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TOPOLOGY IN A CATEGORY: COMPACTNESS

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Dedicated to Guillaume Brümmer on the occasion of his sixtieth birthday

Abstract: In a category with a subobject structure and a closure operator, we provide a categorical theory of compactness and perfectness which yields a number of classical results of general topology as special cases, including the product theorems by Tychonoff and Frolík, the existence of Stone–Čech compactifications, both for spaces and maps, and the Henriksen–Isbell characterization of perfect maps of Tychonoff spaces. Applications to other categories yield, among other things, an alternative proof for the productivity of categorically compact groups.

0 – Introduction

Already the title of Hausdorff's book "Grundzüge der Mengenlehre" indicates that the development of General Topology was intimately linked to progress in Set Theory. In fact the two fields continued to interact throughout this century to such an extent that it is widely believed that fundamental notions and results can only be formulated and obtained in a set-theoretic setting. In this article we wish to present quite a different approach, showing that the topological themes of Hausdorff separation and of compactness and perfectness allow for a purely categorical treatment which covers the basic elements of the theory in a most

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economical fashion. Topologically-motivated ideas may therefore be applied directly to categories of other branches of mathematics, but we hope that even readers only interested in point-set topology may benefit from implementing the categorical approach.

It has been observed by several authors that the Kuratowski–Mrówka characterization [30], [32] of compact topological spaces as those spaces X with closed projection $X \times X \to Y$ for every other space Y opens the way for a categorical treatment of compactness (see, for example, Manes [31] and Herrlich, Salicrup and Strecker [22]). Our approach follows this line but distinguishes itself from its predecessors in at least three essential aspects.

First of all, we combine the topologically-oriented setting of [31] (having subobjects and closures as the basic structure) with the generality of [22] (working in an arbitrary category) while avoiding the unnecessary restrictions of these papers (being $\mathcal{S}$et-based in the case of [31], and dealing only with closed subobjects rather than with arbitrary subobjects in the case of [22], thus losing much of the topological intuition). In fact, we surpass the generality and applicability of [22] substantially since we do not assume the existence of a (dense, closed)-factorization structure a priori but work with an arbitrary closure operator in the sense of [9].

Secondly, the categorical theory presented here includes Tychonoff's crucial product theorem, which then allows us to construct the Stone–Čech compactification of Hausdorff objects. The proof given here improves the first categorical proof presented in [6] since it allows for a clearer explanation of the choice-based topological result and the choice-free localic theorem as given by [27]. (For a thorough investigation of the role of the Axiom of Choice and compactness in topology we refer to the recent paper [19].)

Thirdly and most importantly, by passing from the given category to its "slices", we are able to take full advantage of the categorical approach and obtain a theory of compact morphisms (also perfect [13] or proper morphisms [2]) almost for free, just by re-interpreting the compactness results for objects in the comma categories of the given category. In particular,

- the Frolík–Bourbaki product theorem [16], [2] for compact maps is in fact just a "sliced" Tychonoff Theorem;
- the existence of a Stone–Čech compactification of a morphism (see [33]) is obtained exactly as its object-counterpart and leads to the (antiperfect, perfect)-factorization studied by [21], [35] and others at different levels of generality;

- the Henriksen–Isbell characterization [24], [23] of perfect maps between Tychonoff spaces is fully available in our general setting, but its proof has become almost a triviality. In fact, the statement that a compact map cannot be extended to any proper dense Hausdorff extension is just the "sliced version" of the easily established categorical fact that any morphism from a compact object to a Hausdorff object must preserve the closure. Likewise, the characteristic property that the extension of the given map to the Stone–Čech compactification of its domain and codomain preserves the remainders becomes an easy categorical observation.

The categorical theory presented here therefore leads to a much shorter presentation of the key elements on compactness and perfectness even if we restrict our attention only to topological spaces. On the other hand, the generality of the categorical approach allows for a variety of interesting applications, only few of which we can mention here. One of them is the observation that the categorical Tychonoff Theorem leads to a new result in the category of topological groups which was established independently with non-categorical methods only recently by Dikranjan and Uspenskij [12]. For further applications we refer to [8] (topology) and to [14] and [7] (algebra).

1 – Subobjects and surjections

1.1. For simplicity, throughout the paper, we consider *a complete category $\mathcal{X}$ with a proper $(\mathcal{E},\mathcal{M})$-factorization system for morphisms* (cf. [15]). Hence $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ is a class of monomorphisms in $\mathcal{X}$, both containing the isomorphisms of $\mathcal{X}$, such that every morphism in $\mathcal{X}$ has an $(\mathcal{E},\mathcal{M})$-factorization and the $(\mathcal{E},\mathcal{M})$-diagonalization property holds. Of the resulting properties for $\mathcal{M}$ (and dually for $\mathcal{E}$), we mention that $\mathcal{M}$ is closed under composition and under limits; it contains the regular monomorphisms of $\mathcal{X}$ and is stable under pullback and left-cancelable (so that $m \cdot n \in \mathcal{M}$ implies $n \in \mathcal{M}$).

For every object X, the class $\mathrm{sub}(X)$ of $\mathcal{M}$-morphisms with codomain X is preordered by

$$n \le n \quad\Leftrightarrow\quad (\exists\, j)\ n \cdot j = m\ ;$$

we write $m \cong n$ if $m \le n$ and $n \le m$. Under the given assumptions, $\mathrm{sub}(X)$ has all set-indexed infima, given by multiple pullback in $\mathcal{X}$. But since $\mathrm{sub}(X)$ may be large, one often assumes $\mathcal{X}$ to have multiple pullbacks of arbitrarily large families of morphisms in $\mathcal{M}$ with common codomain, with the pullbacks belonging to $\mathcal{M}$. In the terminology of [36], this means that $\mathcal{X}$ is *$\mathcal{M}$-complete* or, equivalently, that the $(\mathcal{E},\mathcal{M})$-factorization system for morphisms extends to an $(\mathbb{E},\mathcal{M})$-factorization system for arbitrarily large sinks (cf. [1]). In particular, $\mathrm{sub}(X)$ then has all infima and suprema, with the latter being obtained by $(\mathbb{E},\mathcal{M})$-factoring the given sink of $\mathcal{M}$-morphisms.

We frequently refer to $\mathrm{sub}(X)$ as the *subobject lattice* of X (although only $\mathrm{sub}(X)/\cong$ actually has the structure of a meet-semilattice). Note that $\mathcal{X}$ is automatically $\mathcal{M}$-complete when $\mathcal{X}$ is $\mathcal{M}$-wellpowered, i.e., if each subobject lattice has a small skeleton.

1.2. Every morphism $f\colon X \to Y$ in $\mathcal{X}$ induces a pair of adjoint functors

$$f(-) \dashv f^{-1}(-)\colon \mathrm{sub}(Y) \to \mathrm{sub}(X)\ ,$$

with $f^{-1}(n)$ the pullback of $n \in \mathrm{sub}(Y)$ along f, and with $f(m)$ the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of $f\cdot m$ for $m \in \mathrm{sub}(X)$; in case $f \in \mathcal{M}$, $f(m)$ is simply the composite $f\cdot m$. One always has $m \le f^{-1}(f(m))$, with "$\cong$" holding in case $f \in \mathcal{M}$, as well as $f(f^{-1}(n)) \le n$, with "$\cong$" holding in case $f \in \mathcal{E}$ and $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms. More precisely, one has:

1.3 Proposition. *The following conditions are equivalent:*

(**1**) *For every morphism $f\colon X \to Y$ in $\mathcal{E}$ and all $n \in \mathrm{sub}(Y)$, $f(f^{-1}(n)) \cong n$.*

(**2**) *$\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms.*

(**3**) *The Frobenius Reciprocity Law holds, that is:*

$$f(m \wedge f^{-1}(n)) \cong f(m) \wedge n$$

for all $f\colon X \to Y$ in $\mathcal{X}$, $m \in \mathrm{sub}(X)$ and $n \in \mathrm{sub}(Y)$.

Proof: (**1**)$\Rightarrow$(**2**) is obvious.

(**2**)$\Rightarrow$(**3**): In the diagram below, all the left, the right and the back face are pullback diagrams, hence the front face is a pullback diagram. Consequently, by hypothesis, f' belongs to $\mathcal{E}$, which implies the derived formula.

$$\begin{tikzcd}
 & f^{-1}(N) \arrow[rr] \arrow[dd] & & N \arrow[dd, "n"] \\
M \wedge f^{-1}(N) \arrow[ur] \arrow[rr, "f'"] \arrow[dd] & & f(M) \wedge N \arrow[ur] \arrow[dd] & \\
 & X \arrow[rr, "f"] & & Y \\
M \arrow[ur, "m"] \arrow[rr] & & f(M) \arrow[ur] &
\end{tikzcd}$$

(3)$\Rightarrow$**(1)**: For $f \in \mathcal{E}$ one has $f(1_X) \cong 1_Y$. Hence one applies the Frobenius Reciprocity Law in case $m \cong 1_X$. ∎

1.4. The $(\mathcal{E}, \mathcal{M})$-factorization system is said to satisfy the *Beck-Chevalley Property* if for every pullback diagram

$$\begin{tikzcd}
W \arrow[r, "\phi"] \arrow[d, "\psi"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and all $m \in \mathrm{sub}(X)$, one has

$$\phi(\psi^{-1}(m)) \cong g^{-1}(f(m)) \ .$$

Taking $g = n$ to be in $\mathcal{M}$, one readily sees that the Frobenius Reciprocity Law is a particular case of he Beck–Chevalley Property. In fact, one has (cf. [26]):

