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SPLINE APPROXIMATION AND GENERALIZED TURÁN QUADRATURES

M.A. Kovačević and G.V. Milovanović

Abstract: In this paper, which is connected with our previous work [16], we consider the problem of approximating a function f on the half-line by a spline function of degree m with n (variable) knots (multiplicities of the knots are greater or equal than one). In the approximation procedure we use the moments of the function $r \mapsto f(r)$ and its derivatives at the origin $r = 0$. If the approximation exists, we show that it can be represented in terms of the generalized Turán quadrature relative to a measure depending on f. Also the error in the spline approximation formula is expressed by the error term in the corresponding quadrature formula. A numerical example is included.

1 – Introduction

A spline function of degree $m \geq 1$ on the interval $0 \leq r < +\infty$, vanishing at $r = +\infty$, with the variable positive knots r_ν, $\nu = 1, ..., n$, and multiplicity k_ν ($\leq m$), $\nu = 1, ..., n$ ($n > 1$), respectively, can be represented in the form

$$(1.1) \qquad S_{n,m}(r) = \sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu - 1} \alpha_{\nu,i} (r_\nu - r)_+^{m-i} , \qquad 0 \leq r < +\infty ,$$

where $\alpha_{\nu,i}$ are real numbers and the plus sign on the right is the cutoff symbol, $t_+ = t$ if $t > 0$ and $t_+ = 0$ if $t \leq 0$.

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Using the following conditions

$$(1.2)\qquad \int_0^{+\infty} r^{j+d-1}\, S_{n,m}(r)\, dr = \int_0^{+\infty} r^{j+d-1}\, f(r)\, dr\,, \qquad j=0,1,...,2(s+1)n-1\ ,$$

we [16] considered the problem of approximating a function $f(r)$ of the radial distance $r=\|x\|$, $0 \le r < +\infty$ in $\mathbb{R}^d$, $d \ge 1$, by the spline function (1.1), where $k_\nu = 2s+1$, $\nu = 1,...,n$, $s \in \mathbb{N}_0$. The work on this subject was initiated in computational plasma physics ([1], [13]) and continued in mathematics (see [4–10], [12], [14], [16–17]).

In this paper we discuss two similar problems of approximating a function $f(r)$, $0 \le r < +\infty$, by the spline function (1.1). (Let N denote the sum of the variable knots r_ν, $\nu = 1,...,n$, of the spline function (1.1), counting multiplicities, i.e., $N = k_1 + \cdots + k_n$.)

Problem 1. Determine $S_{n,m}$ in (1.1) such that

$$(1.3)\qquad S_{n,m}^{(k)}(0) = f^{(k)}(0)\,, \qquad k=0,1,...,N+n-1, \quad m \ge N+n-1\ .$$

Problem 2. Determine $S_{n,m}$ in (1.1) such that

$$(1.4)\qquad S_{n,m}^{(k)}(0) = f^{(k)}(0)\,, \qquad k=0,1,...,l \quad (l \le m)$$

and

$$(1.5)\qquad \int_0^{+\infty} r^j\, S_{n,m}(r)\, dr = \int_0^{+\infty} r^j\, f(r)\, dr\,, \qquad j=0,1,...,N+n-l-2\ .$$

In Section 2 we give solutions of these problems as well as the approximation errors. Some remarks on the generalized Gauss–Turán quadratures are given in Section 3. Finally, a numerical example is analyzed in Section 4. An analogous problem to Problem 2 for approximation of a function f by defective spline functions on the finite interval $[0,1]$ has been studied by Gori and Santi [9] and solved by means of monosplines.

2 – Spline approximation

We first consider the Problem 2.

Theorem 2.1. *Let $f \in C^{m+1}[0,+\infty)$ and*

$$\int_0^{+\infty} r^{N+n-l+m}\,|f^{(m+1)}(r)|\,dr < +\infty \ . \tag{2.1}$$

Then a spline function $S_{n,m}$ of the form (1.1) with positive knots r_ν, that satisfies (1.4) and (1.5), exists and is unique if and only if the measure

