PORTUGALIAE MATHEMATICA
Vol. 53 Fasc. 3 – 1996

PENTAGONAL NUMBERS IN THE LUCAS SEQUENCE

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Abstract: In this paper we have proved that the only pentagonal number in the Lucas sequence $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$ is $L_1 = 1$, the only generalized pentagonal numbers in this sequence are $L_0 = 2$, $L_1 = 1$ and $L_{\pm 4} = 7$.

1 – Introduction

It is well known that for positive integers m, the numbers of the form $\frac{1}{2}m(3m-1)$ are called pentagonal numbers. In the paper [1], the author had proved that $F_{\pm 1} = F_2 = 1$ and $F_{\pm 5} = 5$ are the only pentagonal numbers in the Fibonacci sequence $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, where n is an integer. The Lucas sequence $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$ is closely related to the Fibonacci sequence. The object of this paper is to show that the only pentagonal number in this sequence is $L_1 = 1$. In fact, the result obtained is more general. Using the method similar to [2] and [3], we can prove that $24\,L_n + 1$ is a perfect square only for $n = 0, 1$ or ± 4. It follows that only L_0, L_1 and $L_{\pm 4}$ can be of the form of $\frac{1}{2}m(3m-1)$ with m integral, not necessarily positive, i.e., so-called generalized pentagonal numbers [4].

2 – Cases $n = 0, 1, \pm 4 \pmod{672}$

To prove our result, we shall use the following well known properties concerning the Lucas numbers (refer. [5] and [6])

$$L_{-n} = (-1)^n\,L_n\ , \tag{1}$$

Received: June 30, 1995; *Revised*: October 27, 1995.

$$L_{3n} = L_n(L_{2n} - (-1)^n) \ , \tag{2}$$

$$2 \mid L_n \quad \text{iff} \ \ 3 \mid n \ . \tag{3}$$

For even m, let

$$\mathcal{L}_m = \begin{cases} L_m & \text{if } m \equiv \pm 2 \pmod 6, \\ \frac{1}{2} L_m & \text{if } m \equiv 0 \pmod 6 \ , \end{cases}$$

then the congruence

$$L_{n+2km} \equiv (-1)^k \, L_n \pmod{\mathcal{L}_m} \tag{4}$$

holds, where k is an integer.

In this paper we shall also use the Jacobi symbol $(\frac{24\,L_n+1}{P})$ to prove that $24\,L_n + 1$ is not a perfect square provided that for some positive odd P the value of this symbol is -1.

Lemma 1. *If $m \equiv 0 \pmod{24}$ and $n \neq 0$, then $24\,L_n + 1$ is not a perfect square.*

Proof: Put $n = (12k \pm 4)\,m$ such that $m = 2 \cdot 3^r$ with $r \geq 1$, then, by (4) and (1),

$$L_n \equiv L_{\pm 4m} \equiv -L_{\mp 2m} \equiv -L_{2m} \pmod{\tfrac{1}{2} L_{3m}} \ .$$

Since (2) implies $\frac{1}{2} L_{3m} = \frac{1}{2} L_m(L_{2m} - 1)$, so that

$$24\,L_n + 1 \equiv -24\,L_{2m} + 1 \pmod{(L_{2m} - 1)} \ .$$

Thus we have

$$\left(\frac{24\,L_n + 1}{L_{2m} - 1}\right) = \left(\frac{-24\,L_{2m} + 1}{L_{2m} - 1}\right) = \left(\frac{-23}{L_{2m} - 1}\right) = \left(\frac{L_{2m} - 1}{23}\right) \ .$$

The residue sequence of $\{L_n\}$ modulo 23 has period 48. Note that $2m \equiv \pm 12 \pmod{48}$, which imply $L_{2m} \equiv 0 \pmod{23}$, so that

$$\left(\frac{24\,L_n + 1}{L_{2m} - 1}\right) = \left(\frac{-1}{23}\right) = -1 \ ,$$

$24\,L_n + 1$ is not a perfect square. ∎

Lemma 2. *If $n \equiv 1 \pmod{32}$ and $n \neq 1$, then $24\,L_n + 1$ is not a perfect square.*