1.5 Proposition. *The $(\mathcal{E}, \mathcal{M})$-factorization system satisfies the Beck–Chevalley Property if and only if $\mathcal{E}$ is stable under pullback.*

Proof: One follows the same argumentation as in 1.3 (2)$\Rightarrow$(3)$\Rightarrow$(1), using the diagram

$$\begin{tikzcd}
& W \arrow[rr, "\phi"] \arrow[dd, "\psi"] & & Z \arrow[dd, "g"] \\
\psi^{-1}(N) \arrow[ur] \arrow[rr] \arrow[dd] & & g^{-1}(f(M)) \arrow[ur] \arrow[dd] & \\
& X \arrow[rr, "f"] & & Y \\
M \arrow[ur, "m"] \arrow[rr] & & f(M) \arrow[ur] &
\end{tikzcd}$$

1.6. Stability of $\mathcal{E}$ under pullback is guaranteed in particular when $\mathcal{E}$ is a *surjectivity class*, that is: if there exists a class $\mathcal{P}$ of objects in $\mathcal{X}$ such that a morphism $f\colon X \to Y$ in $\mathcal{X}$ belongs to $\mathcal{E}$ exactly when every $P \in \mathcal{P}$ is projective w.r.t. f (so that every morphism $y\colon P \to Y$ factors as $f \cdot x = y$).

1.7. For every $Y \in \mathcal{X}$, the comma category (or sliced category) $\mathcal{X}/Y$ of morphisms with codomain Y (of which $\mathrm{sub}(Y)$ is a full subcategory) inherits the factorization system from $\mathcal{X}$: let $\mathcal{E}_Y$ and $\mathcal{M}_Y$ denote the class of morphisms in $\mathcal{X}/Y$ whose underlying $\mathcal{X}$-morphisms belong to $\mathcal{E}$ and $\mathcal{M}$, respectively; then $\mathcal{X}/Y$ has a proper $(\mathcal{E}_Y, \mathcal{M}_Y)$-factorization system and is $\mathcal{M}_Y$-complete whenever $\mathcal{X}$ is $\mathcal{M}$-complete. Furthermore, if $\mathcal{E}$ is a surjectivity class in $\mathcal{X}$, then $\mathcal{E}_Y$ is a surjectivity class in $\mathcal{X}/Y$: the needed class $\mathcal{P}_Y$ of objects in $\mathcal{X}/Y$ is given by all morphisms in $\mathcal{X}$ with codomain Y and with domain in $\mathcal{P}$.

Note that the subobject lattice $\mathrm{sub}(f)$ of $f \in \mathcal{X}/Y$ (with $f\colon X \to Y$ in $\mathcal{X}$) is isomorphic to the subobject lattice $\mathrm{sub}(X)$ of $X \in \mathcal{X}$.

2 – Closure operators

2.1. (Cf. [9]) A *closure operator* c of $\mathcal{X}$ with respect to $\mathcal{M}$ is given by a family of maps $c_X : \mathrm{sub}(X) \to \mathrm{sub}(X)$ $(X \in \mathcal{X})$ such that

1. c is extensive ($m \le c_X(m)$ for all $m \in \mathrm{sub}(X)$);

2. c is monotone ($m \le n \Rightarrow c_X(m) \le c_X(n)$ for all $m, n \in \mathrm{sub}(X)$);

3. every morphism $f\colon X \to Y$ is *c-continuous*, that is: $f(c_X(m)) \le c_Y(f(m))$ for all $m \in \mathrm{sub}(X)$, or, equivalently, $c_X(f^{-1}(n)) \le f^{-1}(c_Y(n))$ for all $n \in \mathrm{sub}(Y)$.

Due to 1, every $m\colon M \to X$ in $\mathcal{M}$ factors as

$$M \xrightarrow{\gamma_m} c_X(M) \xrightarrow{c_X(m)} X \ ,$$

and due to 2 and 3, this factorization is functorial; hence, whenever one has $f \cdot m = n \cdot f'$ in $\mathcal{X}$ with $m, n \in \mathcal{M}$, then there is a unique morphism f'' rendering the diagram

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d] & N \arrow[d] \\ c_X(M) \arrow[r, "f''"] \arrow[d] & c_Y(N) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutative. In other words, with $2 = \{\cdot \longrightarrow \cdot\}$ and $\mathcal{M}^2$ the full subcategory of $\mathcal{X}^2$ with object class $\mathcal{M}$, a closure operator is equivalently described by a pointed endofunctor of $\mathcal{M}^2$ which commutes with codomain functor $\mathcal{M}^2 \to \mathcal{X}$.

2.2. A subobject $m \in \mathrm{sub}(X)$ is *c-closed* if $m \cong c_X(m)$ (so that γ_m is iso), and it is *c-dense* if $c_X(m)$ is iso. More generally, a morphism $f\colon X \to Y$ is c-dense if $f(1_X)$ is c-dense. With $\mathcal{M}^c$ and $\mathcal{E}^c$ we denote the class of all c-closed subobjects and of all c-dense morphisms, respectively. It follows from the functoriality of the closure operator that the $(\mathcal{E}^c, \mathcal{M}^c)$-diagonalization property holds in $\mathcal{X}$, but morphisms in $\mathcal{X}$ may not have $(\mathcal{E}^c, \mathcal{M}^c)$-factorizations (cf. [9]).

$\mathcal{M}^c$ and $\mathcal{E}^c$ have the same closedness and stability properties which we have mentioned for $\mathcal{M}$ and $\mathcal{E}$ in 1.1 *except* closedness under composition. (Also, in general, $\mathcal{M}^c$ is left-cancelable only w.r.t. monomorphisms, so that $m{\cdot}n \in \mathcal{M}^c$ with m monic implies $n \in \mathcal{M}^c$.) Failure of closedness under composition is directly linked to failure of $(\mathcal{E}^c, \mathcal{M}^c)$-factorizability, as we shall see next.

2.3. A closure operator c is *idempotent* if $c_X(m)$ is c-closed for every $m \in \mathrm{sub}(X)$, and it is *weakly hereditary* if $\gamma_X(m)$ is c-dense for every $m \in \mathrm{sub}(X)$, $X \in \mathcal{X}$. The following properties are shown in [9] and [11]:

1. If c is idempotent, then $\mathcal{E}^c$ is closed under composition, and if c is weakly hereditary, then $\mathcal{M}^c$ is closed under composition.

2. One may have both $\mathcal{E}^c$ and $\mathcal{M}^c$ closed under composition, but with c neither idempotent nor weakly hereditary.

3. The following are equivalent:

i. morphisms can be $(\mathcal{E}^c, \mathcal{M}^c)$-factored,

ii. c is idempotent and $\mathcal{M}^c$ is closed under composition,

iii. c is weakly hereditary and $\mathcal{E}^c$ is closed under composition,

iv. c is idempotent and weakly hereditary.

2.4. If $z: Z \to X$ has a retraction (so that there is a morphism p with $p \cdot z = 1$), then the closure $c_Z(m)$ of any $m \in \mathrm{sub}(Z)$ can be computed as $z^{-1}(c_X(z \cdot m))$. One calls a closure operator c *hereditary* if $c_Z(m) \cong z^{-1}(c_X(z \cdot m))$ holds for all $m: M \to Z$ and $z: Z \to X$ in $\mathcal{M}$. It was shown in [11] that c is hereditary if and only if c is weakly hereditary and the class $\mathcal{E}^c$ is left-cancelable w.r.t. $\mathcal{M}$, i.e., $m \cdot f \in \mathcal{E}^c$ with $m \in \mathcal{M}$ holds only if $f \in \mathcal{E}^c$.

2.5. Closure operators w.r.t. $\mathcal{M}$ are preordered pointwise: $c \leq d$ iff $c_X(m) \leq d_X(m)$ for all $m \in \mathrm{sub}(X)$, $X \in \mathcal{X}$. Infima and suprema of families of closure operators exist to the extent infima and suprema of subobjects exist and are formed "pointwise". Since idempotency is stable under meet and since weak hereditariness is stable under join, a closure operator c has an *idempotent hull* $\hat{c}$ and a *weakly hereditary core* $\check{c}$ whenever $\mathcal{X}$ is $\mathcal{M}$-complete. One can prove: $\mathcal{M}^{\hat{c}} = \mathcal{M}^c$, $\mathcal{E}^{\check{c}} = \mathcal{E}^c$, $\hat{c}$ is weakly hereditary if c is, and $\check{c}$ is idempotent if c is (cf. [11]). If $\mathcal{X}$ is $\mathcal{M}$-wellpowered, then $\hat{c}$ and $\check{c}$ can be obtained by transfinite iteration of c (see [9] for details); one usually writes c^∞ and c_∞ in this case.

2.6. For a small family $(X_i)_{i \in I}$ of objects in $\mathcal{X}$ and any subset $J \subseteq I$, let $X_J := \prod_{i \in J} X_i$ and denote by $\pi_J: X_I \to X_J$ the obvious projection. A closure operator c is said to satisfy the *finite structure property of products* (FSPP) if for all $m, x \in \mathrm{sub}(X_I)$ one has $x \leq c_{X_I}(m)$ whenever $\pi_F(x) \leq c_F(\pi_F(m))$ holds for every finite subset $F \subseteq I$ (cf. [11]). Equivalently this means that

$$(*) \qquad c_X(m) \cong \bigvee_{F \text{ finite}} \pi_F^{-1}(c_{X_F}(\pi_F(m)))$$

for every $m \in \mathrm{sub}(X)$.