$$d\lambda(r) = \frac{(-1)^{m+1}}{m!}\, r^{m-l}\, f^{(m+1)}(r)\, dr \tag{2.2}$$

admits a generalized Gauss–Turán quadrature

$$\int_0^{+\infty} g(r)\,d\lambda(r) = \sum_{\nu=1}^{n}\sum_{k=0}^{k_\nu-1} A_{\nu,k}^{(n)}\, g^{(k)}(r_\nu^{(n)}) + R_n(g;d\lambda) \ , \tag{2.3}$$

with n distinct positive nodes $r_\nu^{(n)}$, where $R_n(g;d\lambda)=0$ for all $g \in \mathcal{P}_{N+n-1}$. The knots in (1.1) are given by $r_\nu = r_\nu^{(n)}$, and the coefficients $\alpha_{\nu,i}$ by the following triangular system:

$$A_{\nu,k}^{(n)} = \sum_{i=k}^{k_\nu - i} \frac{(m-i)!}{m!}\binom{i}{k}\Big[\mathrm{D}^{i-k} r^{m-l}\Big]_{r=r_\nu} \alpha_{\nu,i} \qquad (k=0,1,\dots,k_\nu-1) \ , \tag{2.4}$$

where D is the standard differentiation operator.

Proof: Let $j \le N+n-l-2$. Because of (2.1), the integral

$$\int_0^{+\infty} r^{j+m+2}\, f^{(m+1)}(r)\,dr$$

exists and $\lim_{r\to+\infty} r^{j+m+2} f^{(m+1)}(r) = 0$. Then, L'Hospital's rule implies

$$\lim_{r\to+\infty} r^{j+m+1}\, f^{(m)}(r) = 0 \ .$$

Continuing in this manner, we find that

$$\lim_{r\to+\infty} r^{j+\mu+1}\, f^{(\mu)}(r) = 0\,, \qquad \mu = m, m-1, \dots, 1, 0 \ . \tag{2.5}$$

By Taylor's formula, one has for any $b>0$,

$$f^{(k)}(r) = f^{(k)}(b) + f^{(k+1)}(b)\,\frac{(r-b)}{1!} + \cdots + f^{(m)}(b)\,\frac{(r-b)^{m-k}}{(m-k)!}$$
$$+ \frac{1}{(m-k)!}\int_b^r (r-t)^{m-k}\, f^{(m+1)}(t)\,dt \ ,$$

for $k = 0, 1, ..., m$. Letting $b \to +\infty$ and noting (2.5), we obtain

$$f^{(k)}(r) = \frac{(-1)^{m-k+1}}{(m-k)!} \int_r^{+\infty} (t-r)^{m-k} f^{(m+1)}(t)\, dt\,, \qquad k = 0, 1, ..., m\ , \tag{2.6}$$

and, for $r = 0$,

$$f^{(k)}(0) = \frac{(-1)^{m-k+1}}{(m-k)!} \int_0^{+\infty} t^{m-k} f^{(m+1)}(t)\, dt\,, \qquad k = 0, 1, ..., m\ . \tag{2.7}$$

On the other hand, differentiating (1.1), we obtain

$$S_{n,m}^{(k)}(0) = (-1)^k \sum_{\nu=1}^{n} \sum_{i=0}^{s_\nu} \frac{(m-i)!}{(m-i-k)!}\, r_\nu^{m-i-k}\, \alpha_{\nu,i}\,, \qquad k = 0, 1, ..., m\ , \tag{2.8}$$

where $s_\nu = \min(m-k, k_\nu - 1)$, $\nu = 1, ..., n$.

Substituting (2.7) and (2.8) in (1.4), we find

$$\sum_{\nu=1}^{n} \sum_{i=0}^{s_\nu} \frac{(m-i)!}{m!}\, \alpha_{\nu,i}\, \frac{(m-k)!}{(m-i-k)!}\, r_\nu^{m-i-k} = \int_0^{+\infty} \frac{(-1)^{m+1}}{m!}\, r^{m-k} f^{(m+1)}(r)\, dr$$

or

$$\sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu - 1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i r^{m-k}\Big]_{r=r_\nu} = \int_0^{+\infty} \frac{(-1)^{m+1}}{m!}\, r^{m-k} f^{(m+1)}(r)\, dr\ , \tag{2.9}$$

for $k = 0, 1, ..., l$, where D is the standard differentiation operator.

