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DIFFERENTIAL OPERATORS WITH GENERALIZED CONSTANT COEFFICIENTS

S. Pilipović and D. Scarpalézos

Abstract: The classical method of solving the equation $P(D)\,g = f$ is adapted to a method of solving the family of equations with respect to ε with a prescribed growth rate. More precisely, the equation $P_\varepsilon(D)\,U_\varepsilon = H_\varepsilon$ where H_ε is Colombeau's moderate function ($H \in \mathcal{E}_M(\mathbb{R}^n)$) and $P_\varepsilon(D)$ is a differential operator with moderate coefficients in Colombeau's sense, is solved. If $P^j(D) \to P(D)$, $j \to \infty$, in the sense that the coefficients converge in the sharp topology, then there is a sequence E^j of solutions of $P^j(D)\,U = H$ which converges in the sharp topology to a solution E of $P(D)\,U = H$ in $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$. The moderate functions $E^j_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ which converge sharply to $E_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$, such that $P^j_\varepsilon(D)(E^j_\varepsilon|_\Omega) = H_\varepsilon|_\Omega$ and $P_\varepsilon(D)(E_\varepsilon|_\Omega) = H_\varepsilon|_\Omega$, where Ω is a bounded open set, are constructed.

The main problems in presented investigations are the estimates with respect to ε which makes this theory a non-trivial generalization of the classical one.

1 – Introduction

Let $P_\varepsilon(D) = \sum_{|\alpha|\le m} a_{\alpha,\varepsilon}\, D^\alpha$ be a family of constant coefficients differential operators and H_ε, $\varepsilon \in (0,1)$, be a family of smooth functions which satisfy power order estimates with respect to ε. Our purpose is to find a family of solutions E_ε of $P_\varepsilon(D)\,U_\varepsilon = H_\varepsilon$ which satisfy the prescribed power order estimates with respect to ε. Such solutions are intersetting because they solve effective problems of applied mathematics where we have differential operators with coefficients which depend on a parameter ε.

More precisely, we solve the equation $P_\varepsilon(D)\,U_\varepsilon = H_\varepsilon$ where H_ε is a given Colombeau's moderate function and $P_\varepsilon(D)$ is a differential operator with moderate coefficients in Colombeau's sense.

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Moreover, we prove that if $P^j(D) \to P(D)$, $j \to \infty$, in the sense that coefficients converge in the sharp topology ([8]; see also [2]) and H is a given Colombeau's generalized function, then there is a sequence E^j of solutions of $P^j(D)\,U = H$ which converge in the sharp topology to a solution E of $P(D)\,U = H$ in $\mathcal{G}(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $H_\varepsilon \in \mathcal{E}_M$ be a representative of $H \in \mathcal{G}(\mathbb{R}^n)$. We construct a sequence $E^j_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ which satisfies

$$P^j_\varepsilon(D)(E^j_\varepsilon|_\Omega) = H_\varepsilon|_\Omega$$

and E^j_ε converges sharply to E_ε in $\mathcal{E}_M(\mathbb{R}^n)$, where $E_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ satisfies $P_\varepsilon(D)(E_\varepsilon|_\Omega) = H_\varepsilon|_\Omega$.

Our construction of solutions is similar to the construction of a classical distributional solution of $P(D)\,u = h$ which is given in [5]; see also [6] for the general theory. Our main problems appear in proving the necessary estimates with respect to ε.

The computations and proofs will be made in the most simplified model of Colombeau's theory — the space of simplified generalized function. The translation of proofs and results to other models of Colombeau's generalized function spaces is straightforward but with the more complicated notation.

We refer to [1–4] and [9] for the general theory of Colombeau's spaces and its applications to non-linear problems.

2 – Basic definitions

Let V be a topological vector space whose topology is given by a countable set of seminorms μ_k, $k \in \mathbb{N}$. We will define "the polynomially generalized extension" of V, $\mathcal{G}_V$ and its "sharp topology" ([8], see also [2]).

We define $\mathcal{E}_{M,V}$ as the set of locally bounded function $R(\varepsilon) = R_\varepsilon \colon (0,1) \to V$ such that for every $k \in \mathbb{N}$ there exists $a \in \mathbb{R}$ such that

$$\mu_k(R(\varepsilon)) = \mathcal{O}(\varepsilon^a)\ ,$$

where $\mathcal{O}(\varepsilon^a)$ means that the left side is smaller than $C\varepsilon^a$ for some $C > 0$ and every $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$.

The upper bound of such reals a will be called the k-valuation of R_ε and it is denoted by $v_k(R_\varepsilon)$.

We denote by $\mathcal{N}_V$ the space of all elements $H_\varepsilon \in \mathcal{E}_{M,V}$ with the property that for any $k \in \mathbb{N}$ and for any $a \in \mathbb{R}$, $\mu_k(H_\varepsilon) = \mathcal{O}(\varepsilon^a)$.

Note, $\mathcal{N}_V$ is the space of elements H_ε whose all valuations $v_k(H_\varepsilon)$, $k \in \mathbb{N}$, are equal to $+\infty$.

The quotient space $\mathcal{G}_V = \mathcal{E}_{M,V}/\mathcal{N}_V$ is called the polynomially generalized extension of V. The elements of $\mathcal{N}_V$ are called "negligible".

Since $v_k(R_\varepsilon) = v_k(R'_\varepsilon)$ for every $k \in \mathbb{N}_0$ if $R_\varepsilon - R'_\varepsilon \in \mathcal{N}_V$, we can speak of the k-valuation of a class $R = [R_\varepsilon]$, where brackets will be used to denote the equivalence class in the quotient space.

If we put $d_k(F_\varepsilon, G_\varepsilon) = \exp(-v_k(F_\varepsilon - G_\varepsilon))$, we obtain a semi-metric on $\mathcal{E}_{M,V}$, $k \in \mathbb{N}$ and the corresponding metric on $\mathcal{G}_V$. The countable set of those semi-metrics (resp. metrics) defines a uniform structure on $\mathcal{E}_{M,V}$ (resp. $\mathcal{G}_V$) which is called the sharp uniform structure. The induced topology is called the sharp topology. It is proved in [8] that $\mathcal{G}_V$ is complete even when this is not the case for V.

If the space V is an algebra whose products are continuous for all the semi-norms, then $\mathcal{N}_V$ is an ideal of the algebra $\mathcal{E}_{M,V}$ and $\mathcal{G}_V$ becomes a Hausdorff topological ring.

If $V = \mathbb{C}$, then $\mathcal{G}_V$ is called the algebra of generalized constants and it is denoted by $\overline{\mathbb{C}}$; $\mathcal{E}_{M,V}$ is denoted by $\mathcal{E}^0$ and $\mathcal{N}_V$ is denoted by $\mathcal{N}^0$.

If $V = \mathbb{C}^\infty(\Omega)$, then $\mathcal{G}_V$ is called the algebra of generalized functions on Ω and it is denoted by $\mathcal{G}(\Omega)$; $\mathcal{E}_{M,V}$ is denoted by $\mathcal{E}_M(\Omega)$ and $\mathcal{N}_V$ is denoted by $\mathcal{N}(\Omega)$. Let us remind that in order to define a set of seminorms on $C^\infty(\Omega)$ we consider an exhaustive sequence of open sets Ω_k such that $\bigcup_{k=0}^\infty \Omega_k = \Omega$, $\Omega_k \subset\subset \Omega_{k+1}$, $k \in \mathbb{N}_0$. We put

$$\mu_k(f) = \sum_{|\alpha| \le k} \Big(\sup_{x \in \overline{\Omega}_k} |\partial^\alpha f(x)| \Big), \qquad k \in \mathbb{N}_0 \ .$$

The uniform structure on $C^\infty(\Omega)$, defined by this family of seminorms does not depend on the choice of the sequence Ω_k.

It is easy to verify that $\mathcal{G}(\Omega)$ is a differential topological ring where derivations (∂_x) are continuous for its sharp topology and $\overline{\mathbb{C}}$ can be considered as a subalgebra of $\mathcal{G}(\Omega)$.

In order to embed $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ we must first recall the fundamental lemma of Colombeau's theory:

Lemma 1. *Let $\varphi \in C_c^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}(\phi) = \widehat{\phi} \in \mathcal{D}(\mathbb{R}^n)$ and $\widehat{\phi} \equiv 1$ on a neighbourhood of zero. Put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\, \phi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^n$, $\varepsilon \in (0,1)$. Then, $N_\varepsilon(x) = (\varphi * \phi_\varepsilon(x) - \varphi(x))$, $x \in \Omega$, belongs to $\mathcal{N}(\Omega)$.*

The proof follows by using Taylor's expansion, since $\int \phi(x)\, dx = 1$ and $\int x^\alpha\, \phi(x)\, dx = 0$ for $|\alpha| \ge 1$.

We fix once for all the function ϕ of previous lemma and put $I_\phi(\varphi) = [\varphi * \phi_\varepsilon]$. Using this lemma it can be easily verified that if φ and ψ belong to $\mathcal{D}(\mathbb{R}^n)$, then

$$I_\phi(\varphi \cdot \psi) = I_\phi(\varphi) \cdot I_\phi(\psi) \ .$$

If $T \in \mathcal{D}'_c(\Omega) \subset \mathcal{D}'_c(\mathbb{R}^n)$ we put $I_\phi(T) = [T * \phi_\varepsilon(x)]$.