2.7. Keeping the notation of 2.6 we observe that the morphisms π_F present the product X_I as an inverse limit of the finite products X_F. The purpose of FSPP is to guarantee that this presentation is also available for c-closures, in the following sense. Given $m: M \to X$ in $\mathcal{M}$, c-continuity of π_F gives a morphism τ_F rendering the diagram

$$\begin{tikzcd} c_{X_I}(M) \arrow[r] \arrow[d, "\tau_F"'] & X_I \arrow[d, "\pi_F"] \\ c_{X_F}(\pi_F(M)) \arrow[r] & X_F \end{tikzcd}$$

commutative, for every finite $F \subseteq I$, naturally in F; now one easily shows:

2.8 Proposition. *A closure operator c satisfies FSPP if and only if*

$$(**) \qquad c_{X_I}(M) \cong \varprojlim_F c_{X_F}(\pi_F(M))$$

for all $m\colon M \to X_I$ in $\mathcal{M}$.

Proof: Given a compatible family $(\alpha_F : A \to c_{X_F}(\pi_F(M)))$, one first obtains an arrow $b\colon A \to X_I \cong \varprojlim X_F$ with $\pi_F \cdot b = c_{X_F}(\pi_F(m)) \cdot \alpha_F$ for all F and then morphisms $\beta_F : A \to \pi_F^{-1}(c_{X_F}(\pi_F(M)))$ through which b factors. Hence $(*)$ implies $(**)$. Conversely, given a morphism b which factors through every $\pi_F^{-1}(c_{X_F}(\pi_F(M)))$ by (a necessarily unique arrow) β_F, the family (β_F) is automatically compatible. Now $(**)$ shows that b must factor through $c_{X_I}(M)$. ∎

2.9. A closure operator c of $\mathcal{X}$ w.r.t. $\mathcal{M}$ induces a closure operator c^Y w.r.t. $\mathcal{M}_Y$ for every $Y \in \mathcal{X}$: for every $m : h \to f$ in $\mathcal{M}_Y$ with $f : X \to Y$, let $c^Y_f(m) := c_X(m) : f \cdot c_X(m) \to f$, i.e., closures are formed as in $\mathcal{X}$. If c is idempotent or (weakly) hereditary, the same is true for c^Y. Also FSPP is inherited, provided c is hereditary, as we shall show next.

2.10 Proposition. *For a hereditary closure operator c of $\mathcal{X}$ with FSPP and for every $Y \in \mathcal{X}$, also the induced closure operator c^Y of $\mathcal{X}/Y$ has FSPP.*

Proof: For a small family $(f_i : X_i \to Y)_{i\in I}$ of objects in $\mathcal{X}/Y$ and any subset $J \subseteq I$, the product of $(f_i)_{i\in J}$ in $\mathcal{X}/Y$ is given by its multiple pullback $g_J : U_J \to Y$ in $\mathcal{X}$, and this can be constructed by first forming the product $(q^J_i\colon X_J \to X_i)_{i\in J}$ in $\mathcal{X}$ and then the joint equalizer $u_J\colon U_J \to X_J$ of $(f_i \cdot q^J_i)_{i\in J}$ in $\mathcal{X}$; one then has $g_J = f_i \cdot q^J_i \cdot u_J$ for all $i \in J$. For every finite subset $F \subseteq I$, the projection $v_F\colon g_I \to g_F$ in $\mathcal{X}/Y$ is given by the $\mathcal{X}$-morphism $v_F\colon U_I \to U_F$ with $u_F \cdot v_F = \pi_F \cdot u_I$ and π_F as in 2.6.

Let us now assume $v_F(x) \le c_{U_F}(v_F(m))$ for $x, m \in \mathrm{sub}(U_I)$. One then has:

$$\pi_F(u_I \cdot x) \cong u_F \cdot v_F(x) \le u_F \cdot c_{U_F}(v_F(m)) \le c_{X_F}(u_F \cdot v_F(m)) \cong c_{X_F}(\pi_F(u_I \cdot m))\ .$$

FSPP for c yields $u_I \cdot x \le c_{X_I}(u_I \cdot m)$, hence

$$x \le u_I^{-1}(c_{X_I}(u_I \cdot m)) \cong c_{U_I}(m)$$

since c is hereditary. This proves FSPP for c^Y. ∎

2.11. Our standard example is the category $\mathcal{X} = \mathcal{T}\mathrm{op}$ of topological spaces with its (surjective, embedding)-factorization structure and its natural Kuratowski closure operator, which is idempotent and hereditary and satisfies FSPP.

3 – Closure-preserving morphisms

3.1. Let c be a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$. A morphism $f\colon X \to Y$ in $\mathcal{X}$ is *c-preserving* if $f(c_X(m)) \cong c_Y(f(m))$ for all $m \in \mathrm{sub}(X)$. In this case $f(-)$ maps c-closed subobjects to c-closed subobjects, and this property is equivalent to c-preservation in case c is idempotent, but not in general.

3.2 Proposition.

(**1**) *Every isomorphism of $\mathcal{X}$ is c-preserving, and c-preserving morphisms are closed under composition.*

(**2**) *Let the composite $g \cdot f$ be c-preserving. Then f is c-preserving if $g \in \mathcal{M}$, and g is c-preserving if $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms.*

(**3**) (**a**) *Every c-preserving morphism in $\mathcal{M}$ is a c-closed subobject.*

(**b**) *If c is weakly hereditary, then every c-closed subobject is a c-preserving morphism; in fact, weak hereditariness is not needed if the subobject has a retraction.*

(**c**) *If c is idempotent and if every c-closed subobject is a c-preserving morphism, then c is weakly hereditary.*

(**4**) *If c is hereditary and if $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms, then every pullback of a c-preserving map along an $\mathcal{M}$-morphism is c-preserving; in fact, hereditariness is not needed if the pullback of the given $\mathcal{M}$-morphism has a retraction.*

Proof: (**1**) is trivial. (**2**) For $f\colon X \to Y$, $g\colon Y \to Z$ with $g \cdot f$ c-preserving, in case $g \in \mathcal{M}$ one has

$$\begin{aligned} c_Y(f(m)) &\cong c_Y(g^{-1}(g(f(m)))) \\ &\le g^{-1}(c_Z(g(f(m)))) \\ &\cong g^{-1}(g(f(c_X(m)))) \cong f(c_X(m)) \end{aligned}$$

for all $m \in \mathrm{sub}(X)$, as desired. In case $f \in \mathcal{E}$ with $\mathcal{E}$ as in 1.3 one concludes similarly

$$\begin{aligned} c_Z(g(n)) &\cong c_Z(g(f(f^{-1}(n)))) \\ &\cong g(f(c_X(f^{-1}(n)))) \\ &\le g(f(f^{-1}(c_Y(n)))) \cong g(c_Y(n)) \ . \end{aligned}$$

(**3**) Let $m : M \to X$ be in $\mathcal{M}$. If m is c-preserving, then it preserves in particular the closure of 1_M, hence $m \cong c_X(m)$. Conversely, let m be c-closed and assume c to be weakly hereditary. Then, for every $k : K \to M$, in the factorization

$$K \xrightarrow{\gamma_{m\cdot k}} c_X(K) \xrightarrow{c_X(m\cdot k)} X \ ,$$

the morphism $\gamma_{m\cdot k}$ is c-dense, with $c_X(m\cdot k) \le c_X(m) \cong m$. Hence the functoriality of c shows $c_X(m\cdot k) \le m \cdot c_M(k)$, as desired. If m has a retraction, the use of weak hereditariness of c can be avoided, due to the first statement of 2.4. This completes the proof of (a) and (b). Finally, under the hypotheses of (c), the morphism $n := c_X(m) \colon N \to X$ is c-preserving, hence

$$n \cdot c_N(\gamma_m) \cong c_X(n \cdot \gamma_m) = c_X(m) = n \ .$$

Consequently, $c_N(\gamma_m)$ is iso, and the proof of the weak hereditariness is complete.

(**4**) For $f : X \to Y$ in $\mathcal{X}$ and $n : N \to Y$ in $\mathcal{M}$, we must show that $f' : f^{-1}(N) \to N$ is c-preserving. But for every $k \colon K \to f^{-1}(N)$ in $\mathcal{M}$, we may apply the Frobenius Reciprocity Law to $m := c_X(n' \cdot k)$ with $n' = f^{-1}(n)$ and obtain

$$\begin{aligned} c_N(f'(k)) &\cong n^{-1}(n(c_N(f'(k)))) \\ &\le n^{-1}(c_Y(f(n'\cdot k))) \\ &\cong n^{-1}(f(c_X(n'\cdot k))) \\ &\cong f'((n')^{-1}(m)) \ . \end{aligned}$$

If c is hereditary or if n' has a retraction, $(n')^{-1}(m) \cong c_{f^{-1}(N)}(k)$, and this completes the proof. ∎

We shall apply these rules in Section 5 below.

4 – Hausdorff separation

4.1. Let c be a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$. An object $A \in \mathcal{X}$ is *c-Hausdorff* if for all $u, v\colon X \to A$ in $\mathcal{X}$ and $m \in \mathrm{sub}(X)$ with $u \cdot m = v \cdot m$, also $c_X(m) \cdot u = c_X(m) \cdot v$. By $\mathcal{H}\mathrm{aus}(c)$ we denote the full subcategory of c-Hausdorff objects in $\mathcal{X}$.