The conditions (2.9) can be represented in the form

$$\sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu - 1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i (r^{m-l} r^j)\Big]_{r=r_\nu} = \int_0^{+\infty} \frac{(-1)^{m+1}}{m!}\, r^{m-l} f^{(m+1)}(r)\, r^j\, dr, \qquad j = 0, 1, ..., l\ ,$$

or, after the application of Leibniz's formula to the i-th derivative,

$$\sum_{\nu=1}^{n} \sum_{k=0}^{k_\nu - 1} A_{\nu,k}^{(n)} \Big[\mathrm{D}^k r^j\Big]_{r=r_\nu} = \int_0^{+\infty} r^j\, d\lambda(r)\,, \qquad j = 0, 1, ..., l\ , \tag{2.10}$$

where $A_{\nu,k}^{(n)}$ and $d\lambda(r)$ are given by (2.4) and (2.2).

Now, we consider the conditions (1.5).

Using (1.1) and observing that $r_\nu > 0$, we have

$$\int_0^{+\infty} r^j\, S_{n,m}(r)\, dr = \sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu-1} \alpha_{\nu,i} \int_0^{r_\nu} r^j\, (r_\nu - r)^{m-i}\, dr\ .$$

Changing variables, $r = tr_\nu$, in the integral on the right, we obtain the well-known beta integral, which can be expressed in terms of factorials. So we find

$$\int_0^{+\infty} r^j\, S_{n,m}(r)\, dr = \sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu-1} \frac{j!(m-i)!}{(j+m-i+1)!}\, \alpha_{\nu,i}\, r_\nu^{j+m-i+1}$$

or

$$\int_0^{+\infty} r^j\, S_{n,m}(r)\, dr = \frac{j!}{(j+m+1)!} \sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu-1} (m-i)!\, \alpha_{\nu,i} \Big[\mathrm{D}^i r^{j+m+1}\Big]_{r=r_\nu}\ . \tag{2.11}$$

Through $m+1$ integrations by parts and noting (2.5), the integral on the right of (1.5) can be transformed to

$$\begin{aligned} \int_0^{+\infty} r^j\, f(r)\, dr &= \frac{(-1)^{m+1}}{(j{+}1)\,(j{+}2)\cdots(j{+}m{+}1)} \int_0^{+\infty} r^{j+m+1}\, f^{(m+1)}(r)\, dr \\ &= \frac{(-1)^{m+1} j!}{(j+m+1)!} \int_0^{+\infty} r^{j+m+1}\, f^{(m+1)}(r)\, dr\ . \end{aligned} \tag{2.12}$$

The conditions (1.5) now become

$$\sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu-1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i r^{m+j+1}\Big]_{r=r_\nu} = \int_0^{+\infty} \frac{(-1)^{m+1}}{m!}\, r^{m+j+1}\, f^{(m+1)}(r)\, dr\ ,$$

i.e.,

$$\sum_{\nu=1}^{n} \sum_{i=0}^{k_\nu-1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i r^{m-l} r^{j+l+1}\Big]_{r=r_\nu} = \int_0^{+\infty} \frac{(-1)^{m+1}}{m!}\, r^{m-l}\, f^{(m+1)}(r)\, r^{j+l+1}\, dr\ ,$$

where $j = 0, 1, ..., N+n-l-2$. After the application of Leibniz's formula to the i-th derivative on the left side of the above equation, we get

$$\sum_{\nu=1}^{n} \sum_{k=0}^{k_\nu-1} A_{\nu,k}^{(n)} \Big[\mathrm{D}^k r^{j+l+1}\Big]_{r=r_\nu} = \int_0^{+\infty} r^{j+l+1}\, d\lambda(r)\ , \tag{2.13}$$

where $j = 0, 1, ..., N + n - l - 2$, and $A_{\nu,k}^{(n)}$ and $d\lambda(r)$ are given by (2.4) and (2.2), respectively.

Finally, (2.10) and (2.13) yield

$$\sum_{\nu=1}^{n} \sum_{k=0}^{k_\nu - 1} A_{\nu,k}^{(n)} \left[\mathrm{D}^k r^j\right]_{r=r_\nu} = \int_0^{+\infty} r^j \, d\lambda(r) \,, \qquad j = 0, 1, ..., N+n-1 \; . \tag{2.14}$$

Hence, we conclude that Eqs. (1.4) and (1.5) are equivalent to Eqs. (2.14). These are precisely the conditions for r_ν to be the nodes of the generalized Gauss–Turán quadrature formula (2.3) ($r_\nu = r_\nu^{(n)}$) and $A_{\nu,k}^{(n)}$, determined by (2.14), their coefficients. ∎

Remark 2.1. If we let $l = N + n - 1$, the Theorem 2.1 gives the solution of Problem 1. Namely, equating (2.7) and (2.8), for $k = 0, 1, ..., N + n - 1$ ($m \geq N + n - 1$), we obtain (2.14), where $l = N + n - 1$.