After proving that the presheaf $U \to \mathcal{G}(U)$ (U is open in $\mathbb{R}^n$) is a sheaf, one can prove that the above embeddings can be extended to embeddings of $C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ (cf. [9]). The support of a generalized function H is defined as the complement of the largest open subset $\Omega' \subset \Omega$ such that $H|_{\Omega'} = 0$. This notion is coherent with the embedding I_ϕ because if $T \in \mathcal{D}'(\Omega)$, then $\operatorname{supp} T = \operatorname{supp}(I_\phi T)$.

If G is a generalized function with compact support $K \subset\subset \Omega$ ($G \in \mathcal{G}_c(\Omega)$) and $G_\varepsilon(x)$ is a representative of G, then its integral is defined by

$$\int G\,dx = \Big[\int \psi(x)\, G_\varepsilon(x)\, dx\Big] \ ,$$

where $\psi \in C_0^\infty(\Omega)$, $\psi = 1$ on K. This definition does not depend on ψ.

Likewise, one defines the Fourier transform of $G \in \mathcal{G}_c(\Omega)$ as being the class in $\mathcal{G}(\mathbb{R}^n)$ of the Fourier transform of $G_\varepsilon(x)\,\psi(x)$.

Let $G, F \in \mathcal{G}(\Omega)$. It is said that they are equal in the distribution sense, $G \overset{\mathcal{D}'}{=} H$, if

$$\int \Big(G(x) - F(x)\Big)\, \varphi(x)\, dx = 0, \qquad \text{for any} \quad \varphi \in C_c^\infty(\Omega) \ .$$

3 – Tempered generalized functions

Let us first remind the definitions of $\mathcal{E}_{t,M}(\mathbb{R}^n)$, the space of moderate (tempered) families of C^∞-functions. It is said that R_ε belongs to $\mathcal{E}_{t,M}(\mathbb{R}^n)$ if for any $k \in \mathbb{N}_0$ there exist $a \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that:

$$\sup_{|\alpha| \le k} \left(\sup_{\mathbb{R}^n} \left|\frac{\partial^\alpha R_\varepsilon(x)}{(1+|x|^2)^{m/2}}\right|\right) = \mathcal{O}(\varepsilon^a) \ .$$

The upper bound of such a, $v_{k,m}(R_\varepsilon)$, is called the valuation of R_ε.

The space of elements H_ε of $\mathcal{E}_{t,M}(\mathbb{R}^n)$ with the property that for every $k \in \mathbb{N}_0$ there exists $m \in \mathbb{N}$ such that for every $a \in \mathbb{R}$,

$$\sup_{|\alpha| \le k} \left(\sup_{\mathbb{R}^n} \left|\frac{\partial^\alpha R_\varepsilon(x)}{(1+|x|^2)^{m/2}}\right|\right) = \mathcal{O}(\varepsilon^a) \ ,$$

is denoted by $\mathcal{N}_t(\mathbb{R}^n)$. It is an ideal of $\mathcal{E}_{t,M}(\mathbb{R}^n)$. The quotient space $\mathcal{G}_t(\mathbb{R}^n) = \mathcal{E}_{t,M}(\mathbb{R}^n)/\mathcal{N}_t(\mathbb{R}^n)$ is called the space of tempered generalized functions. It is not a metric space but we can define its sharp topology by using the valuations $v_{k,m}$ and a procedure of injective and projective limits (cf. [8]). Note that $\mathcal{G}_t$ is not a subspace of $\mathcal{G}$ because $\mathcal{N}(\mathbb{R}^n) \cap \mathcal{E}_{t,M}(\mathbb{R}^n) \neq \mathcal{N}_t(\mathbb{R}^n)$ ([4]), but there is a canonical mapping $\mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$ (cf. [9]).

In order to embed $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}_t(\mathbb{R}^n)$ we consider as before the mapping $I_\phi\colon T \to [T * \phi_\varepsilon]$. This embedding respects the products of elements of $\mathcal{O}_M(\mathbb{R}^n)$.

If $G, F \in \mathcal{G}_t(\mathbb{R}^n)$ then they are equal in the sense of tempered distributions, $G \overset{\mathcal{S}'}{=} F$, if

$$\int \Big(G_\varepsilon(x) - F_\varepsilon(x)\Big)\, \psi(x)\, dx \ \in \ \mathcal{N}^0 \quad \text{ for any } \ \psi \in \mathcal{S}(\mathbb{R}^n) \ ,$$

G_ε and F_ε being the representatives of G and F, respectively.

Let $G \in \mathcal{G}_t(\mathbb{R}^n)$ be represented by G_ε. We define the Fourier and inverse Fourier transformations $\mathcal{F}(G)(\xi)$, $\xi \in \mathbb{R}^n$ and $\mathcal{F}^{-1}(G)(t)$, $t \in \mathbb{R}^n$ by the representatives

$$\int e^{ix\xi}\, G_\varepsilon(x)\, \phi(\varepsilon x)\, dx \quad \text{ and } \quad (2\pi)^{-n} \int e^{-it\xi}\, G_\varepsilon(\xi)\, \phi(\varepsilon\, \xi)\, d\xi \ ,$$

respectively. Though in general, $\mathcal{F}^{-1}\, \mathcal{F}\, G \neq G$, we can prove that $\mathcal{F}^{-1}\, \mathcal{F}\, G \overset{\mathcal{S}'}{=} G$.

4 – Generalized polynomials

A formal generalized polynomial in n real variables is an element of $\overline{\mathbb{C}}[x_1, ..., x_n]$. A generalized polynomial function, in short, a generalized polynomial, is a tempered generalized function of the form

$$\sum_{|\alpha| \le m} a_\alpha\, x^\alpha\,, \qquad a_\alpha \in \overline{\mathbb{C}} \ .$$

We say that such a generalized function is of degree m if $a_\alpha = 0$ for $|\alpha| > m$ and there exists β, $|\beta| = m$, such that $a_\beta \neq 0$. The corresponding differential operator is of the form

$$P(D) = \sum_{|\alpha| \le m} a_\alpha\, D^\alpha, \quad a_\alpha \in \overline{\mathbb{C}}, \qquad \text{where } \ D^\alpha = (i\, \partial)^\alpha \ . \tag{1}$$

If $\sum_{|\beta| \le m} [b_{\beta,\varepsilon}]\, x^\beta = N_\varepsilon(x) \in \mathcal{N}_t(\mathbb{R}^n)$, then by making successive derivations and by putting $x = 0$ we obtain that $b_{\beta,\varepsilon} \in \mathcal{N}^0$, $|\beta| \le m$. This implies:

Lemma 2. *If* $[H^i_\varepsilon(x)] = \sum_{|\alpha|\le m}[a^i_{\alpha,\varepsilon}]\,x^\alpha$, $i=1,2$, *are representatives of a generalized polynomial then*

$$a^1_{\alpha,\varepsilon} - a^2_{\alpha,\varepsilon} \;\in\; \mathcal{N}^0\,, \qquad |\alpha| \le m\ .$$

Let $\mathcal{P}_m$ be the set of generalized polynomials P of degree at most m ($\deg P \le m$). If P^j is a sequence of generalized polynomials belonging to $\mathcal{P}_m$, we say that P^j m-sharply converges to $P \in \mathcal{P}_m$ ($j \to \infty$) if P^j_ε, $j \in \mathbb{N}$ and P_ε being polynomial representatives of P^j and P, respectively, satisfy the following condition:

(*) For any $a>0$ there exists $j_0 \in \mathbb{N}$ such that for every $j > j_0$,

$$\sup_{x\in\mathbb{R}^n}\left|\frac{P^j_\varepsilon(x) - P_\varepsilon(x)}{(1+|x|^2)^{m/2}}\right| \stackrel{\text{def}}{=} \mu_m\Big(P^j_\varepsilon(x) - P_\varepsilon(x)\Big) = \mathcal{O}(\varepsilon^a)\ .$$

Note that we can choose the representatives P^j_ε and P_ε such that for every $a>0$ there exist $j(a)\in\mathbb{N}$ and $\varepsilon(a)>0$ such that $\mu_m(P^j_\varepsilon - P_\varepsilon) < \varepsilon^a$ for $j > j(a)$ and $\varepsilon < \varepsilon(a)$.

More generally, we say that a sequence P^j of generalized polynomials converges sharply to a generalized polynomial P if there exists some m such that (*) holds.

This convergence implies the sharp convergence of coefficients:

Lemma 3. *Let* P^j *converge sharply to* P, *i.e. let (*) hold. Moreover, let* $\deg P \le m$. *Then there exists* j_0 *such that* $\deg(P^j) \le m$ *for* $j > j_0$ *and the coefficients* $a^j_{\alpha,\varepsilon}$ *of* P_j *converge sharply to the corresponding coefficients* $[a_{\alpha,\varepsilon}]$, $|\alpha| \le m$, *of* P.

Proof: By assumptions, there exists $j_0\in\mathbb{N}$ such that for every $j>j_0$ there exists $\varepsilon_j > 0$ such that

$$\frac{|P^j_\varepsilon(x) - P_\varepsilon(x)|}{(1+|x|^2)^{m/2}} < 1, \qquad \text{for all}\ \ x\in\mathbb{R}^n\ \ \text{and}\ \ \varepsilon < \varepsilon_j\ . \tag{2}$$

Suppose that the degree d of $[H^j_\varepsilon] = [P^j_\varepsilon - P_\varepsilon]$ is greater than m and take $\varepsilon_1 < \varepsilon_j$ and $j_1 > j_0$. Let $x_0 \in \mathbb{R}^n$ be such that the homogeneous part of degree d of $H^{j_1}_\varepsilon$, is not zero at x_0. There exist $C>0$ and $t_0>0$ such that

$$\left|\frac{P^{j_1}_{\varepsilon_1}(t\,x_0) - P_{\varepsilon_1}(t\,x_0)}{(1+|x_0|^2)^{m/2}}\right| > C\,t^{d-m}\,, \qquad t > t_0\ .$$

This contradicts assumption (2) if $d > m$.