Recall that an arbitrary family $\sigma = (p_i : A \to A_i)_{i \in I}$ of morphisms in $\mathcal{X}$ with common domain is *monic* (or a *mono-source*) if for all $u, v\colon X \to A$ in $\mathcal{X}$, $p_i \cdot u = p_i \cdot v$ for all $i \in I$ always implies $u = v$. A full subcategory $\mathcal{A}$ of $\mathcal{X}$ is *closed under mono-sources* if $A_i \in \mathcal{A}$ for all $i \in I$ implies $A \in \mathcal{A}$, for every mono-source σ in $\mathcal{X}$.

4.2 Proposition. *(Cf. [3])*

(**1**) *$\mathcal{H}\mathrm{aus}(c)$ is closed under mono-sources in $\mathcal{X}$, in particular under all limits in $\mathcal{X}$. Consequently, $\mathcal{H}\mathrm{aus}(c)$ is extremally-epireflective in $\mathcal{X}$ if $\mathcal{X}$ is $\mathcal{E}$-cowellpowered.*

(**2**) *An object A of $\mathcal{X}$ is c-Hausdorff if and only if the diagonal morphism $\delta_A\colon A \to A \times A$ is c-closed.*

(**3**) *$\mathcal{H}\mathrm{aus}(c) = \mathcal{H}\mathrm{aus}(\hat{c})$ if $\mathcal{X}$ is $\mathcal{M}$-complete.*

Proof: (**1**) For a mono-source σ as in 4.1, and for $u, v : X \to A$ and $m \in \mathrm{sub}(X)$, assume $u \cdot m = v \cdot m$. Then $(p_i \cdot u) \cdot m = (p_i \cdot v) \cdot m$ implies $(p_i \cdot u) \cdot c_X(m) = (p_i \cdot v) \cdot c_X(m)$ whenever $A_i \in \mathcal{H}\mathrm{aus}(c)$, hence $u \cdot c_X(m) = v \cdot c_X(m)$ since σ is monic.

Reflectivity of $\mathcal{H}\mathrm{aus}(c)$ in $\mathcal{X}$ follows from the General Adjoint Functor Theorem (see [1] and 7.2 below).

(**2**) The diagonal δ_A is the equalizer of the product projections $p_1, p_2 : A \times A \to A$. If A is c-Hausdorff, one has $p_1 \cdot c(\delta_A) = p_2 \cdot c(\delta_A)$, hence $c(\delta_A) \le \delta_A$ by the equalizer property. Consequently, δ_A is c-closed.

Conversely, let δ_A be c-closed and assume $u \cdot m = v \cdot m$ for $u, v\colon X \to A$ and $m \in \mathrm{sub}(X)$. Then the diagram

$$\begin{tikzcd} M \arrow[r, "u \cdot m"] \arrow[d, "m"'] & A \arrow[d, "\delta_A"] \\ X \arrow[r, "<u, v>"'] & A \times A \end{tikzcd}$$

commutes, and the functoriality of c (cf. 2.1) yields a morphism $t\colon c_X(M) \to A$ with $\delta_A \cdot t = \langle u, v\rangle \cdot c_X(m)$, hence $u \cdot c_X(m) = t = v \cdot c_X(m)$.

(**3**) Follows from (2) since $\mathcal{M}^c = \mathcal{M}^{\hat{c}}$ (cf. 2.5). ∎

4.3. A morphism $f\colon A \to B$ of $\mathcal{X}$ is *c-Hausdorff* if f is c^B-Hausdorff as an object of $\mathcal{X}/B$. This simply means that for all $u, v\colon X \to A$ in $\mathcal{X}$ and $m \in \mathrm{sub}(X)$ with $u \cdot m = v \cdot m$ and $f \cdot u = f \cdot v$, one has $u \cdot c_X(m) = v \cdot c_X(m)$; equivalently, the diagonal morphism $\delta_f\colon A \to \mathrm{Ker}\, f = A \times_B A$ is c-closed. Trivially, if A is c-Hausdorff, every $f\colon A \to B$ is c-Hausdorff, and every monomorphism of $\mathcal{X}$ is c-Hausdorff. Moreover:

4.4 Lemma. *For the commutative diagrams*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_i"'] & B \arrow[d, "q_i"] \\ A_i \arrow[r, "f_i"] & B_i \end{tikzcd} \qquad (i \in I)$$

let the extended source $(f, p_i)_{i\in I}$ be monic. Then, if each f_i is c-Hausdorff, also f is c-Hausdorff.

Proof: For $u, v\colon X \to A$ and $m \in \mathrm{sub}(X)$ with $f \cdot u = f \cdot v$ and $u \cdot m = v \cdot m$ one has $f_i \cdot p_i \cdot u = f_i \cdot p_i \cdot v$, hence $p_i \cdot u \cdot c_X(m) = p_i \cdot v \cdot c_X(m)$ for each $i \in I$. Since also $f \cdot u \cdot c_X(m) = f \cdot v \cdot c_X(m)$, with the mono-assumption one concludes that $u \cdot c_X(m) = v \cdot c_X(m)$. ∎

4.5 Proposition. *The class of c-Hausdorff morphisms in $\mathcal{X}$ is left-cancelable, closed under limits and stable under (multiple) pullback. It is also closed under composition if c is weakly hereditary.*

Proof: For the first statement, apply the lemma to the diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1"'] & B \arrow[d, "g"] \\ A \arrow[r, "g \cdot f"] & C \end{tikzcd} \quad \begin{tikzcd} \lim A_i \arrow[r, "f"] \arrow[d] & \lim B_i \arrow[d] \\ A_i \arrow[r, "f_i"] & B_i \end{tikzcd} \quad \begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \begin{tikzcd} P \arrow[r, "p"] \arrow[d, "p_i"'] & B \arrow[d, "1"] \\ A_i \arrow[r, "f_i"] & B \end{tikzcd}$$

For the second statement, consider c-Hausdorff morphisms $f\colon A \to B$ and $g\colon B \to C$ and assume $g \cdot f \cdot u = g \cdot f \cdot v$ and $u \cdot m = v \cdot m$ for $u, v\colon X \to A$ and $m \in \mathrm{sub}(X)$. Since g is c-Hausdorff, $f \cdot u \cdot c_X(m) = f \cdot v \cdot c_X(m)$ follows.

With $\gamma_m\colon\ M \to Y := c_X(M)$ as in 2.1 one then concludes $u \cdot c_X(m) \cdot c_Y(\gamma_m) = v \cdot c_X(m) \cdot c_Y(\gamma_m)$ since f is c-Hausdorff. But when c is weakly hereditary, $c_Y(\gamma_m)$ is iso, and $u \cdot c_X(m) = v \cdot c_X(m)$ follows. ∎

4.6. In the standard example $\mathcal{X} = \mathcal{T}\mathrm{op}$ with the usual closure, the c-Hausdorff objects are the Hausdorff spaces, and a continuous map $f\colon\ A \to B$ is c-Hausdorff if for any $u, v \in A$ with $u \neq v$ and $f(u) = f(v)$ one has disjoint neighbourhoods U, V of u, v, respectively. Hence we have the same notion of T_2-separation for maps as the one used in [25] and in the recent paper [29].

5 – Compactness and perfectness

5.1. Let c be a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$. An object $X \in \mathcal{X}$ is *c-compact* if the product projection $p_Y\colon\ X \times Y \to Y$ is c-preserving for every object $Y \in \mathcal{X}$. By $\mathcal{C}\mathrm{omp}(c)$ we denote the full subcategory of c-compact objects in $\mathcal{X}$, and we put

$$\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c) = \mathcal{C}\mathrm{omp}(c) \cap \mathcal{H}\mathrm{aus}(c)\ .$$

5.2 Proposition.

(**1**) *If X in $\mathcal{X}$ is c-compact and $m\colon\ M \to X$ in $\mathcal{M}$ is c-closed, with c weakly hereditary, then M is c-compact; in fact, weak hereditariness is not needed if m has a retraction.*

(**2**) *For X c-compact and Y c-Hausdorff, every morphism $f\colon\ X \to Y$ is c-preserving.*

(**3**) *For $f\colon\ X \to Y$ in $\mathcal{E}$, with $\mathcal{E}$ stable under pullback, if X is c-compact, so is Y.*

(**4**) *$\mathcal{C}\mathrm{omp}(c)$ is closed under finite products in $\mathcal{X}$.*

(**5**) *If c is weakly hereditary, then $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is closed under finite limits in $\mathcal{X}$.*

(**6**) *$\mathcal{C}\mathrm{omp}(c) \subseteq \mathcal{C}\mathrm{omp}(\hat{c})$ if $\mathcal{X}$ is $\mathcal{M}$-complete.*

Proof: (**1**) The projection $M \times Y \to Y$ decomposes as

$$M \times Y \xrightarrow{m\times 1} X \times Y \longrightarrow Y\ ,$$

with both morphisms c-preserving; in fact, $m \times 1$ is c-closed (cf. 2.2), hence c-preserving by 3.2(3)(b).