Remark 2.2. The case $k_1 = k_2 = \cdots = k_n = 1$, $l = -1$, of Theorem 2.1 has been obtained in [8].

Similarly as in [16], we can prove the following result regarding the approximating error.

Theorem 2.2. *Let f be given as in Theorem* 2.1 *and such that the measure $d\lambda$ in* (2.2) *admits a generalized Gauss–Turán quadrature formula* (2.3) *with distinct positive nodes $r_\nu = r_\nu^{(n)}$. Define*

$$\sigma_r(t) = t^{-(m-l)} (t - r)_+^m \; .$$

Then the error of the spline approximation (1.1), (1.3) ($l = N + n - 1$) *or* (1.1), (1.4), (1.5), *is given by*

$$f(r) - S_{n.m}(r) = R(\sigma_r(t); d\lambda(t)) \,, \qquad r > 0 \,, \tag{2.15}$$

where $R(\sigma_r(t); d\lambda(t))$ is the remainder term in the formula (2.2)–(2.3)

$$\int_0^{+\infty} g(t) \, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{k=0}^{k_\nu - 1} A_{\nu,k}^{(n)} \, g^{(k)}(r_\nu^{(n)}) + R(g(t); d\lambda(t)) \; . \tag{2.16}$$

Proof: Using (2.6) for $k = 0$, we find

$$f(r) = \frac{(-1)^{m+1}}{m!} \int_r^{+\infty} (t - r)^m \, f^{(m+1)}(t) \, dt = \frac{(-1)^{m+1}}{m!} \int_0^{+\infty} (t - r)_+^m \, f^{(m+1)}(t) \, dt \;,$$

i.e.,

$$f(r) = \int_0^{+\infty} \sigma_r(t)\, d\lambda(t) \ . \tag{2.17}$$

On the other hand, we consider the sum

$$F_\nu(r) = \sum_{k=0}^{k_\nu-1} A_{\nu,k}^{(n)} \Big[\mathrm{D}^k \sigma_r(t)\Big]_{t=r_\nu} \ ,$$

where $A_{\nu,k}^{(n)}$ are the coefficientes of the generalized Gauss–Turán quadrature (2.16). By (2.4) and Leibniz's formula, we obtain

$$\begin{aligned}
F_\nu(r) &= \sum_{k=0}^{k_\nu-1} \Big[\mathrm{D}^k \sigma_r(t)\Big]_{t=r_\nu} \sum_{i=k}^{k_\nu-1} \frac{(m-i)!}{m!} \binom{i}{k} \Big[\mathrm{D}^{i-k} t^{m-l}\Big]_{t=r_\nu} \alpha_{\nu,i} \\
&= \sum_{i=0}^{k_\nu-1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \sum_{k=0}^{i} \binom{i}{k} \Big\{[\mathrm{D}^k \sigma_r(t)][\mathrm{D}^{i-k} t^{m-l}]\Big\}_{t=r_\nu} \\
&= \sum_{i=0}^{k_\nu-1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i (t^{m-l} \sigma_r(t))\Big]_{t=r_\nu} \\
&= \sum_{i=0}^{k_\nu-1} \frac{(m-i)!}{m!}\, \alpha_{\nu,i} \Big[\mathrm{D}^i (t-r)_+^m\Big]_{t=r_\nu} \\
&= \sum_{i=0}^{k_\nu-1} \alpha_{\nu,i} (r_\nu - r)_+^{m-i} \ ,
\end{aligned}$$

i.e.,

$$\sum_{\nu=1}^{n} F_\nu(r) = S_{n,m}(r) \ . \tag{2.18}$$

Finally, using (2.17) and (2.18), we obtain (2.15). ∎

3 – On the generalized Gauss-Turán quadratures

The generalized Gauss–Turán quadratures with a given nonnegative measure $d\lambda(r)$ on the real line $\mathbb{R}$ (with compact or infinite support for which all moments $\mu_i = \int_{\mathbb{R}} r^i\, d\lambda(r)$, $i = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$),