Let us note that the vector space of classical polynomials of degree at most m has finite dimension. Thus, all the norms on this space are equivalent. Since

$$\mu_m(P) = \sup_{x\in\mathbb{R}^n} \left| \frac{P(x)}{(1+|x|^2)^{m/2}} \right|$$

is a norm, there exists a constant $C > 0$ such that for every polynomial h of degree $\leq m$, $\|h\| \leq C\,\mu_m(h)$, where

$$\|h\| = \sup_{|\alpha|\leq m} |a_\alpha| , \qquad h = \sum_{|\alpha|\leq m} a_\alpha\, x^\alpha .$$

Thus, $\mu_m(P^j_\varepsilon - P_\varepsilon) = \mathcal{O}(\varepsilon^n)$ for every $a > 0$, is equivalent to

$$|a^j_{\alpha,\varepsilon} - a_{\alpha,\varepsilon}| = \mathcal{O}(\varepsilon^a), \qquad \text{for every} \ \ a > 0 \ . \ \blacksquare$$

5 – The solution of $P(D)\,U = H$

Theorem 1, in the setting of distribution theory, is a famous classical result. In its proof we follow the ideas of Friedman's proof, [5], Ch. 11, Sec. 5. Our main problems are estimations with respect to ε.

Theorem 1. *Let P be a differential operator with coefficients in $\overline{\mathbb{C}}$ of the form (1) such that*

$$\left| \sum_{|\alpha|=m} a_{\alpha,\varepsilon}\, c^\alpha \right| > C\,\varepsilon^r , \qquad \varepsilon \in (0,1) \ , \tag{3}$$

holds for some $c \in \mathbb{R}^n$, $C > 0$ and $r \in \mathbb{R}$. Then for every $H_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ there exists a solution $E_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ of the equation

$$P_\varepsilon(D)\,U_\varepsilon = H_\varepsilon \ .$$

In particular, $[E_\varepsilon]$ is the solution of $P(D)\,U = H$.

Proof: *Case I* – Let

$$P_\varepsilon\left(i\,\frac{\partial}{\partial x}\right) = P_\varepsilon(D) = a_{m,\varepsilon}\, D_1^m + \sum_{k=0}^{m-1} P_{k,\varepsilon}(D')\, D_1^k \ .$$

Therefore,

$$P_\varepsilon(s_1, s') = a_{m,\varepsilon}\, s_1^m + \sum_{k=0}^{m-1} P_{k,\varepsilon}(s')\, s_1^k \,, \qquad s = (s_1, s') \in \mathbb{C}^n \ , \tag{4}$$

and condition (3) means that there exist $C>0$ and $r>0$ such that

$$|a_{m,\varepsilon}| > C\,\varepsilon^r\,, \qquad \varepsilon\in(0,1)\ . \tag{5}$$

Let h_ν be a C^∞ partition of unity

$$h_\nu = \frac{g(x-\nu)}{\sum_{\mu\in\mathbf{Z}^n} g(x-\mu)}\,, \qquad \nu\in\mathbf{Z}^n\ ,$$

where $\mathbf{Z}$ is the set of integers and $g\in C^\infty$ such that $g(x)\ge 0$, $g(x)=1$ if $|x_j|\le\frac12$, $j=1,...,n$, $g(x)=0$ if $|x_j|\ge 1$, $j=1,...,n$. Put

$$H_{\nu,\varepsilon} = h_\nu\,H_\varepsilon\,, \qquad \nu\in\mathbf{Z}^n,\quad \varepsilon\in(0,1)\ .$$

For every $\alpha\in\mathbb{N}_0^n$ and $\nu\in\mathbf{Z}^n$, there exist $C_{\alpha,\nu}>0$ and $r_{\alpha,\nu}>0$ such that

$$\sup\Big\{|D^\alpha H_{\nu,\varepsilon}(x)|;\ |x_k-\nu_k|\le 1,\ k=1,...,n\Big\} \le \frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}}\,, \qquad \varepsilon\in(0,1)\ .$$

Put $\Lambda_\alpha=\{\nu\in\mathbb{R}^n;\ \max|\nu_i|\ge|\alpha|,\ i=1,...,n\}$. Let $h\in\mathbb{N}$ and $\varepsilon\in(0,1)$. Define

$$B_{h,\varepsilon} = 3+\max\left\{\frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}};\ 0\le\alpha_k\le h,\ |\nu_k|\le h,\ k=1,...,n\right\}\ ,$$

$$\delta_{\nu,\varepsilon} = \delta_{\nu_1,\varepsilon}\cdots\delta_{\nu_n,\varepsilon},\quad \nu\in\mathbf{Z}^n, \qquad \text{where}\quad \delta_{j,\varepsilon} = \frac{1}{B_{|j|,\varepsilon}(1+|j|^2)},\quad j\in\mathbf{Z}\ .$$

Put $t_{j,\varepsilon}=-\log\delta_{j,\varepsilon}$, $j\in\mathbf{Z}$. Note that for $\nu=(\nu_1,...,\nu_n)\in\Lambda_\alpha$,

$$\frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}}\,\delta_{\nu,\varepsilon} \le \frac{1}{(1+|\nu_1|^2)\cdots(1+|\nu_n|^2)}\,, \qquad \varepsilon\in(0,1)\ .$$

Let $r_\alpha=\max\{r_{\alpha,\nu};\ \max\{|\nu_1|,...,|\nu_n|\}<|\alpha|\}$. Then, there is $C_\alpha>0$ such that

$$\sum_{\nu\in\mathbf{Z}^n}\frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}}\,\delta_{\nu,\varepsilon} < \frac{C_\alpha}{\varepsilon^{r_\alpha}}\,, \qquad \varepsilon\in(0,1)\ .$$

Fix $\nu=(\nu_1,...,\nu_n)\in\mathbf{Z}^n$, and $\varepsilon\in(0,1)$. Put

$$\tau_{k,\varepsilon,\nu_k} = t_{\nu_k,\varepsilon}\,\mathrm{sgn}\,\nu_k\,, \qquad k=1,...,n\ ,$$

where

$$\mathrm{sgn}\,t = \begin{cases} 1, & t\ge 0,\\ -1, & t<0\ . \end{cases}$$

Fix $(\sigma_2, ..., \sigma_n) = \sigma'$ and $\tau_{\nu,\varepsilon} = (\tau_{1,\nu,\varepsilon}, ..., \tau_{n,\nu,\varepsilon})$. We have

$$(6) \quad P_\varepsilon\Big(s_1 + i\tau_{1,\nu,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\nu,\varepsilon}\Big) = a_{m,\varepsilon} \prod_{k=1}^{m} \Big(s_1 - s_1^{(k)}(\sigma', \tau_{\nu,\varepsilon})\Big)\ , \qquad s_1 \in \mathbb{C}^1 .$$

Let $f(\sigma', \tau_{\nu,\varepsilon})$ be a step function with values in the set $\{\frac{1}{m}, \frac{2}{m}, ..., 1\}$, if $\nu_1 \geq 0$ and in $\{-1, -\frac{m-1}{m}, ..., -\frac{1}{m}\}$, if $\nu_1 < 0$, such that for $s_1 = \sigma_1 + if(\sigma', \tau_{\nu,\varepsilon}) + i\tau_{1,\nu,\varepsilon}$

$$(7) \qquad |\operatorname{Im}(s_1 - s_1^{(k)})| \geq \frac{1}{2m}\ , \qquad k = 1, ..., m, \quad \sigma_1 \in \mathbb{C}\ .$$

Thus, (6) implies that there is $C > 0$ such that

$$\Big|P_\varepsilon\Big(\sigma_1 + if(\sigma', \tau_{\nu,\varepsilon}) + i\tau_{1,\nu,\varepsilon},\ \sigma_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\nu,\varepsilon}\Big)\Big| \geq C\,\varepsilon^r, \qquad \text{for} \quad \sigma_1 \in \mathbb{R}\ .$$

If we take $\widetilde{\sigma}'$ instead of σ' (with the same ν) we have

$$P_\varepsilon\Big(s_1 + i\tau_{1,\nu,\varepsilon},\ \widetilde{\sigma}_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \widetilde{\sigma}_n + i\tau_{n,\nu,\varepsilon}\Big) = a_{m,\varepsilon} \prod_{k=1}^{m} \Big(s_1 - s_1^{(k)}(\widetilde{\sigma}', \tau_{\nu,\varepsilon})\Big)\ , \quad s \in \mathbb{C}\ .$$

There exists $\delta(\varepsilon, \nu)$ such that

$$\Big|s_1^{(k)}(\widetilde{\sigma}', \tau_{\nu,\varepsilon}) - s_1^{(k)}(\sigma', \tau_{\nu,\varepsilon})\Big| \leq 2^{-(m+1)} \qquad \text{for} \quad |\widetilde{\sigma}' - \sigma'| < \delta(\varepsilon, \nu)\ .$$

This implies that there is a cube $\prod_{\nu,\sigma',\varepsilon}$ (open) around σ' contained in the ball with radius $\delta(\varepsilon, \nu)$ such that if $\sigma_1 \in \mathbb{R}$ and $(\widetilde{\sigma}_2, ..., \widetilde{\sigma}_n) \in \prod_{\nu,\sigma',\varepsilon}$, then by (7)

$$\Big|\operatorname{Im}\Big(s_1 - s_1^{(k)}(\widetilde{\sigma}', \tau_{\nu,\varepsilon})\Big)\Big| \geq \frac{1}{4m}\ , \qquad k = 1, ..., m\ ,$$

where $s_1 = \sigma_1 + i\tau_{1,\nu,\varepsilon} + if(\sigma', \tau_{\nu,\varepsilon})$, $\sigma_1 \in \mathbb{R}$. This implies that there exists $C > 0$ (which depends on m but not on ν and ε) such that

$$\Big|P_{\nu,\varepsilon}\Big(\sigma_1 + if(\sigma', \tau_{\nu,\varepsilon}) + i\tau_{1,\nu,\varepsilon},\ \widetilde{\sigma}_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \widetilde{\sigma}_n + i\tau_{n,\nu,\varepsilon}\Big)\Big| \geq C\,\varepsilon^r\ ,$$

where $s_1 = \sigma_1 + i\tau_{1,\nu,\varepsilon} + if(\sigma', \tau_{\nu,\varepsilon})$, $\sigma_1 \in \mathbb{R}$.