(**2**) The morphism f factors as

$$X \stackrel{\langle 1_X, f\rangle}{\longrightarrow} X \times Y \longrightarrow Y$$

with both morphisms c-preserving; in fact, the graph of f is c-closed since it is the pullback of the diagonal of Y which is c-closed (cf. 4.2(2)):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "< 1, f >"'] & Y \arrow[d, "\delta_Y =< 1, 1 >"] \\ X \times Y \arrow[r, "f \times 1"] & Y \times Y \end{tikzcd}$$

(**3**) For every object Z, the projection $X \times Z \to Z$ factors as

$$X \times Z \stackrel{f\times 1}{\longrightarrow} Y \times Z \longrightarrow Z \ .$$

As a pullback of f, the morphism $f \times 1$ belongs to $\mathcal{E}$. Hence the cancellation rule 3.2(2) yields c-preservation of the projection $Y \times Z \to Z$.

(**4**) For X, Y c-compact and any object Z, the projection $(X \times Y) \times Z \to Z$ factors as

$$X \times (Y \times Z) \longrightarrow Y \times Z \longrightarrow Z \ ,$$

with both factors c-preserving. The terminal object T of $\mathcal{X}$ is trivially c-compact since the projection $T \times Z \to Z$ is iso for every Z.

(**5**) According to (4) and 4.2(1), $\mathcal{C}$omp$\mathcal{H}$aus(c) is closed under finite products in $\mathcal{X}$. For every c-Hausdorff object A, in any equalizer diagram

$$M \stackrel{m}{\longrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} A$$

of $\mathcal{X}$, one has $f \cdot c_X(m) = g \cdot c_X(m)$, hence $m \cong c_X(m)$ is c-closed. Consequently, if $X \in \mathcal{C}$omp$\mathcal{H}$aus(c), also $M \in \mathcal{C}$omp$\mathcal{H}$aus(c), by (1) and 4.2(1).

(**6**) is trivial. ∎

5.3. A morphism $f : A \to B$ in $\mathcal{X}$ is *c-compact* if f is c^B-compact as an object of $\mathcal{X}/B$. Since products in $\mathcal{X}/B$ are given by pullback, c-compactness of f simply means that f is *stably c-preserving*, i.e., in every pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{f} & B \end{array}$$

in $\mathcal{X}$, f' is c-preserving. The morphism f is called *c-perfect* if it is c-compact and c-Hausdorff.

There is an immediate connection between c-compact objects and c-compact morphisms:

5.4 Proposition. *If in the pullback diagram above Y and f are c-compact, then also X is c-compact. In particular, a c-compact morphism is c-preserving and has c-compact fibres.*

Proof: When crossing the pullback diagram above with 1_Z for any object Z, we obtain the pullback diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{f' \times 1} & Y \times Z \\ {\scriptstyle h' \times 1}\downarrow & & \downarrow{\scriptstyle h \times 1} \\ A \times Z & \xrightarrow{f \times 1} & B \times Z \end{array}$$

Since $f \times 1$ is a pullback of f, also $f' \times 1$ is a pullback of f and therefore c-preserving. When composing it with the c-preserving projection $Y \times Z \to Z$, we see that also the projection $X \times Z \to Z$ is c-preserving, as desired.

Fibres of f are pullbacks as above with Y the terminal object, which is c-compact. ∎

Next we collect some basic properties of c-compact morphisms.

5.5 Proposition.

(1) *Every isomorphism is c-compact, and every c-closed morphism in $\mathcal{M}$ is c-compact if c is weakly hereditary.*

(2) *The class of c-compact morphisms in $\mathcal{X}$ is stable under pullback and closed under composition and under the formation of finite direct products.*

(3) *Let the composite $g \cdot f$ be c-compact. Then:*

(a) *f is c-preserving if g is c-Hausdorff;*

(**b**) *f is c-compact if g is a monomorphism;*

(**c**) *g is c-compact if $f \in \mathcal{E}$, with $\mathcal{E}$ stable under pullback.*

Proof: (**1**) For the second statement, observe that $\mathcal{M}^c$ is stable under pullback and apply 3.2(3).

(**2**) Since pullback diagrams compose, the first two statements are trivial. The direct product $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ of two c-compact morphisms f_1, f_2 in $\mathcal{X}$ can be presented as the (fibred) product of $f'_1, f'_2 \in \mathcal{X}/Y_1 \times Y_2$, with f'_i the pullback of f_i along the product projection $Y_1 \times Y_2 \to Y_i$ $(i = 1, 2)$

$$\begin{tikzcd}
X_1 \times X_2 \arrow[rr] \arrow[dd, "f_1 \times f_2"'] \arrow[dr] & & X_i \arrow[dd, "f_i"] \\
& P_i \arrow[ur] \arrow[dl, "f'_i"] & \\
Y_1 \times Y_2 \arrow[rr] & & Y_i
\end{tikzcd}$$

Since f'_1, f'_2 are c-compact, 5.2(4) (applied to $\mathcal{X}/Y_1 \times Y_2$) gives the c-compactness of $f_1 \times f_2$.

(**3**)(**a**) and (**c**) follow from 5.2(2) and (3), while (**b**) follows from the pullback stability of c-compact morphisms:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "1"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "g \cdot f"] & \bullet
\end{tikzcd}$$

∎

5.6 Corollary. *The class of c-perfect morphisms in $\mathcal{X}$ contains all isomorphisms, even all c-closed morphisms of $\mathcal{M}$ if c is weakly hereditary. It is stable under pullback, left-cancelable w.r.t. monomorphisms and closed under the formation of finite direct products. It is also closed under composition if c is weakly hereditary.*

Proof: Combine 4.5 and 5.5. ∎

5.7 Corollary. *For $f\colon X \to Y$ in $\mathcal{E}$ c-compact, with $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms, $X \in \mathcal{H}\mathrm{aus}(c)$ implies $Y \in \mathcal{H}\mathrm{aus}(c)$.*

Proof: Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta_X"'] & Y \arrow[d, "\delta_Y"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd}$$

in which δ_X and $f \times f$ are c-preserving due to 4.2(2), 3.2(3) and 5.5(2). Now apply 3.2 again to conclude that δ_Y is c-closed. ∎

We conclude this section by giving two criteria which are useful when checking c-compactness for morphisms and for objects in concrete cases.

5.8 Proposition. *Let c be hereditary and $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms. Then a morphism $f\colon X \to Y$ is c-compact if and only if $f \times 1_Z\colon X \times Z \to Y \times Z$ is c-preserving for every object Z.*

Proof: The pullback $f'\colon W \to Z$ of f along a morphism $h\colon Z \to Y$ may be obtained in two steps, as follows:

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "<h', f'>"'] & Z \arrow[d, "<h, 1>"] \\ X \times Z \arrow[r, "f \times 1"] \arrow[d] & Y \times Z \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The upper diagram is a pullback diagram since the lower and the whole diagram are pullbacks. Hence, if $f \times 1$ is c-preserving, also f' is c-preserving, by 3.2(4). ∎

5.9 Corollary. *Under the hypotheses of 5.8, every morphism $f\colon X \to Y$ with X c-compact and Y c-Hausdorff is c-compact.*

Proof: The morphism f in $\mathcal{X}$ can be considered a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & T & \end{tikzcd}$$

in $\mathcal{X}/T$, with T the terminal object of $\mathcal{X}$ and with u c-compact and v c-Hausdorff (see 1.7). As pullbacks of u and v in $\mathcal{X}$, the projections $p: X \times Z \to Z$ and $q: Y \times Z \to Z$ are c-compact and c-Hausdorff, respectively, for every $Z \in \mathcal{X}$. By 5.2(2), the morphism

$$\begin{tikzcd} X \times Z \arrow[rr, "f \times 1"] \arrow[dr, "p"'] & & Y \times Z \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

is c-preserving in $\mathcal{X}/Y$, hence also in $\mathcal{X}$. Now 5.8 applies. ∎

5.10. For c idempotent, let us call a sink $(g_i: G_i \to Y)_{i \in I}$ in $\mathcal{X}$ *c-final* if any $m \in \operatorname{sub}(Y)$ is c-closed whenever each $g_i^{-1}(m)$ is c-closed. A class $\mathcal{G}$ of objects in $\mathcal{X}$ is *c-generating* in $\mathcal{X}$ if for every object Y there is a c-final sink with codomain Y and with all domains in $\mathcal{G}$.

5.11 Proposition. *Let c be idempotent, let $\mathcal{E}$ be stable under pullback, and let $\mathcal{G}$ be a c-generating class of objects in $\mathcal{X}$. Then an object X in $\mathcal{X}$ is c-compact if the projection $X \times G \to G$ is c-preserving for every $G \in \mathcal{G}$.*

Proof: Let Y be in $\mathcal{X}$ and pick a c-final sink $(g_i: G_i \to Y)_{i \in I}$. For every $i \in I$, we have a pullback diagram

$$\begin{tikzcd} X \times G_i \arrow[r, "p_i"] \arrow[d, "1 \times g_i"'] & G_i \arrow[d, "g_i"] \\ X \times Y \arrow[r, "p"] & Y \end{tikzcd}$$

By the Beck–Chevalley Property 1.5, for every $m \in \operatorname{sub}(X \times Y)$, $g_i^{-1}(p(m)) \cong p_i((1 \times g_i)^{-1}(m))$. Therefore, since each p_i is c-preserving and since (g_i) is c-final, c-closedness of m yields c-closedness of $p(m)$. Since c is idempotent, this suffices to conclude c-preservation of p. ∎

5.12. In case c is also hereditary, it is sufficient in 5.11 to assume $\mathcal{G}$ to be *c-subgenerating*, in the following sense: for every object $Y \in \mathcal{X}$ there is a morphism $Y \to Z$ in $\mathcal{M}$ and a c-final sink with codomain Z and domains in $\mathcal{G}$. This follows immediately from 5.11 and 3.2(4).