Proof: Put $n = 1 + 2km$ such that $m = 2^r$, $r \geq 4$ and $2 \nmid k$, then $m \equiv \pm 16 \pmod{48}$. Now (4) gives

$$24\,L_n + 1 \equiv -24\,L_1 + 1 \equiv -23 \pmod{L_m} \ .$$

Since the residue sequence of $\{L_n\}$ modulo 23 has period 48 and $m \equiv \pm 16 \pmod{48}$ imply $L_m \equiv -1 \pmod{23}$, so that

$$\left(\frac{24\,L_n+1}{L_m}\right) = \left(\frac{-23}{L_m}\right) = \left(\frac{L_m}{23}\right) = \left(\frac{-1}{23}\right) = -1 \ .$$

Hence $24\,L_n+1$ is not a perfect square. ∎

Lemma 3. *If $n \equiv \pm 4 \pmod{224}$ and $n \neq \pm 4$, then $24\,L_n+1$ is not a perfect square.*

Proof: Put $n = \pm 4 + 2km$ such that $2 \not| k$ and $m = 7 \cdot 2^r$ with $r \geq 4$, then it is easy to check $m \equiv \pm 112 \pmod{336}$. By (4) we get

$$24\,L_n+1 \equiv -24\,L_{\pm 4}+1 \equiv -167 \pmod{L_m} \ ,$$

and

$$\left(\frac{24\,L_n+1}{L_m}\right) = \left(\frac{-167}{L_m}\right) = \left(\frac{L_m}{167}\right) \ .$$

The residue sequence of $\{L_n\}$ modulo 167 has period 336 and $m \equiv \pm 112 \pmod{336}$ imply $L_m \equiv -1 \pmod{167}$. Thus

$$\left(\frac{24\,L_n+1}{L_m}\right) = \left(\frac{-1}{167}\right) = -1 \ ,$$

so that $24\,L_n+1$ is not a perfect square. ∎

Corollary. *If $n \equiv 0, 1, \pm 4 \pmod{672}$ then $24\,L_n+1$ is a perfect square only for $n = 0, 1, \pm 4$.*

Proof: Note that the least common multiple of the moduli in the above three lemmas is 672, then the necessary for $n = 0, 1, \pm 4$ follows immediately. In fact, $24\,L_0+1 = 7^2$, $24\,L_1+1 = 5^2$, $24\,L_{\pm 4}+1 = 13^2$, which complete the proof. ∎

3 – Cases $n \not\equiv 0, 1, \pm 4 \pmod{672}$

Lemma 4. *If $n \not\equiv 0, 1, \pm 4 \pmod{672}$, then $24\,L_n+1$ is not a perfect square.*

Proof: We prove this lemma by showing that $24\,L_n+1$ is a quadratic nonresidue modulo some prime for each residue class of n modulo 672 except for $n \equiv 0, 1, \pm 4 \pmod{672}$. For brevity, we let $H_n = 24\,L_n+1$ and the calculations will be carried out directly to the sequence $\{H_n\}$, which satisfies recurrent relation $H_{n+2} = H_{n+1} + H_n - 1$, $H_0 = 49$, $H_1 = 25$.

i) Modulo 29. The sequence of residues of $\{H_n\}$ has period 14. We can exclude $n \equiv \pm 2, 3, \pm 6, 9, 11 \pmod{14}$ since they imply respectively $H_n \equiv 15, 10, 27, 27, 21 \pmod{29}$, all of which are quadratic nonresidues modulo 29. Hence there remain $n \equiv 0, 1, \pm 4, 5, 7, 13 \pmod{14}$.

To obtain the desired period $4k$, we usually take a prime factor of L_k or F_k as the modulo.

ii) Modulo 13. We get the residue sequence of $\{H_n\}$ with period 28. Since $n \equiv 5, 7, \pm 10, 14, 21 \pmod{28}$ imply respectively $H_n \equiv 5, 8, 2, 5, 7 \pmod{13}$, which are quadratic nonresidues modulo 13, then may be excluded. Thus there remain $n \equiv 0, 1, \pm 4, 13, 15, 19, 27 \pmod{28}$, which are equivalent to $n \equiv 0, 1, \pm 4, 13, 15, 19, \pm 24, 27, \pm 28, 29, 41, 43, 47, \pm 32, 55, 57, 69, 71, 75, 83 \pmod{84}$.

iii) Modulo 421. The period of the residue sequence of $\{H_n\}$ is 84. When $n \equiv \pm 24, \pm 28, 43, 47, 55, 71, 83 \pmod{84}$, $H_n \equiv 259, 398, 398, 158, 127, 127, 398 \pmod{421}$ respectively, all of which are quadratic nonresidues modulo 421, so that these values of n may be excluded.