We cover $\mathbb{R}^{n-1}$ by $\prod_{\nu,\sigma',\varepsilon}$, $\sigma' \in \mathbb{R}^{n-1}$. The Heine–Borel theorem implies that this covering admit a refinement $\Delta_{\nu,k,\varepsilon}$, $k \in \mathbb{N}$, consisting of cubes. Put

$$\Gamma_{\nu,1,\varepsilon} = \overline{\Delta_{\nu,1,\varepsilon}}, \quad \Gamma_{\nu,2,\varepsilon} = \overline{\Delta_{\nu,2,\varepsilon} \backslash \Delta_{\nu,1,\varepsilon}}, \quad ..., \quad \Gamma_{\nu,k,\varepsilon} = \overline{\Delta_{\nu,k,\varepsilon} \backslash \Big(\bigcup_{i=1}^{k-1} \Delta_{\nu,i,\varepsilon}\Big)}, \quad k \in \mathbb{N}\ ,$$

$$T_{\nu,k,\varepsilon} = \Big\{s \in \mathbb{C}^n;\ s_1 = \sigma_1 + i(\tau_{1,\nu,\varepsilon} + f),\ \sigma_1 \in \mathbb{R}^1,\ s_2 = \sigma_2 + i\tau_{2,\nu,\varepsilon},\ ..., \\ s_n = \sigma_n + i\tau_{n,\nu,\varepsilon},\ (\sigma_2, ..., \sigma_n) \in \Gamma_{\nu,k,\varepsilon}\Big\}\ ,$$

where f takes values in $\{\frac{1}{m}, \frac{2}{m}, ..., 1\}$, if $\nu_1 \geq 0$ and in $\{-1, -\frac{m-1}{m}, ..., -\frac{1}{m}\}$ if $\nu_1 < 0$ and is such that (7) holds.

Let $T_{\nu,\varepsilon} = \bigcup_{k=1}^{\infty} T_{\nu,k,\varepsilon}$. We put

$$\begin{aligned} E_\varepsilon(x) &= \left(\frac{1}{2\pi}\right)^n \sum_{\nu\in\mathbf{Z}^n} \int_{T_{\nu,\varepsilon}} \frac{e^{-ixs}\,\widehat{H}_{\nu,\varepsilon}(s)}{P_\varepsilon(s)}\,ds \\ &= \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}^n} \sum_{k=1}^{\infty} \int_{\mathbb{R}^1} d\sigma_1 \int_{\Gamma_{\nu,k,\varepsilon}} \frac{e^{-i[x_1(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f))+\ldots+x_n(\sigma_n+i\tau_{n,\nu,\varepsilon})]}}{P_\varepsilon\Big(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f),\ \ldots,\ \sigma_n+i\tau_{n,\nu,\varepsilon}\Big)} \\ &\quad \cdot H_{\nu,\varepsilon}\Big(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f),\ \ldots,\ \sigma_n+i\tau_{n,\nu,\varepsilon}\Big)\,d\sigma_2\cdots d\sigma_n\,, \qquad x\in\mathbb{R}^n\ . \end{aligned}$$

One can prove by straightforward computations that $P_\varepsilon(D)\,E_\varepsilon = H_\varepsilon$. We are going to prove that $E_\varepsilon \in \mathcal{E}_M$.

Let $\alpha \in \mathbb{N}_0^n$ and $K \subset\subset \mathbb{R}^n$ be given. We will show that there exist $C = C(K,\alpha)$ and $r = r(K,\alpha)$ such that

$$\sup_{x\in K} |E_\varepsilon^{(\alpha)}(x)| \leq \frac{C}{\varepsilon^r}\,, \qquad \varepsilon\in(0,1)\ .$$

Since $H_{\nu,\varepsilon}\in C_0^\infty$, the Paley–Wiener theorem implies that there is $C>0$ such that

$$|s^\alpha\,\widehat{H}_{\nu,\varepsilon}(s)| \leq C\,\frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}}\exp\Big(-\sum_{i=1}^{n} t_{\nu_i,\varepsilon}(|\nu_i|-1)\Big) \qquad \text{for}\quad s\in T_{\nu,\varepsilon}\ .$$

(The Paley–Wiener type theorem for Colombeau's generalized functions are investigated in [7].) Let $\beta = [2]+\alpha$, where $[p] = (p,p,...,p)$. Then

$$|s^\alpha\widehat{H}_{\nu,\varepsilon}(s)| \leq C\,\frac{C_{\beta,\nu}}{\varepsilon^{r_{\beta,\nu}}}\,\frac{1}{(\sigma_1^2+(\tau_{1,\nu,\varepsilon}+f)^2)\prod_{i=2}^n(\sigma_i^2+\tau_{i,\nu,\varepsilon}^2)}\exp\Big(-\sum_{j=1}^{n} t_{\nu_j,\varepsilon}(|\nu_j|-1)\Big).$$

One can simply verify that $(\tau_{1,\nu,\varepsilon}+f)^2\geq 1$, $\tau_{j,\nu,\varepsilon}^2 \geq 1$, $\varepsilon\in(0,1)$, $j=2,...,n$. This implies that there is $C>0$ which does not depend on ν such that

$$\sup_{\varepsilon\in(0,1)} \sum_{k=1}^{\infty}\int_{\mathbb{R}^1}\int_{\Gamma_{\nu,k,\varepsilon}} \frac{d\sigma_1\cdots d\sigma_n}{(\sigma_1^2+(\tau_{1,\nu,\varepsilon}+f)^2)\prod_{i=2}^n(\sigma_i^2+\tau_{i,\nu,\varepsilon}^2)} < C\ .$$

Since

$$\mathrm{Re}(-i\,x\,s) \leq \sum_{i=1}^{n} |x_i|\,(|t_{\nu_i,\varepsilon}|+1), \qquad \text{for}\quad s\in T_{\nu,\varepsilon}\ ,$$

it follows

$$|E_\varepsilon^{(\alpha)}(x)| \le C \sum_{\nu\in\mathbf{Z}} \frac{C_{\alpha+[2],\nu}}{\varepsilon^{r_{\alpha+[2],\nu}}} \cdot \exp\Big(\sum_{i=1}^n |x_i|\,(t_{\nu_i,\varepsilon}+1) - t_{\nu_i,\varepsilon}(|\nu_i|-1)\Big)$$
$$\le C \sum_{\nu\in\mathbf{Z}} \frac{C_{\alpha+[2],\nu}}{\varepsilon^{r_{\alpha+[2],\nu}}} \prod_{i=1}^n \delta_{\nu_i,\varepsilon}^{|\nu_i|-|x_i|-1}\,, \qquad x \in K\ .$$

The set Λ_K of indices $\nu = (\nu_1, ..., \nu_n)$ with the property

$$\max_{x\in K}\Big(|\nu_i| - |x_i| - 1\Big) \le 1$$

is finite. We divide the above sum into two parts over $\nu\in\mathbf{Z}^n\setminus\bigcup\Lambda_K$ and $\nu\in\bigcup\Lambda_K$. The second one satisfies the estimate of the form C/ε^r. It remains to bound the second part. In this case the exponent in $(\delta_{\nu_i,\varepsilon})^a$ satisfies $a \ge 1$ and, as $\delta_{\nu_i,\varepsilon} \le 1$, it follows $(\delta_{\nu_i,\varepsilon})^a \le \delta_{\nu_i,\varepsilon}$.

This implies that there exist $C = C(K,\alpha) > 0$ and $r = r(K,\alpha) > 0$ such that

$$\sup_{x\in K} |E_\varepsilon^{(\alpha)}(x)| \le \frac{C}{\varepsilon^r}\,, \qquad \varepsilon \in (0,1)\ .$$

Case II – Let $P_\varepsilon(D)$ be a representative of $P(x)$ such that its principal symbol $P_{m,\varepsilon} = \sum_{|\alpha|=m} a_{\alpha,\varepsilon}\, x^\alpha$ satisfies (3) for some $C > 0$ and $r > 0$.