5.13. In the standard example $\mathcal{X} = \mathcal{T}\mathrm{op}$ with c the usual closure, c-compact objects are the usual compact spaces, due to the Kuratowski–Mrówka Theorem (cf. [13]). As stably-closed maps, the c-compact morphisms are the proper maps in the sense of Bourbaki [2], which are equivalently described as closed maps with compact fibres; such maps are called compact in [33] and [29]. Many authors call such maps perfect, while Engelking [13] requires Hausdorffness of the domain of the map. Hence the notion of perfectness adopted here lies in between the no-separation-at-all notion and Engelking's terminology. The properties proved here in general give the standard properties for compact spaces and perfect maps, with the important exceptions of the product theorems by Tychonoff and Frolík, which follow next.

5.14 Remark. In general, c-compactness of a morphism is not equivalently described by the necessary conditions of being c-preserving and having c-compact fibres (cf. 5.4). Simply consider any c-preserving $f\colon X \to Y$ which is not c-compact and find any morphism $h\colon Y \to Z$ *not* in $\mathcal{E}$. Then the $\mathcal{X}/Z$ morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

is c-preserving, and the condition of having c-compact fibres is vacuously satisfied, since a fibre of h would have to be given by a section of h, forcing h to belong to $\mathcal{E}$.

Not even the hypothesis that pullbacks of morphisms $V \to Y$ with V c-compact along f have again c-compact domain would make a c-preserving morphism $f : X \to Y$ c-compact, in general. To see this, again pass to the sliced category $\mathcal{X}/Z$, with $\mathcal{X} = \mathcal{T}\mathrm{op}$, Z a two-point indiscrete space, and h a constant map.

6 – Tychonoff's and Frolík's Theorem

6.1. We continue to consider a closure operator c of $\mathcal{X}$ w.r.t. $\mathcal{M}$ as well as objects Y, $X_i \in \mathcal{X}$, $i \in I$. Keeping the notation of 2.6/2.7, for every $m\colon M \to X_I \times Y$ in $\mathcal{M}$, we have a commutative diagram

$$\begin{tikzcd}
 & & c(M) \arrow[d] \arrow[dddll, "\overline{\tau}_F"'] \arrow[dddrr, "e"] & & \\
 & & X_I \times Y \arrow[dl, "\overline{\pi}_F"'] \arrow[dr, "p"] & & \\
 & X_F \times Y \arrow[rr, "p_F"] & & Y & \\
c(\overline{\pi}_F(M)) \arrow[ur] \arrow[rrrr, "e_F"] & & & & c(p(M)) \arrow[ul]
\end{tikzcd}$$

with projections p, p_F, $\overline{\pi}_F = \pi_F \times 1_Y$ and induced morphisms e, e_F, $\overline{\tau}_F$ for every finite subset $F \subseteq I$. We observe that the morphisms $\overline{\pi}_F$ present $X_I \times Y$ as an inverse limit of the objects $X_F \times Y$, and the morphisms $\overline{\tau}_F$ assume the same role under FSPP (see 2.8).

If all objects X_i are c-compact, then also X_F is c-compact for every finite set $F \subseteq I$ by 5.2(4), hence p_F is c-preserving. But the latter property means equivalently that e_F belongs to $\mathcal{E}$. We say that c and $\mathcal{E}$ have the *inverse-limit stability property of products* (ILSPP) if $e_F \in \mathcal{E}$ for all $F \subseteq I$ implies $e \in \mathcal{E}$. In this terminology, one obviously has:

6.2 Proposition. *If c and $\mathcal{E}$ have FSPP and ILSPP, then $\mathcal{C}\mathrm{omp}(c)$ is closed under direct products in $\mathcal{X}$.* ∎

In the presence of FSPP and the Axiom of Choice, ILSPP can be obtained as follows:

6.3 Lemma. *Let $\mathcal{E}$ be a surjectivity class (cf. 1.6). Then, under the Axiom of Choice, FSPP implies ILSPP.*

Proof: We may assume that the indexing system I is given by an ordinal number κ; for notational purposes we distinguish between κ and the set $\overline{\kappa} = \{\beta\colon \beta < \kappa\}$. Extending the notation of 6.1 naturally, for all $\alpha \leq \gamma \leq \kappa$ one has a commutative diagram

$$\begin{tikzcd}
 & & c(\overline{\pi}_\gamma(M)) \arrow[dddll, "\overline{\tau}^\gamma_\alpha"'] \arrow[d] \arrow[dddrr, "e_\gamma"] & & \\
 & & X_{\overline{\gamma}} \times Y \arrow[dl, "\overline{\pi}^\gamma_\alpha"'] \arrow[dr, "p_\gamma"] & & \\
 & X_{\overline{\alpha}} \times Y \arrow[rr, "p_\alpha"] & & Y & \\
c(\overline{\pi}_\alpha(M)) \arrow[ur] \arrow[rrrr, "e_\alpha"] & & & & c(p(M)) \arrow[ul]
\end{tikzcd}$$

For every $y: P \to c(p(M))$, with $P \in \mathcal{P}$ and $\mathcal{P}$ as in 1.6, and every $\gamma \leq \kappa$, we shall construct a morphism y_γ with $e_\gamma \cdot y_\gamma = y$; for $\gamma = \kappa$, this then shows $e = e_\kappa \in \mathcal{E}$ and finishes the proof.

In case $\gamma = 0$, e_γ is (like p_γ) an isomorphism, hence $y_0 = e_0^{-1} \cdot y$ does the job. In case $\gamma = \alpha + 1$ is a successor, the projection

$$\overline{\pi}^\gamma_\alpha \colon X_{\overline{\gamma}} \times Y \cong X_\alpha \times X_{\overline{\alpha}} \times Y \to X_{\overline{\alpha}} \times Y$$

is c-preserving (due to the c-compactness of X_α), hence $\overline{\tau}^\gamma_\alpha$ belongs to $\mathcal{E}$; consequently, the existing morphism y_α factors as $y_\alpha = \overline{\tau}^\gamma_\alpha \cdot y_\gamma$, hence $e_\gamma \cdot y_\gamma = e_\alpha \cdot \overline{\tau}^\gamma_\alpha \cdot y_\gamma = e_\alpha \cdot y_\alpha = y$. In case γ is a limit ordinal, the existing morphisms y_α induce a morphism $x \colon P \to X_{\overline{\gamma}} \times Y$ such that

$$p_\gamma \cdot x = c(p(m)) \cdot y \quad \text{ and } \quad q^\gamma_\beta \cdot x = q^\alpha_\beta \cdot c(\overline{\pi}_\alpha(m)) \cdot y_\alpha$$

for all $\beta < \gamma$, with $\alpha = \beta + 1$ and with $q^\gamma_\beta \colon X_{\overline{\gamma}} \times Y \to X_\beta$ denoting a product projection. We now want to show that x factors through $c(\overline{\pi}_\gamma(m))$, by a morphism y_γ with $e_\gamma \cdot y_\gamma = y$. For that it suffices to show $\widehat{x} := x(1_P) \leq c(\overline{\pi}_\gamma(m))$. In fact, since c satisfies FSPP, it is enough to consider a finite set $F \subseteq \overline{\gamma}$ and show

$$\overline{\pi}^\gamma_F(\widehat{x}) \leq c_{X_F}(\overline{\pi}^\gamma_F(\overline{\pi}_\gamma(m))) \cong c_{X_F}(\overline{\pi}_F(m)) \ .$$

But since γ is a limit ordinal, one can find a successor ordinal $\alpha < \gamma$ with $F \subseteq \overline{\alpha}$.

Continuity of $\overline{\pi}^{\alpha}_{F}$ then gives

$$\begin{aligned}\overline{\pi}^{\gamma}_{F}(\hat{x}) \cong \overline{\pi}^{\alpha}_{F}(\overline{\pi}^{\gamma}_{\alpha}(\hat{x})) &\cong \overline{\pi}^{\alpha}_{F}(c(\overline{\pi}_{\alpha}(m)) \cdot y_{\alpha}(1_P)) \\ &\leq \overline{\pi}^{\alpha}_{F}(c(\overline{\pi}_{\alpha}(m))) \\ &\leq c_{X_F}(\overline{\pi}^{\alpha}_{F}(\overline{\pi}_{\alpha}(m))) \cong c_{X_F}(\overline{\pi}_F(m)) \ ,\end{aligned}$$

as desired. This completes the proof. ∎

6.2 and 6.3 give immediately:

6.4 Tychonoff's Theorem. *If $\mathcal{E}$ is a surjectivity class and c has FSPP, under the Axiom of Choice $\mathcal{C}\mathrm{omp}(c)$ is closed under direct products in $\mathcal{X}$.* ∎

With 4.2(1) and 5.2(5) we conclude:

6.5 Corollary. *Under the assumptions of 6.2 or of 6.4, $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is closed under limits in $\mathcal{X}$ whenever c is weakly hereditary.*

We can now apply 6.4 to the sliced category $\mathcal{X}/Y$ and obtain:

6.6 Frolík's Theorem. *Let $\mathcal{E}$ be a surjectivity class and c be a hereditary closure operator with FSPP. Then the direct product $\prod_i f_i : \prod_i X_i \to \prod_i Y_i$ of a family $f_i : X_i \to Y_i$ $(i \in I)$ of c-compact (c-perfect) morphisms is c-compact (c-perfect).*

Proof: We simply extend the proof of 5.5(2) given in the finite case to infinite families, as follows. The pullback $f_i' \colon P_i \to Y = \prod_j Y_j$ of f_i along the projection $p_i : Y \to Y_i$ is c-compact. Proposition 2.10 allows us to apply 6.4 to the sliced category $\mathcal{X}/Y$ and c^Y in lieu of $\mathcal{X}$ and c. Hence the fibred product of $(f_i')_{i\in I}$ is c-compact; but this is exactly the morphism $\prod_i f_i$. The assertion for c-perfect morphisms follows with 4.5. ∎

6.7. The classical theorems in $\mathcal{T}\mathrm{op}$ on products of compact spaces and perfect maps follow immediately from the categorical theorems presented in this section.

