$\begin{aligned}&\beta''(t)+\beta(t)+f(t,\beta(t)) =\\ &\qquad = x_0''(t)+x_0(t)+f(t,x_0(t))+\overline{w}''(t)+\overline{w}(t)+f(t,\beta(t))-f(t,x_0(t))\\ &\qquad \le \mu_0(\nu_0)+\nu_0\,\varphi(t)-A+Bt+L\,|\overline{w}(t)|^\alpha\\ &\qquad \le \mu+\nu\,\varphi(t)+\Big(\mu_0(\nu_0)-\mu-\varepsilon^\alpha\Big)+\Big(\nu_0-\nu+B\,\frac{t}{\varphi(t)}\Big)\,\varphi(t)\ .\end{aligned}$$

On $]t_1, \tau[$ we have

$$
\begin{aligned}
&\beta''(t) + \beta(t) + f(t, \beta(t)) \leq \\
&\qquad \leq \mu_0(\nu_0) + \nu_0\, \varphi(t) - \varepsilon + L\varepsilon^\alpha \\
&\qquad \leq \mu + \nu\, \varphi(t) + \Big(\mu_0(\nu_0) - \mu - \varepsilon\Big) + \left(\nu_0 - \nu + L\, \frac{\varepsilon^\alpha}{\varphi(t_1)}\right) \varphi(t) \ ,
\end{aligned}
$$

and for $t \in]\tau, \pi - \tau[$,

$$
\begin{aligned}
&\beta''(t) + \beta(t) + f(t, \beta(t)) \leq \\
&\qquad \leq \mu_0(\nu_0) + \nu_0\, \varphi(t) - \varepsilon + \frac{\nu - \nu_0}{2}\, \varphi(\tau) \\
&\qquad \leq \mu + \nu\, \varphi(t) + \Big(\mu_0(\nu_0) - \mu - \varepsilon\Big) + (\nu_0 - \nu)\, (\varphi(t) - \varphi(\tau)) \ .
\end{aligned}
$$

On $[\pi - \tau, \pi]$ estimates are similar to those on $[0, \tau]$. It is then easy to see that we can choose $\varepsilon > 0$ and next $\mu_0(\nu_0) - \mu > 0$ both small enough so that

$$\beta''(t) + \beta(t) + f(t, \beta(t)) \leq \mu + \nu\, \varphi(t) \ . \ \blacksquare$$

This result extends an idea due to C. Fabry [8].

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