Put $s_j = \sum c_{jk}\, t_k$, $s = Mt$, where the members of the matrix M c_{jk} are chosen such that $c_1 = c_{11}$, ..., $c_n = c_{n1}$ and $\det M \ne 0$. Recall, $c \in \mathbb{R}^n$ is from (3). Then,

$$P_\varepsilon(s) = \widetilde{P}_\varepsilon(t) = a_{m,\varepsilon}\, t_1^m + \text{lower order terms}\ ,$$

where

$$|a_{m,\varepsilon}| = |P_{m,\varepsilon}(c)| \ge C\,\varepsilon^r\,, \qquad \varepsilon \in (0,1)\ .$$

Let $\widetilde{E}_\varepsilon$ be a solution of

$$\widetilde{P}_\varepsilon(D)\,\widetilde{E}_\varepsilon(x) = H_\varepsilon({}^{\rm t}M^{-1}x)$$

(${}^{\rm t}M$ is the transpose matrix) constructed in the same way as in Case 1. Then, by using

$$\mathcal{F}(R(Ax))(s) \stackrel{\mathcal{S}'}{=} \frac{1}{\det A}\,\mathcal{F}(R(x))\,({}^{\rm t}A^{-1}s)\,, \qquad s \in \mathbb{R}^n, \quad R \in \mathcal{S}'\ ,$$

where A is a regular linear transformation and $\widetilde{P}_\varepsilon(t) = P_\varepsilon(s)$, we obtain that the solution of $P_\varepsilon(D)\,E_\varepsilon = H_\varepsilon$ is $E_\varepsilon(x) = \widetilde{E}_\varepsilon({}^{\rm t}Mx)$, $x \in \mathbb{R}^n$. ∎

6 – On the sequence of equations $P^j(D)\,U = H$

Let P^j be a sequence of polynomials in $\mathcal{G}(\mathbb{R}^n)$ which m-sharply converges to P as $j \to \infty$, where m is the order of P. For the next theorem we have to analyze the behaviour of zeros of P^j and P. Let $\sigma' = (\sigma_2, ..., \sigma_n)$ and $\tau_\varepsilon = (\tau_{1,\varepsilon}, ..., \tau_{n,\varepsilon})$, $\varepsilon \in (0,1)$, be fixed and P and P^j be written in the canonical form

$$P_\varepsilon\Big(s_1 + i\tau_{1,\varepsilon},\ \sigma_2 + i\tau_{2,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\varepsilon}\Big) = a_{m,\varepsilon} \prod_{p=1}^{m} \Big(s_1 - s_1^{(p)}(\sigma', \tau_\varepsilon)\Big)\ , \tag{8}$$

$$\begin{aligned} P^j_\varepsilon\Big(s_1 + i\tau_{1,\varepsilon},\ \sigma_2 + i\tau_{2,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\varepsilon}\Big) &= a^j_{m,\varepsilon} \prod_{p=1}^{m} \Big(s_1 - s_1^{j(p)}(\sigma', \tau_\varepsilon)\Big) = \\ &= a^j_{m,\varepsilon}\, s_1^m + C^j_{m-1,\varepsilon}(\sigma', \tau_\varepsilon)\, s_1^{m-1} + ...\ , \end{aligned} \tag{9}$$

where $s_1 \in \mathbb{C}$, $j \in \mathbb{N}$, $\varepsilon \in (0,1)$.

First we prove the following lemma.

Lemma 4. *For every $\overline{p} \in \{1, ..., m\}$ and $a > 0$ there exist $C_0 > 0$, $\varepsilon_0 > 0$ and $j_0 \in \mathbb{N}$ such that for every $j > j_0$ and $\varepsilon < \varepsilon_0$ there exists $p(j,\varepsilon) \in \{1, ..., m\}$ such that*

$$\Big|s_1^{j(p(j,\varepsilon))}(\sigma', \tau_\varepsilon) - s_1^{(\overline{p})}(\sigma', \tau_\varepsilon)\Big| \le C_0\, \varepsilon^a (1 + |\tau_\varepsilon|^m)\,(1 + |\sigma'|^m)\,, \qquad \sigma' \in \mathbb{R}^{n-1}. \tag{10}$$

This lemma implies that there exists j_0 such that for every $j > j_0$, $\varepsilon < \varepsilon_0$ and $\sigma' \in \mathbb{R}^{n-1}$ we may rearrange zeros $s_1^{j(p(j,\varepsilon))}(\sigma', \tau_\varepsilon)$ such that on the $\overline{p}$-th place in the above representation of P^j_ε stands $s_1^{j(p(j,\varepsilon))}(\sigma', \tau_\varepsilon)$.

Proof: Let us first prove the following assertion for a polynomial of one variable $g(x) = b_m\, x^m + b_{m-1}\, x^{m-1} + ... + b_0$. There holds

$$g(x) \ne 0 \quad \text{if}\ \ |x| \ge m \sup\left\{ \sqrt[k]{\left|\frac{b_{m-k}}{b_m}\right|},\ k \in \{1, ..., m\} \right\}\ .$$

(All the zeros are inside the ball of radius $m \cdot \sup\{\sqrt[k]{\frac{b_{m-k}}{b_m}},\ k \in \{1, ..., m\}\}$ and the centre at 0.) To see this we just have to write

$$\begin{aligned} |g(x)| &= \left| b_m\, x^m \left[1 + \frac{b_{m-1}}{b_m}\, \frac{1}{x} + ... + \frac{b_0}{b_m}\, \frac{1}{x^m} \right] \right| \\ &\ge |b_m\, x^m| \left[1 - \left(\left|\frac{b_{m-1}}{b_m}\right| \left|\frac{1}{x}\right| + ... + \left|\frac{b_0}{b_m}\right| \frac{1}{|x|^m} \right) \right] > 0\ . \end{aligned}$$

Fix $\overline{p} \in \{1, ..., m\}$. By using [5], Lemma 4.1.1, in one dimension, (for fixed σ') we obtain that for every $a_1 > 0$ there are $C_1 > 0$, $\varepsilon_1 > 0$ and $j_1 \in \mathbb{N}$ such that for $j > j_1$ and $\varepsilon < \varepsilon_1$ there holds

$$\left|s_1^{j(p(j,\varepsilon))}(\sigma', \tau_\varepsilon) - s_1^{(\overline{p})}(\sigma', \tau_\varepsilon)\right| < C_1\, \varepsilon^{a_1} \left(1 + |s_1^{j(p(j,\varepsilon))}|\right)^m ,$$

where C_1 does not depend on j; a_1 is a constant which will be chosen later as big as we need.

The first part of the proof implies that for $p = 1, ..., n$,

$$|s_1^{j(p)}(\sigma', \tau_\varepsilon)| \leq m \sup\left\{ \sqrt[k]{\left|\frac{C^j_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)}{a^j_{m,\varepsilon}}\right|},\ k \in \{1, ..., m\} \right\} .$$

Since the coefficients of P^j converge sharply to the corresponding ones of P, it follows that the polynomials $C^j_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)$ converge $(m-k)$-sharply to $C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)$ and $a^j_{m,\varepsilon}$ converge sharply to $a_{m,\varepsilon}$ as $j \to \infty$.

Note if $C^j_{m-k,\varepsilon}$ is written as a polynomial with respect to σ', then the corresponding coefficients are polynomial functions of coefficients of P^j and powers of τ_ε. By extracting from all of these coefficients $(1 + |\tau_\varepsilon|^m)$, we obtain that for every $a_2 > 0$ there exist $j_2 \in \mathbb{N}$, $\varepsilon_2 > 0$ and $C_2 > 0$ such that for $j > j_2$ and $\varepsilon < \varepsilon_2$,

$$\left|C^j_{m-k,\varepsilon}(\sigma', \tau_\varepsilon) - C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)\right| \leq C_2 (1 + |\tau_\varepsilon|^m)\, \varepsilon^{a_2} (1 + |\sigma'|^m),$$
$$\sigma' \in \mathbb{R}^{n-1}, \quad k = 1, ..., m-1 ,$$

where a_2 will be chosen later as big as we need.

Likewise, for every $d > 0$ there exist $j_d \in \mathbb{N}$ and $\varepsilon_d > 0$ such that

$$|a^j_{m,\varepsilon} - a_{m,\varepsilon}| < \varepsilon^d \quad \text{for} \quad j > j_d, \quad \varepsilon < \varepsilon_d ,$$

where $d > r$ will be chosen later as big as we need. Thus, there are $j_3 \in \mathbb{N}$, $\varepsilon_3 > 0$ and $C_3 > 0$ such that if $j > j_3$ and $\varepsilon < \varepsilon_3$, then $|a^j_m| > C_3\, \varepsilon^r$ and

$$I_k = \sqrt[k]{\left|\frac{C^j_{m-k}(\sigma', \tau_\varepsilon)}{a^j_{m,\varepsilon}}\right|} < \left(\sqrt[k]{\frac{|C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon) - C^j_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)| + |C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)|}{|a^j_{m,\varepsilon}|}}\right)$$
$$\leq C_3^{-1/k}\, \varepsilon^{-r/k} \sqrt[k]{C_2\, \varepsilon^{a_2} (1 + |\tau_\varepsilon|^m)\, (1 + |\sigma'|^m) + |C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)|}\,, \qquad \sigma' \in \mathbb{R}^{n-1}.$$

There exist $N \in \mathbb{N}$ and $C_N > 0$ such that for any $k \in \{1, ..., m\}$

$$|C_{m-k,\varepsilon}(\sigma', \tau_\varepsilon)| \leq \frac{1}{\varepsilon^N}\, C_N (1 + |\tau_\varepsilon|^m)\, (1 + |\sigma'|^m) .$$