We remark that in $\mathcal{T}\mathrm{op}$ inverse limits commute with the (usual) closure c (ILCC), i.e.,

> if $(X, (\pi_i : X \to X_i))$ is the inverse limit of (X_i) and $m : M \to X$ belongs to $\mathcal{M}$, then

$$c_X(\mathcal{M}) \cong \varprojlim c_{X_i}(\pi_i(M)) \ ,$$

(see [13], Prop. 2.5.6); hence FSPP is a particular instance of ILCC (see 2.8).

By contrast, the following generalization of ILSPP to arbitrary inverse limits:

> if $(X, (\pi_i : X \to X_i))$ is an inverse limit and $(e_i : X_i \to Y)$ is a compatible family of morphisms in $\mathcal{E}$, then the induced morphism $e\colon X \to Y$ belongs to $\mathcal{E}$,

which we call the inverse-limit stability property (ILSP), does no longer hold true in $\mathcal{T}$op. But ILSP remains true in $\mathcal{L}$oc (see 9.6 below and [38], 2.3). Consequently, since ILCC and ILSP for any closure operator c of $\mathcal{X}$ make $\mathcal{C}$omp(c) closed under inverse limits in $\mathcal{X}$, this property holds in $\mathcal{L}$oc (see [38] for a slightly different argumentation), while it fails for $\mathcal{T}$op.

7 – Stone–Čech compactification for objects and morphisms

Throughout this section *let c be an idempotent and weakly hereditary closure operator with FSPP, and $\mathcal{E}$ is assumed to be a surjectivity class* (but see Remark 7.6 below). We assume the Axiom of Choice.

7.1. A full subcategory $\mathcal{A}$ of $\mathcal{X}$ is called *c-cowellpowered* if every $A \in \mathcal{A}$ has only a small set of non-isomorphic c-dense morphisms with domain A and codomain in $\mathcal{A}$; in other words, if the comma category $A/(\mathcal{E}^c \cap \mathrm{Mor}\,\mathcal{A})$ has a small skeleton.

An easy application of the "General Adjoint Functor Theorem" gives:

7.2 Theorem. *If $\mathcal{H}$aus(c) is c-cowellpowered, then $\mathcal{C}$omp$\mathcal{H}$aus(c) is c-dense-reflective in $\mathcal{H}$aus(c).*

Proof: For $H \in \mathcal{H}$aus(c), consider a representative system of non-isomorphic c-dense morphisms $f_i\colon X \to Y_i$ with $Y_i \in \mathcal{C}$omp$\mathcal{H}$aus(c) $(i \in I)$ and form the induced map $f\colon X \to \prod_{i \in I} Y_i$, which factors through the c-closed subobject

$$\beta X := c(f(X)) \to \prod_{i \in I} Y_i$$

by a c-dense map $\beta_X : X \to \beta X$, since c is weakly hereditary and idempotent. Note that βX is c-compact and c-Hausdorff, by 6.5 and 5.2(1).

An arbitrary morphism $g : X \to A \in \mathcal{C}omp\mathcal{H}aus(c)$ factors through the c-dense morphism $g'\colon X \to c(g(X))$, which must be isomorphic to some f_i and must therefore factor through β_X. The resulting factorization of g is unique since β_X is c-dense and A is c-Hausdorff.

$$\begin{tikzcd}
X \arrow[rr, "\beta_X"] \arrow[ddd, "g"'] \arrow[rrd, "f"] \arrow[dddr, "g'"] \arrow[dddrr, "f_i"] & & \beta X \arrow[d] \\
& & \prod_j Y_j \arrow[dd, "p_i"] \\
& & \\
A & c(g(X)) \arrow[l] & Y_i \arrow[l, "\simeq"']
\end{tikzcd}$$

∎

7.3. A class $\mathcal{H}$ of morphisms in $\mathcal{X}$ is called *c-cowellpowered* if, for every object $Y \in \mathcal{X}$, the full subcategory $\mathcal{H}/Y$ of the comma category $\mathcal{X}/Y$ with objects in $\mathcal{H}$ is c^Y-cowellpowered.

A morphism $g\colon A \to B$ in $\mathcal{X}$, is called *c-antiperfect* if it is orthogonal to every c-perfect morphism, that is: if for every commutative square

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
U \arrow[r, "k"] & V
\end{tikzcd}$$

with k c-perfect there is a unique morphism w with $w \cdot g = u$ and $k \cdot w = v$.

7.4 Theorem. *If c is hereditary and if the class of c-Hausdorff morphisms is c-cowellpowered, then every c-Hausdorff morphism $f\colon X \to Y$ factors as*

$$X \xrightarrow{\beta_f} \overline{X} \xrightarrow{\beta f} Y$$

with βf c-perfect and β_f c-antiperfect. The restriction to c-Hausdorff morphisms can be dropped if $\mathcal{X}$ is $\mathcal{E}$-cowellpowered.

Proof: With $\mathcal{H}$ and $\mathcal{K}$ the classes of c-Hausdorff and c-perfect maps in $\mathcal{X}$, respectively, an application of 7.2 gives that, for every $Y \in \mathcal{X}$, $\mathcal{K}/Y$ is (c^Y-dense-)reflective in $\mathcal{H}/Y$. Hence, with βf the $(\mathcal{K}/Y)$-reflection, we need to show only that the reflexion morphism β_f is in fact c-antiperfect.

Given a commutative square as in 7.3, with g replaced by β_f, we first form the pullback P of k and v and obtain an induced morphism t rendering the diagram

$$\begin{tikzcd}
X \arrow[rr, "\beta_f"] \arrow[dd, "u"'] \arrow[dr, "t"] & & \overline{X} \arrow[dd, "v"] \\
& P \arrow[ur, "k'"'] \arrow[dl, "v'"] & \\
U \arrow[rr, "k"] & & V
\end{tikzcd}$$

commutative. With k also k' is c-perfect, hence also $l := (\beta f) \cdot k' : P \to Y$ is c-perfect. The reflection property of β_f then gives a unique morphism $s\colon \overline{X} \to P$ with $s \cdot \beta_f = t$ and $l \cdot s = \beta f$, hence $k' \cdot s = 1$.

$$\begin{tikzcd}
f \arrow[r, "\beta_f"] \arrow[dr, "t"'] & \beta f \arrow[d, "s"] \\
& l
\end{tikzcd}$$

This implies $v' \cdot s \cdot \beta_f = v' \cdot t = u$ and $k \cdot v' \cdot s = v \cdot k' \cdot s = v$, hence $v' \cdot s$ is a suitable "diagonal" for the original square.

Let now $d\colon \overline{X} \to U$ be any morphism with $d \cdot \beta_f = u$ and $k \cdot d = v$. The pullback property then yields a morphism $e\colon \overline{X} \to P$ with $v' \cdot e = d$ and $k' \cdot e = 1$. We must show $e = s$ in order to conclude $d = v' \cdot s$. But e satisfies the defining equations for s: trivially, $l \cdot e = \beta f \cdot k' \cdot e = \beta f$, and from

$$k' \cdot e \cdot \beta_f = \beta_f = k' \cdot t \quad \text{and} \quad v' \cdot e \cdot \beta_f = d \cdot \beta_f = u = v' \cdot t$$

one concludes $e \cdot \beta_f = t$.

If $\mathcal{X}$ is $\mathcal{E}$-cowellpowered, then $\mathcal{K}/Y$ is $\mathcal{E}_Y$-reflective in $\mathcal{X}/Y$ by 4.2(1), hence $\mathcal{K}/Y$ is reflective in $\mathcal{X}/Y$, and we can proceed as before. ∎

7.5. For $\mathcal{X} = \mathcal{T}\mathrm{op}$ with c the Kuratowski closure operator, Theorem 7.2 gives the Stone–Čech compactification of a Hausdorff space: the condition of c-cowellpoweredness is trivially satisfied since the size of a Hausdorff space Z with a dense subset X cannot exceed $2^{2^{\mathrm{card}\,X}}$. More generally, the class of c-Hausdorff morphisms in $\mathcal{T}\mathrm{op}$ is c-cowellpowered. In fact, if $f : X \to Y$ is the restriction of a c-Hausdorff morphism $g\colon Z \to Y$ to a dense subspace X of Z, then $\mathrm{card}\,Z \le \mathrm{card}\,Y \times 2^{2^{\mathrm{card}\,X}}$; simply note that there is an injective map

$$\phi\colon Z \to Y \times PPX$$

which assigns to z the pair with first component $g(z)$ and second component $\{U \cap X \colon U \text{ nhod of } z \text{ in } Z\}$. Hence Theorem 7.4 gives the Stone–Čech compactification of a continuous map, see [33], [29].