Modulo 211. The period is 42, and $n \equiv 15, 27, 29, \pm 32 \pmod{42}$ imply respectively $H_n \equiv 32, 181, 157, 210 \pmod{211}$, all of which are quadratic nonresidues modulo 211, so that $n \equiv 15, 27, 29, \pm 32, 57, 69 \pmod{84}$ may be excluded.

Thus there remain $n \equiv 0, 1, \pm 4, 13, 19, 41, 75 \pmod{84}$, which are equivalent to $n \equiv 0, 1, \pm 4, 13, 19, 41, 75, \pm 80, 84, 85, 97, 103, 125, 159 \pmod{168}$.

iv) Modulo 281. The residue sequence of $\{H_n\}$ has period 56. Since $n \equiv 19, 28, 29, 41 \pmod{56}$ imply respectively $H_n \equiv 139, 234, 258, 142 \pmod{281}$, which are quadratic nonresidues modulo 281, then may be excluded. Hence we can exclude $n \equiv 19, 41, 75, 84, 85, 97 \pmod{168}$.

Modulo 83. The residue sequence of $\{H_n\}$ has period 168. We can exclude $n \equiv 13, \pm 80, 103, 125 \pmod{168}$ since they imply $H_n \equiv 55, 82, 57, 79 \pmod{83}$ respectively, all of which are quadratic nonresidues modulo 83.

Modulo 1427. The period is also 168, and $n \equiv 159 \pmod{168}$ implies $H_n \equiv 1031 \pmod{1427}$, which is a quadratic nonresidue modulo 1427. Therefore $n \equiv 159 \pmod{168}$ may be excluded.

Thus there remain $n \equiv 0, 1, \pm 4 \pmod{168}$, i.e., $n \equiv 0, 1, \pm 4, \pm 164, 168, 169 \pmod{336}$.

v) Modulo 7. The residue sequence of $\{H_n\}$ has period 16. We can exclude $n \equiv 9 \pmod{16}$ since it implies $H_n \equiv 5 \pmod 7$, a quadratic non residue modulo 7. Hence $n \equiv 169 \pmod{336}$ may be excluded.

Modulo 23. We get the period 48, and $n \equiv \pm 20, 24 \pmod{48}$ imply

$H_n \equiv 17, 22 \pmod{23}$ respectively, both of which are quadratic nonresidues modulo 23. Hence we can exclude $n \equiv \pm 164, 168 \pmod{336}$.

Now there remain $n \equiv 0, 1, \pm 4 \pmod{336}$, i.e., $n \equiv 0, 1, \pm 4, \pm 332, 336, 337 \pmod{672}$.

Modulo 1103. The period of the residue sequence of $\{H_n\}$ is 96. If $n \equiv \pm 44, 48, 49 \pmod{96}$, then $H_n \equiv 936, 1056, 1080 \pmod{1103}$ respectively, all of which are quadratic nonresidues modulo 1103. Hence $n \equiv \pm 332, 336, 337 \pmod{672}$ may be excluded.

Finally there remain $n \equiv 0, 1, \pm 4 \pmod{672}$. The proof is complete. ∎

4 – Result

Theorem. *The Lucas number L_n is a generalized pentagonal number only for $n = 0, 1$, or ± 4; a pentagonal number only for $n = 1$.*

Proof: Since L_n is a generalized pentagonal number, i.e., of the form $\frac{1}{2}m(3m-1)$ with m integral, if and only if $24\,L_n + 1 = (6m-1)^2$, then the first part of the theorem follows from Lemma 4 and the corollary in section 2. Moreover, a pentagonal number $\frac{1}{2}m(3m-1)$ means m positive, so that, obviously, only $L_1 = 1$ is in this case. Then the second part of the theorem follows. ∎

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