Thus, by taking d and a_2 big enough, we can estimate I_k by

$$C_4\,(1+|\tau_\varepsilon|^m)^{\frac{1}{k}}\,(1+|\sigma'|^m)^{\frac{1}{k}}\,\varepsilon^{-\frac{N+r}{k}}\ ,$$

where $C_4>0$ is a suitable constant. This implies (with suitable $C_5>0$) that for every $j>j_3$ and $\varepsilon<\varepsilon_3$

$$\left|s_1^{j(p(j,\varepsilon))}(\sigma',\tau_\varepsilon)-s_1^{(\overline{p})}(\sigma',\tau_\varepsilon)\right|\le C_5\,(1+|\tau_\varepsilon|^m)\,\varepsilon^{a_1-(N+r)/m}\,(1+|\sigma'|^m)\,,\quad \sigma'\in\mathbb{R}^{n-1}.$$

If we choose a_1 such that $a_1-(N+r)/m=a$, $\varepsilon_0=\varepsilon_3$ and $j_0=j_3$ we obtain the required inequality. ∎

Theorem 2. *Let $H\in\mathcal{G}(\mathbb{R}^n)$ and P^j be a sequence of generalized polynomials in $\mathcal{G}(\mathbb{R}^n)$ which m-sharply converges to P as $j\to\infty$, where P satisfies condition (3) of Theorem 1. Then there is a sequence E^j of generalized functions satisfying the equations $P^j(D)\,E^j=H$, $j\in\mathbb{N}$, which converges sharply to a solution E of $P(D)\,U=H$.*

Moreover, if Ω is open bounded set and H_ε is a given representative of H, then there exist E_ε and a sequence E^j_ε in $\mathcal{E}_M(\mathbb{R}^n)$ such that

$$P(D)\,E_\varepsilon|_\Omega=H_\varepsilon|_\Omega\,,\qquad P^j(D)\,E^j_\varepsilon|_\Omega=H_\varepsilon|_\Omega\quad (\textit{pointwise equalities})$$

and $E^j_\varepsilon\to E_\varepsilon$, $j\to\infty$, sharply in $\mathcal{E}_M(\mathbb{R}^n)$.

Note that the first part also can be proved with Ω instead of $\mathbb{R}^n$. Since the proof is already technically complicate, we omit this generalization.

Proof: As in Theorem 1, Case II, we transfer the general case to the one when P is of the form (4) such that (5) holds. The sharp convergence of a sequence of polynomials is invariant with respect to the transformation of arguments $s=Mt$, $\det M\ne 0$. Thus, it is enough to prove the theorem in this case.

We will assume that P and P^j are of the form (8), (9) for which (10) holds.

We will use the notation of Theorem 1. Let $h>0$, $\varepsilon\in(0,1)$ and $\nu\in\mathbf{Z}^n$ be fixed. We put

$$B_{h,\varepsilon}=3+\max\left\{\frac{C_{\alpha,\nu}}{\varepsilon^{r_{\alpha,\nu}}};\ 0\le\alpha_k\le h,\ |\nu_k|\le h,\ k=1,...,n\right\}\ ,$$

$$\delta_{j,\varepsilon}=\frac{1}{B_{|j+[m+3]|,\varepsilon}(1+|j|^2)}\,,\qquad j\in\mathbf{Z},\quad \delta_{\nu,\varepsilon}=\prod_{k=1}^{n}\delta_{\nu_k\varepsilon}\,,\quad t_{j,\varepsilon}=-\log\delta_{\nu_k\varepsilon}\,,\quad k=1,...,n.$$

This implies

$$\frac{C_{\beta+[m+3],\nu}}{\varepsilon^{r_{\beta+[m+3],\nu}}}\,\delta_{\nu,\varepsilon}\le\frac{1}{\prod_{i=1}^{n}(1+|\nu_i|^2)},\qquad \text{if}\quad \max_{i=1,\dots,n}|\nu_i|\ge|\beta|+n(m+3)\ .$$

Put

$$\tau_{k,\varepsilon}(\nu_k) = t_{\nu_k,\varepsilon}\,\mathrm{sgn}\,\nu_k\,, \qquad k = 1, ..., n\ .$$

Fix $\sigma' = (\sigma_2, ..., \sigma_n)$ and put

$$\tau_\varepsilon(\nu) = \tau_{\nu,\varepsilon} = (\tau_{1,\nu,\varepsilon}, ..., \tau_{n,\nu,\varepsilon})\,, \qquad \tau'_\varepsilon(\nu) = (\tau_{2,\nu,\varepsilon}, ..., \tau_{n,\nu,\varepsilon})\ .$$

Let $f(\sigma', \tau_{\nu,\varepsilon})$ (resp. $f^j(\sigma', \tau_{\nu,\varepsilon})$) be a step function with values in the set $\{\frac{1}{m}, \frac{2}{m}, ..., 1\}$ if $\nu_1 \geq 0$, and $\{-1, ..., -\frac{1}{m}\}$ if $\nu_1 < 0$, with the property that for $\varepsilon \in (0,1)$

$$s_1 = \sigma_1 + i\Big(\tau_{1,\nu,\varepsilon} + f(\sigma', \tau_{\nu,\varepsilon})\Big) \ \Big(\text{resp. } s_1 = \sigma_1 + i\Big(\tau_{1,\nu,\varepsilon} + f^j(\sigma', \tau_{\nu,\varepsilon})\Big)\Big)\,, \quad \sigma_1 \in \mathbb{R}\,,$$

$$\Big|f(\sigma', \tau_{\nu,\varepsilon}) - \mathrm{Im}(s_1^{(p)}(\sigma', \tau_{\nu,\varepsilon}))\Big| \geq \frac{1}{2m}\,, \quad p = 1, ..., m$$

$$\Big(\text{resp. } \Big|f^j(\sigma', \tau_{\nu,\varepsilon}) - \mathrm{Im}(s_1^{j(p)}(\sigma', \tau_{\nu,\varepsilon}))\Big| \geq \frac{1}{2m}\,, \quad p = 1, ..., m, \ \ j \in \mathbb{N}\Big)\ .$$

Note, for given $a > 0$ there are $j_a \in \mathbb{N}$ and $\varepsilon_a > 0$ such that $|a^j_{m,\varepsilon} - a_{m,\varepsilon}| < \varepsilon^a$ if $j \geq j_a$ and $\varepsilon < \varepsilon_a$ (Lemma 3). By taking $a > r$, this implies that there are $C > 0$, j_0 and ε_0 such that for $\sigma_1 \in \mathbb{R}$ and $\varepsilon < \varepsilon_0$,

$$\Big|P_\varepsilon\Big(\sigma_1 + i(f(\sigma', \tau_{\nu,\varepsilon}) + \tau_{1,\nu,\varepsilon}),\ \sigma_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\nu,\varepsilon}\Big)\Big| \geq C\,\varepsilon^r\ ,$$

(resp., for $j > j_0$, $\varepsilon < \varepsilon_0$,

$$\Big|P^j_\varepsilon\Big(\sigma_1 + i(f^j(\sigma', \tau_{\nu,\varepsilon}) + \tau_{1,\nu,\varepsilon}) + \tau_{1,\nu,\varepsilon},\ \sigma_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\nu,\varepsilon}\Big)\Big| \geq C\,\varepsilon^r\ .)$$

We consider solutions E^j_ε of $P^j_\varepsilon(D)\,U_\varepsilon = H_\varepsilon$ (resp. E_ε of $P_\varepsilon(D)\,U_\varepsilon = H_\varepsilon$) which are of the form

$$E^j_\varepsilon(x) = \frac{1}{(2\pi)} \sum_{\nu\in\mathbf{Z}^n} \sum_{k=1}^{\infty} \int_{\mathbb{R}^1} \int_{\Gamma^j_{\nu,k,\varepsilon}} \frac{e^{-ixs}\,\widehat{H}_{\nu,\varepsilon}(s)}{P^j_\varepsilon(s)}\, d\sigma_1 \cdots d\sigma_n$$

(resp.

$$E_\varepsilon(x) = \frac{1}{(2\pi)} \sum_{\nu\in\mathbf{Z}^n} \sum_{k=1}^{\infty} \int_{\mathbb{R}^1} \int_{\Gamma_{\nu,k,\varepsilon}} \frac{e^{-ixs}\,\widehat{H}_{\nu,\varepsilon}(s)}{P_\varepsilon(s)}\, d\sigma_1 \cdots d\sigma_n \quad)\,, \qquad x \in \mathbb{R}^n\ ,$$

where $s = (\sigma_1 + i(f^j(\sigma', \tau_{\nu,\varepsilon}) + \tau_{1,\nu,\varepsilon}),\ \sigma_2 + i\tau_{2,\nu,\varepsilon},\ ...,\ \sigma_n + i\tau_{n,\nu,\varepsilon})$, $\sigma_1 \in \mathbb{R}$, $\sigma' \in \Gamma^j_{\nu,k,\varepsilon}$ and $\Gamma^j_{\nu,k,\varepsilon}$ are disjunct cubes which cover $\mathbb{R}^{n-1}$ (in the case E_ε we omit j), $j \in \mathbb{N}$, $\varepsilon \in (0,1)$.

One can prove that $P_\varepsilon(D)\,E^j_\varepsilon = H_\varepsilon$ and $[E^j_\varepsilon] \in \mathcal{G}$, $j \in \mathbb{N}$, as in the proof of Theorem 1.