$$\int_{\mathbb{R}} g(r)\, d\lambda(r) = \sum_{\nu=1}^{n} \sum_{k=0}^{m-1} A_{\nu,k}\, g^{(k)}(r_\nu) + R_n(g; d\lambda) \tag{3.1}$$

is exact for all polynomials of degree at most $(m+1)n-1$, if m is odd, i.e., $m=2s+1$ (see [19]). The nodes r_ν, $\nu=1,...,n$, are the zeros of the (monic) polynomial π_n minimizing

$$\int_{\mathbb{R}} [\pi_n(r)]^{2s+2}\, d\lambda(r) \ . \tag{3.2}$$

Such polynomials are known as power-orthogonal (s-orthogonal or s-self associated) polynomials with respect to the measure $d\lambda(r)$. For a given n and s, the minimization of the integral (3.2) leads to the *"orthogonality conditions"*

$$\int_{\mathbb{R}} \pi_n(r)^{2s+1}\, r^i\, d\lambda(r)\,, \qquad i=0,1,...,n-1 \ , \tag{3.3}$$

which can be interpreted as (see [15])

$$\int_{\mathbb{R}} \pi_\nu^{s,n}(r)\, r^i\, d\mu(r)=0\,, \qquad i=0,1,...,\nu-1 \ , \tag{3.4}$$

where $\{\pi_\nu^{s,n}\}$ is a sequences of monic orthogonal polynomials with respect to the new measure $d\mu(r)=d\mu^{s,n}(r)=(\pi_n^{s,n}(r))^{2s}d\lambda(r)$. As we can see, the polynomials $\pi_\nu^{s,n}$, $\nu=0,1,...$, are implicitly defined because the measure $d\mu(r)$ depends on $\pi_n^{s,n}(=\pi_n(r))$. Of course, we are interested only in $\pi_n^{s,n}(r)$. A stable procedure of constructing such polynomials (s-orthogonal) is given in [15].

A generalization of the formula (3.1) to rules having nodes with arbitrary multiplicities was given, independently, by Chakalov [2–3] and Popoviciu [18].

Let $\sigma=(s_1,s_2,...,s_n)$ be a sequence of nonegative integers. In this case, it is important to assume that the nodes r_ν are ordered, say

$$a\le r_1<r_2<\cdots<r_n\le b\ ,$$

with odd multiplicities $2s_1+1,...,2s_n+1$, respectively. Here $[a,b]$ is the support of the measure $d\lambda(r)$. Then the coresponding quadrature formula

$$\int_{\mathbb{R}} g(r)\, d\lambda(r)=\sum_{\nu=1}^{n}\sum_{k=0}^{2s_\nu} A_{\nu,k}\, g^{(k)}(r_\nu)+R_n(g;d\lambda) \tag{3.5}$$

has the maximum degree of exactness $d_{\max}=2\sum\limits_{\nu=1}^{n} s_\nu+2n-1$, if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n}(r-r_\nu)^{2s_\nu+1}\, r^i\, d\lambda(r)=0\,, \qquad i=0,1,...,n-1 \ . \tag{3.6}$$

The last *"orthogonality conditions"* correspond to (3.3).

If we put

$$\pi_{k,\sigma}^{(n)}(r) = \prod_{\nu=1}^{k} (r - r_\nu^{(k)}) , \qquad a \le r_1^{(k)} < \cdots < r_k^{(k)} \le b ,$$

and

$$d\mu(r) = \prod_{\nu=1}^{n} (r - r_\nu^{(n)})^{2s_\nu}\, d\lambda(r) \qquad (r_\nu^{(n)} \equiv r_\nu,\ \nu = 1,...,n) ,$$

then the *"orthogonality conditions"* (3.6) can be interpreted as

$$\int_{\mathbb{R}} \pi_{k,\sigma}^{(n)}(r)\, r^i\, d\mu(r) = 0 , \qquad i = 0,1,...,k-1 .$$

So we conclude that $\{\pi_{k,\sigma}^{(n)}\}$ is a sequence of (standard) orthogonal polynomials with respect to the measure $d\mu(r)$. The polynomials $\pi_{n,\sigma}^{(n)}$ are called σ-orthogonal polynomials. An algorithm for constructing them is given in [11].