The main idea of the proof is that we will construct inductively sets $\Gamma_{\nu,k,\varepsilon}$ and $\Gamma^j_{\nu,k,\varepsilon}$ in a way that when j tends to infinity we can identify $\Gamma^j_{\nu,k,\varepsilon}$ with $\Gamma_{\nu,k,\varepsilon}$ and f^j with f in increasingly large domains.

Note that the $\Gamma_{\nu,k,\varepsilon}$'s were constructed in such a way that the step function f is constant in a small cube (for given ν and ε) and that for any σ' in such a cube we have, with suitable $C' > 0$

$$\left|P_\varepsilon\Big(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f_\nu),\sigma'+i\tau'_{\nu,\varepsilon}\Big)\right| = |a_{m,\varepsilon}|\prod_{p=1}^{m}\left|\Big(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f_\nu)-\sigma^{(p)}(\sigma',\tau'_{\nu,\varepsilon})\Big)\right| \ge C'\varepsilon^r,$$

where $f_\nu = f(\sigma',\tau_{\nu,\varepsilon})$ in $\Gamma_{\nu,k,\varepsilon}$. It is easy to verify that we can construct the $\Gamma_{\nu,k,\varepsilon}$'s in a way that for any ν, ε there exists an increasing sequence $k(M)$, $M \in \mathbb{N}$, such that

$$\bigcup_{k=1}^{k(M)} \Gamma_{\nu,k,\varepsilon} = \mathcal{B}_\infty(0,1/\varepsilon^M)\ ,$$

where on the right hand side is the ball with respect to the "sup" norm, i.e. $|\sigma'| = \sup_{i\in(2,\dots,n)}|\sigma_i|$. This will be the only norm we will use in the sequel.

We have proved in Lemma 4 that for any a there is $j_{0,a} \in \mathbb{N}$ and $\varepsilon_{0,a} > 0$ such that for $j > j_{0,a}$ and $\varepsilon < \varepsilon_{0,a}$,

$$\left|\sigma_1^{(p)}(\sigma',\tau_\varepsilon) - \sigma_1^{j(p)}(\sigma',\tau_\varepsilon)\right| \le \varepsilon^a(1+|\sigma'|^m)\,(1+|\tau_\varepsilon|^m)\,, \qquad \sigma' \in \mathbb{R}^{n-1}\ .$$

Thus, the left side is $\le C\,\varepsilon^a\,\varepsilon^{-Mm}(1+|\tau_{\nu,\varepsilon}|^m)$ for $\sigma' \in \mathcal{B}_\infty(0,1/\varepsilon^M)$.

By construction, $|\tau_{\nu,\varepsilon}| = \mathcal{O}(|\log\varepsilon|)$. This implies that for every $\nu \in \mathbf{Z}$ there exists $\varepsilon_\nu > 0$ such that

$$(1+|\tau_{\nu,\varepsilon}|^m) \le \varepsilon^{-m} \quad \text{ for } \ \varepsilon < \varepsilon_\nu\ .$$

If $\varepsilon < \varepsilon_{0,a}$ $j > j_{0,a}$ and $\varepsilon < \varepsilon_\nu$, then

$$\left|\sigma_1^{(p)}(\sigma',\tau_{\nu,\varepsilon}) - \sigma_1^{j(p)}(\sigma',\tau_{\nu,\varepsilon})\right| \le C\,\varepsilon^{a-Mm-m}\ .$$

Let $a = mM+m+r'$, with $r' > r$. We can easily conclude that there are $j_M \in \mathbb{N}$ and $\varepsilon_M > 0$ such that if $|\sigma'| \le 1/\varepsilon^M$, $j > j_M$, $\varepsilon < \varepsilon_M$ and $\varepsilon < \varepsilon_\nu$, then

$$\left|P^j\Big(\sigma_1+i(\tau_{1,\nu,\varepsilon}+f_\nu),\ \sigma'+i\tau'_{\nu,\varepsilon}\Big)\right| \ge C\,\varepsilon^r\left(\frac{1}{4m}\right)^m\ .$$

The fact that we have above an infinite number of ν's and that the previous inequality depends on ε_ν could seem to be a serious problem. In order to solve the equations in the frame of $\mathcal{G}(\mathbb{R}^n)$ we overcome the quoted problem by choosing representatives of H such that

$$H_{\nu,\varepsilon} = 0, \quad \text{for } \ \varepsilon > \varepsilon_\nu \ .$$

We can suppose without loss of generality that the sequence j_M is strictly increasing and that the sequence ε_M is strictly decreasing.

Let $j_M < j \le j_{M+1}$ and $\varepsilon_{M+1} \le \varepsilon < \varepsilon_M$. We complete for any ν, the covering $(\Gamma_{\nu,k,\varepsilon})_{k\le k(M)}$, of the ball $\mathcal{B}_\infty(0, 1/\varepsilon^M) \subset \mathbb{R}^{n-1}$ by a covering

$$\Gamma^j_{\nu,\widetilde{k},\varepsilon}, \quad \widetilde{k} \in \widetilde{\Lambda} \subset \mathbb{N}, \quad \text{of the set } \ \overline{\mathbb{R}^{n-1} \backslash \mathcal{B}_\infty(0, 1/\varepsilon^M)} \ .$$

We put

$$f^j(\sigma', \tau_{\nu,\varepsilon}) = f(\sigma', \tau_{\nu,\varepsilon}) = f_\nu \quad \text{ for } \ \sigma' \in \mathcal{B}_\infty(0, 1/\varepsilon^M)$$

and for $\sigma' \notin \mathcal{B}_\infty(0, 1/\varepsilon^M)$ we choose $f^j(\sigma', \tau_{\nu,\varepsilon})$ in such a way that in the corresponding covering the above step functions satisfy the prescription in the construction of the solution of $P^j_\varepsilon(D)\, U^j_\varepsilon = H_\varepsilon$ as it is explained in the proof of Theorem 1.

We split the expressions of E^j_ε and E_ε as follows:

$$\begin{aligned} E^j_\varepsilon(x) &= \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}} \sum_{k=1}^{k(M)} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\Gamma_{\nu,k,\varepsilon}} \frac{e^{-ixs}\, \widehat{H}_{\nu,\varepsilon}(s)}{P^j_\varepsilon(s)}\, d\sigma \\ &\quad + \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}^n} \sum_{\widetilde{k}\in\widetilde{\Lambda}} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\Gamma_{\nu,k,\varepsilon}} \frac{e^{-ixs}\, \widehat{H}_{\nu,\varepsilon}(s)}{P^j_\varepsilon(s)}\, d\sigma \\ &= I^j_{1,\varepsilon}(x) + I^j_{2,\varepsilon}(x)\,, \quad x \in \mathbb{R}^n \ , \end{aligned}$$

and likewise

$$\begin{aligned} E_\varepsilon(x) &= \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}^n} \sum_{k=1}^{k(M)} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\Gamma_{\nu,k,\varepsilon}} \frac{e^{-ixs}\, \widehat{H}_{\nu,\varepsilon}(s)}{P_\varepsilon(s)}\, d\sigma \\ &\quad + \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}^n} \sum_{\widetilde{k}\in\widetilde{\Lambda}} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\Gamma_{\nu,\widetilde{k},\varepsilon}} \frac{e^{-ixs}\, \widehat{H}_{\nu,\varepsilon}(s)}{P_\varepsilon(s)}\, d\sigma \\ &= I_{1,\varepsilon}(x) + I_{2,\varepsilon}(x)\,, \quad x \in \mathbb{R}^n \ . \end{aligned}$$

Note that for $I_{1,\varepsilon}$, $I_{2,\varepsilon}$, $s = (\sigma_1 + i(f + \tau_{1,\nu,\varepsilon}), \sigma' + i\tau'_{\nu,\varepsilon})$, while for $I^j_{1,\varepsilon}$, $I^j_{2,\varepsilon}$, $s = (\sigma_1 + i(f^2 + \tau_{1,\nu,\varepsilon}), \sigma' + i\tau'_{\nu,\varepsilon})$.

We can choose the sequence ε_ν to be strictly decreasing with respect to $|\nu|$. Thus, for a given ε there is a finite set $\mathbf{Z}^n_\varepsilon$ of indices ν for which $\widehat{H}_{\nu,\varepsilon} \ne 0$. (Recall that we have taken $H_{\nu,\varepsilon} = 0$ for $\varepsilon > \varepsilon_\nu$.) Let $\beta \in \mathbb{N}^n$. Then

$$|E_\varepsilon^{j(\beta)}(x) - E_\varepsilon^{(\beta)}(x)| \leq |I_{1,\varepsilon}^{j(\beta)}(x) - I_{1,\varepsilon}^{(\beta)}(x)| + |I_{2,\varepsilon}^{j(\beta)}(x)| + |I_{2,\varepsilon}^{(\beta)}(x)| \ .$$

In order to estimate the right hand side of this inequality we use

$$|s^{\beta+[m+3]}\, \widehat{H}_{\nu,\varepsilon}(s)| \leq \frac{C_{\beta+[m+3],\nu}}{\varepsilon^{r_{\beta+[m+3],\nu}}} \tag{11}$$

for $s = (\sigma_1 + i(\tau_{1,\nu,\varepsilon} + f),\, \sigma' + i\tau'_{\nu,\varepsilon})$ or $s = (\sigma_1 + i(\tau_{1,\nu,\varepsilon} + f^j),\, \sigma' + i\tau'_{\nu,\varepsilon})$ and

$$|t| > \frac{1}{\varepsilon^M} \implies \frac{1}{1+|t|} < \varepsilon^M \ . \tag{12}$$

Let us estimate $|I_2^{j(\beta)}(x)|$ (and likewise $I_2^\beta(x)$), $x \in \mathbb{R}^n$.

We obtain by (11) (as in the proof of Theorem 1) that for ε small enough

$$\left|\frac{e^{-ixs}\, s^\beta\, \widehat{H}_{\nu,\varepsilon}(s)}{P_\varepsilon^j(s)}\right| \leq \frac{1}{C'\,\varepsilon^r} \cdot \frac{C_{\beta+[m+3],\nu}}{\varepsilon^{r_{\beta+[m+3],\nu}}} \cdot \frac{\prod_1^n \delta_{\nu_i}^{|\nu_i|-|x_i|-1}}{(1+|s_1|^3)\,(1+|s_2|)^3 \cdots (1+|s_n|)^3} \ .$$

Then,

$$|I_{2,\varepsilon}^{j(\beta)}(x)| \leq \Bigg(\sum_{\nu\in\mathbf{Z}_\varepsilon^n} \prod_1^n \delta_{\nu_i}^{(|\nu_i|-|x_i|-1)}\, \frac{C_{\beta+[m+3],\nu}}{\varepsilon^{r_{\beta+[m+3],\nu}}}\Bigg) \cdot \int \frac{d\sigma}{(1+|\sigma_1|)^3} \int \frac{d\sigma'}{(1+|\sigma_2|)^3 \cdots (1+|\sigma_n|)^3}$$

(because $1/(1+|s_i|)^3 < 1/(1+|\sigma_i|)^3$).

As in Theorem 1, when x belongs to a compact set, say $x \in \mathcal{B}_\infty(0,p) \subset \mathbb{R}^n$, the quantity in brackets can be estimated by $C_1/\varepsilon^{s_{\beta,p}}$, for ε small enough, where $C_1 > 0$ and $s_{\beta,p} > 0$ are constants which depend on β and p but not on j.

Then, (12) implies that there exists a constant $C_{\beta,p} > 0$ which does not depend upon j such that:

$$\sup_{\mathcal{B}_\infty(0,p)} |I_{2,\varepsilon}^{j(\beta)}(x)| \leq C_{p,\beta}\, \varepsilon^{(n-1)M - s_{\beta,p}} \ .$$

In a similar way one can prove

$$\sup_{\mathcal{B}_\infty(0,p)} |I_{2,\varepsilon}^{(\beta)}(x)| \leq C_{p,\beta}\, \varepsilon^{(n-1)M - s_{\beta,p}} \ ,$$

for $\varepsilon_{M+1} \leq \varepsilon < \varepsilon_M$ and $j_M < j \leq j_{M+1}$.

Let M_p be such that

$$(n-1)M - \sup_{|\beta|\leq p} |s_{\beta,p}| \geq p, \quad \text{for } M \geq M_p \ .$$

We obtain that for every $p \in \mathbb{N}^n$ there are $j_{M(p)} > 0$ and $\varepsilon_{M(p)} > 0$ such that

$$\sup_{|\beta|\le p,\, x\in\mathcal{B}_\infty(0,p)} |I_{2,\varepsilon}^{j(\beta)}(x)| < C_p\,\varepsilon^p, \qquad \text{for } \; j \ge j_{M(p)} \; \text{ and } \; \varepsilon < \varepsilon_{M(p)} \; .$$

A similar inequality holds for $I_{2,\varepsilon}^{(\beta)}$.

Let us estimate $|I_{1,\varepsilon}^{j(\beta)}(x) - I_{1,\varepsilon}^{(\beta)}(x)|$, $x \in \mathbb{R}^n$. We have

$$\left|I_{1,\varepsilon}^{j(\beta)}(x) - I_{1,\varepsilon}^{(\beta)}(x)\right| \le \frac{1}{(2\pi)^n} \sum_{\nu\in\mathbf{Z}^n} \sum_{k=1}^{k(\nu)} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\mathcal{B}_\infty} \left| \frac{s^\beta\, \widehat{H}_{\nu,\varepsilon}(s)\, e^{-ixs}}{P_\varepsilon(s)} \right| \cdot \left| \frac{P_\varepsilon^j(s) - P_\varepsilon(s)}{P_\varepsilon^j(s)} \right| d\sigma \; .$$

We already proved that if $j \ge j_M$ and $\varepsilon \le \varepsilon_M$, $|\sigma'| < 1/\varepsilon^M$, then

$$|P_\varepsilon^j(s)| \ge C''\,\varepsilon^r \quad \text{ where } \; s = \Big(\sigma_1 + i(\tau_{1,\nu,\varepsilon} + f),\, \sigma' + i\tau'_{\nu,\varepsilon}\Big) \; . \tag{13}$$

(C'' depends only on M). On the other hand, since P^j converges m-sharply to P, it follows that for every a there exist $\widetilde{j}_a$ ($\widetilde{j}_a \ge j_M$) and $\widetilde{\varepsilon}_a$ ($\widetilde{\varepsilon}_a \le \varepsilon_M$) such that:

$$|P_\varepsilon^j(s) - P_\varepsilon(s)| \le \varepsilon^a (1+|s|)^m\,, \qquad \varepsilon < \widetilde{\varepsilon}_a, \;\; j > \widetilde{j}_a \; .$$

Thus, (13) implies

$$\left|I_{1,\varepsilon}^{(\beta)}(x) - I_{1,\varepsilon}^{j(\beta)}(x)\right| \le$$
$$\le \varepsilon^{-r+a}\, 2\pi\, C''^{-1} \left(\sum_{\nu\in\mathbf{Z}_\varepsilon^n} \sum_{k=1}^{\infty} \int_{\sigma_1\in\mathbb{R}} \int_{\sigma'\in\Gamma_{\nu,k,\varepsilon}} \left| \frac{e^{-ixs}\, s^\beta (1+|s|)^m}{P_\varepsilon(s)}\, \widehat{H}_{\nu,\varepsilon}(s) \right| d\sigma \right) \; .$$

Now, as in the last part of the proof of Theorem 1, we conclude that the quantity in brackets is $\mathcal{O}(\varepsilon^{-\widetilde{r}_{\beta,p}})$, when x remains in the ball $\mathcal{B}_\infty(0,p)$. Choose

$$a = g(p) = p + r' + \sup_{|\beta|\le p} \widetilde{r}_{\beta,p}$$

with $r' > |r|$. Since $\varepsilon^{-r+a-\widetilde{r}_{\beta,p}} < \varepsilon^{p+(r'-r)}$ it follows

$$\sup_{|\beta|\le p,\, x\in\mathcal{B}_\infty(0,p)} \left| I_{1,2}^{(\beta)}(x) - I_{1,2}^{j(\beta)}(x) \right| = \mathcal{O}(\varepsilon^p) \; .$$

All the above estimates imply that there is a decreasing sequence $\widehat{\varepsilon}_p$ and an increasing sequence $\widehat{j}_p$ such that for $j > \widehat{j}_p$ and $\varepsilon \le \widehat{\varepsilon}_p$

$$\sup_{|\beta|\le p,\, x\in B(0,p)} \left| E_\varepsilon^{j(\beta)}(x) - E_\varepsilon^{(\beta)}(x) \right| < \varepsilon^p \; .$$

For a given p, the sequences ε_M and j_M satisfy $\varepsilon_M < \widehat{\varepsilon}_p$ and $j_M > \widehat{j}_p$ for large enough M. This implies the sharp convergence of E^j to E in $\mathcal{G}(\mathbb{R}^n)$.

Let us prove the second part of Theorem 2. Fix $\kappa \in \mathcal{D}(\mathbb{R}^n)$ such that $\kappa = 1$ on $\overline{\Omega}$. Denote

$$\mathbf{Z}^n_\Omega = \left\{\nu \in \mathbf{Z}^n,\ \operatorname{supp} h_\nu \cap \operatorname{supp} \kappa \neq \emptyset\right\} , \tag{14}$$

where h_ν is the partition of unity used in the proof of Theorem 1. Note that $\mathbf{Z}^n_\Omega$ is a finite set because $\operatorname{supp} \kappa$ is compact.

Put $H_{\Omega,\varepsilon} = \kappa H_\varepsilon$. Clearly, $H_{\Omega,\varepsilon}|_\Omega = H_\varepsilon|_\Omega$. Then, the solution of $P^j_\varepsilon(D)\, U_\varepsilon = H_{\Omega,\varepsilon}$ on Ω is given by restriction to Ω of the solution E^j_ε of $P^j_\varepsilon(D)\, U_\varepsilon = H_{\Omega,\varepsilon}$:

$$P^j_\varepsilon(D)\, E^j_\varepsilon|_\Omega = H_{\Omega,\varepsilon}|_\Omega\,, \qquad j \in \mathbb{N}\ .$$

Since the set in (14) is finite, we construct the solutions E^j_ε and E_ε of $P^j_\varepsilon(D)\, U_\varepsilon = H_{\Omega,\varepsilon}$ and $P_\varepsilon(D)\, U_\varepsilon = H_{\Omega,\varepsilon}$, respectively, as in the previous part of the proof. The sequence E^j_ε converges sharply to E_ε in $\mathcal{E}_M(\mathbb{R}^n)$. ∎

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S. Pilipović,
University of Novi Sad, Faculty of Science, Institute for Mathematics,
Trg D. Obradovića 4, 21000 Novi Sad – YUGOSLAVIA

and

D. Scarpalézos,
U.F.R. de Mathématiques, Université Paris 7,
2 place Jussieu, Paris 5$^{\text{ème}}$, 75005 – FRANCE