If we have $s_\nu = s$, $\nu = 1,...,n$, the above polynomials reduce to the s-orthogonal polynomials.

If we find the nodes (r_ν, $\nu = 1,...,n$) of the generalized Gauss–Turán quadrature formula (3.1) or (3.5) (the zeros of the s-orthogonal polynomial $\pi_n^{s,n}$ or σ-orthogonal polynomial $\pi_{n,\sigma}^{(n)}$, respectively), then their coefficients are determined from the linear system equations (3.1) or (3.5), for $g(r) = r^i$ ($R_n(r^i, d\lambda) = 0$), where $i = 0,1,...,2(s+1)n-1$ or $i = 0,1,...,d_{\max}$, respectively.

4 – Numerical example

If in the spline function (1.1) we take $k_\nu = 2s+1$, $\nu = 1,...,n$, $s \in \mathbb{N}_0$, i.e.,

$$S_{n,m}(r) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} \alpha_{\nu,i} (r_\nu - r)_+^{m-i} , \qquad 0 \le r < +\infty , \tag{4.1}$$

and l is formally replaced by $-d$ in Theorem (2.1), in view of the approximative requirement (1.2), then we get the identical statement as in [16, Theorem 2.1]. Therefore, this fact enables us in this case to use the previously developed software for the problem (1.2). Now, for solving problems (1.3) or (1.4)–(1.5), one can take $d := -l$.

Let $f(r) = e^{-r}$ on $[0, +\infty)$. For this function the measure (2.2) becomes the generalized Laguerre measure

$$d\lambda(r) = \frac{1}{m!} r^{m-l} e^{-r}\, dr , \qquad 0 \le r < +\infty .$$

First, for a given (n, s, m, l), we determine r_ν^n (the zeros of the polynomial $\pi_n^{s,n}$) and the weight coefficients of the Turán quadrature (2.3). Then, the knots in (3.1) are given by $r_\nu = r_\nu^{(n)}$, $\nu = 1, ..., n$, and we find the coefficients of the spline function (4.1) using the triangular system of equations (2.4).

In Tables 3.1 and 3.2 we can see the behavior of approximate values of the resulting maximum absolute errors $e_{n,m}^{(l)} = \max_{0 \le r \le r_n} |S_{n,m}(r) - f(r)|$, for different values of (n, s, m, l). (Numbers in parenthesis indicate decimal exponents.) Clearly, for $r \ge r_n$, the absolute error is equal to $f(r)$.

TABLE 3.1 – Accuracy of the spline approximation for $s = 1$.

	$l=0$			$l=1$			$l=2$	
n	$m=2$	$m=3$	$m=4$	$m=2$	$m=3$	$m=4$	$m=3$	$m=4$
2	1.5(−1)	1.8(−2)	4.9(−3)	1.5(−1)	2.6(−2)	6.4(−3)	3.0(−2)	6.5(−3)
3	8.4(−2)	1.3(−2)	2.5(−3)	6.7(−2)	1.3(−2)	2.3(−3)	1.1(−2)	1.9(−3)
4	5.1(−2)	8.1(−3)	1.2(−3)	4.1(−2)	7.1(−3)	9.2(−4)	4.8(−3)	8.6(−4)
5	3.3(−2)	5.1(−3)	6.2(−4)	3.0(−2)	4.0(−3)	5.2(−4)	4.0(−3)	6.1(−4)

TABLE 3.2 – Accuracy of the spline approximation for $m = 8$.

	$l=0$		$l=4$	
n	$s=1$	$s=2$	$s=1$	$s=2$
6	2.37(−6)	1.24(−6)	2.10(−6)	1.24(−6)
7	1.08(−6)	5.31(−7)	1.00(−6)	6.73(−7)
8	5.62(−7)	2.62(−7)	5.13(−7)	3.59(−7)
9	3.20(−7)	1.88(−7)	2.85(−7)	1.93(−7)
10	2.01(−7)	1.31(−7)	1.80(−7)	1.07(−7)

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Milan A. Kovačević and Gradimir V. Milovanović,
Faculty of Electronic Engineering, Department of Mathematics,
P.O. Box 73, 18000 Niš – YUGOSLAVIA
E-mail: `gradeefnis.elfak.ni.ac.yu`