7.6 Remark. Theorems 7.2 and 7.4 rely on the validity of FSPP and the assumption that $\mathcal{E}$ be a surjectivity class only in so far as these conditions are needed to prove Theorem 6.4. Hence, these conditions can be dropped whenever one is able to prove the productivity assertion 6.4 by some other means such as 6.2.

8 – Henriksen–Isbell characterization of perfect morphisms

8.1. Let c be a closure operator such that $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is reflective in $\mathcal{H}\mathrm{aus}(c)$. Keeping the notation of 7.2, one defines $X \in \mathcal{H}\mathrm{aus}(c)$ to be *c-Tychonoff* if $\beta_X : X \to \beta X$ belongs to $\mathcal{M}$; equivalently, if X is embeddable into some compact Hausdorff object. By $\mathcal{T}\mathrm{ych}(c)$ we denote the full subcategory of $\mathcal{H}\mathrm{aus}(c)$ of c-Tychonoff objects. Categorical routine shows:

8.2 Proposition. *If $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is $\mathcal{E}^c$-reflective in $\mathcal{H}\mathrm{aus}(c)$, then $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is $(\mathcal{E}^c \cap \mathcal{M})$-reflective in $\mathcal{T}\mathrm{ych}(c)$ and $\mathcal{T}\mathrm{ych}(c)$ is $\mathcal{E}$-reflective in $\mathcal{H}\mathrm{aus}(c)$.*

Proof: The first statement holds true by definition of c-Tychonoff object, and for the second statement one just confirms that in the $(\mathcal{E}, \mathcal{M})$-factorization

$$\beta_X = (X \xrightarrow{e} Z \xrightarrow{m} \beta X) ,$$

one has β_Z isomorphic to $m \in \mathcal{M}$, so that e is the $\mathcal{T}\mathrm{ych}(c)$-reflexion of X. ∎

In order to characterize c-perfect morphisms of c-Tychonoff objects, we need the following important lemma, the proof of which demonstrates again the power of being able to switch between $\mathcal{X}$ and its slices:

8.3 Lemma. *A c-compact morphism $f\colon M \to Y$ in $\mathcal{X}$ cannot be factored through a c-dense extension (= subobject) $m\colon M \to X$ with X c-Hausdorff unless m is an isomorphism.*

Proof: Suppose we have a factorization $f = g \cdot m$ with $g : X \to Y$ and $m \in \mathcal{E}^c \cap \mathcal{M}$. Then we have a c-dense morphism

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

in $\mathcal{X}/Y$ with f c-compact and g c-Hausdorff which, by 5.2(2), must also be c-closed. Hence m is an isomorphism. ∎

8.4 Theorem. *Let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms, and let c be a hereditary closure operator such that $\mathcal{C}\mathrm{omp}\mathcal{H}\mathrm{aus}(c)$ is c-dense-reflective in $\mathcal{H}\mathrm{aus}(c)$. For a morphism $f\colon X \to Y$ in $\mathcal{X}$ with $X, Y \in \mathcal{T}\mathrm{ych}(c)$, the following statements are equivalent:*

(i) *f is c-compact;*

(ii) *f is c-perfect;*

(iii) *f cannot be factored through a proper c-dense extension $X \to Z$ with Z c-Hausdorff;*

(iv)

$$\begin{tikzcd} X \arrow[r, "\beta_X"] \arrow[d, "f"'] & \beta X \arrow[d, "\overline{f}"] \\ Y \arrow[r, "\beta_Y"] & \beta Y \end{tikzcd}$$

is a pullback diagram.

Proof: Since every morphism with c-Hausdorff domain is c-Hausdorff, i) and ii) are trivially equivalent.

i)⇒iii) was shown in 8.3.

iii)⇒iv) With $\overline{f}$ the functorial extension of f, form the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "n"] \arrow[d, "g"'] & \beta X \arrow[d, "\overline{f}"] \\ Y \arrow[r, "\beta_Y"] & \beta Y \end{tikzcd}$$

and consider the unique morphism $m\colon X \to Z$ with $g \cdot m = f$ and $n \cdot m = \beta_X$. Since $\beta_X \in \mathcal{M}$ and $\beta_Y \in \mathcal{M}$, also $m \in \mathcal{M}$ and $n \in \mathcal{M}$. Moreover, since β_X is

c-dense with c hereditary, also m is c-dense; in fact,

$$c_Z(m) \cong n^{-1}(c_{\beta X}(\beta_X)) \cong n^{-1}(1_{\beta X}) \cong 1_Z \ .$$

As a pullback in $\mathcal{H}\mathrm{aus}(c)$, Z is c-Hausdorff, so that m must be an isomorphism under hypothesis iii).

iv)⇒i) By 5.9, $\overline{f}$ is c-compact, hence also its pullback f. ∎

8.5. In the standard example $\mathcal{X} = \mathcal{T}\mathrm{op}$ with the usual closure operator, the necessary conditions for c-compact morphisms of being c-preserving and having c-compact fibres (cf. 5.4) are known to be also sufficient. Hence 8.4 gives the classical characterization of perfect maps of Tychonoff spaces as given by [23] and [24].

9 – Examples

9.1. *Topological spaces*

As mentioned previously, an application of the general results in this paper to the usual (Kuratowski-)closure operator of $\mathcal{T}\mathrm{op}$ gives the basic theorems on compact spaces and perfect maps. Other closure operators of $\mathcal{T}\mathrm{op}$ and of some of its supercategories (like the category of pretopological spaces (=Čech topological) spaces) have been considered in [8]. Here we restrict ourselves to some additional observations.

(1) The (Kuratowski-)compact spaces can be characterized as the compact objects with respect to closure operators that are much finer than the usual closure. In fact, with the *compact closures* k and k' defined by

$$k_X(M) = \bigcup \Big\{ \overline{M \cap \overline{K}} \colon\ K \subseteq X \text{ compact} \Big\} \ ,$$

$$k'_X(M) = \bigcup \Big\{ \overline{M \cap K} \colon\ K \subseteq X \text{ compact} \Big\}$$

for all X and $M \subseteq X$, we obtain the following.

Theorem. *For any space X, the following conditions are equivalent:*

(i) *X is compact,*

(ii) *X is k-compact,*

(iii) *X is k'-compact,*

(**iv**) X *is* $\hat{k}$*-compact,*

(**v**) X *is* $\hat{k}'$*-compact.*

Proof: **i**)⇒**ii**) For $M \subseteq X \times Y$ with any space Y, consider $y \in k_Y(p(M))$. We must find $x \in X$ such that $(x,y) \in k_{X\times Y}(M)$. There is a compact subset $K \subseteq Y$ such that $y \in \overline{p(M) \cap \overline{K}}$. Since $X \times K$ is compact, it is sufficient to find $x \in X$ with $(x,y) \in \overline{M \cap \overline{X \times K}}$. If there were no such x, then for all $x \in X$ we would have neighbourhoods U_x, V_x of x, y, respectively, such that

$$(U_x \times V_x) \cap (X \times \overline{K}) \cap M = \emptyset\ .$$

Since X is compact, we may then choose $x_1, ..., x_n \in X$ with $X = \bigcup_{i=1}^n U_{x_i}$. With $V = \bigcap_{i=1}^n V_{x_i} \ni y$ we obtain a point $z \in V \cap p(M) \cap \overline{K}$. Hence there is w with $(w,z) \in M$, which must belong to some U_{x_i}. Consequently,

$$(w,z) \in (U_{x_i} \times V_{x_i}) \cap (X \times \overline{K}) \cap M\ ,$$

a contradiction.

ii)⇒**i**) Assume that X is not compact and let $(x_\alpha)_{\alpha\in A}$ be a net with no cluster point in X. The set $(A, \leq)$ may be assumed to have a first element. Let $Y = A \cup \{\infty\}$ have the topology with base

$$[\alpha, \rightarrow) \cup \{\infty\}\,, \qquad \alpha \in A\ .$$

It makes Y a compact T_0-space. For $M = \{(x_\alpha, \alpha)\colon\ \alpha \in A\} \subseteq X \times Y$ one has $\infty \in k_Y(p(M)) = k_Y(A) = \overline{A}$, but $\infty \notin p(M)$, in particular $\infty \notin p(k_{X\times Y}(M))$, which contradicts ii).

ii)⇒**iv**) follows from 5.2(6).

iv)⇒**i**) Note that in ii)⇒i) we may replace k by any of its ordinal powers (cf. [9]) and therefore by the idempotent hull of k.

The equivalence of i), iii), v) follows similarly. ∎

Note that k_X and k'_X coincide for Hausdorff spaces X.

(2) The θ-closure defined by

$$\theta_X(M) = \Big\{x \in X\colon\ (\forall U \text{ neighbourhood of } x)\ \ \overline{U} \cap M \neq \emptyset\Big\}$$

satisfies FSPP. The θ-compact objects are precisely the H-closed spaces in the sense of Alexandroff and Urysohn (which we treat here without any separation condition; for Hausdorff spaces, the equivalence of i) and iii) below was shown in [28]):

Theorem. *For any space X, the following conditions are equivalent:*

(i) *X is H-closed (i.e., every open cover $(A_i)_{i\in I}$ of X admits a finite subcollection $(A_{i_k})_{k=1,\dots,n}$ such that $\bigcup\{\overline{A_{i_k}};\ k=1,\dots,n\}=X$);*

(ii) *X is θ-compact;*

(iii) *X is $\widehat{\theta}$-compact;*

(iv) *every filter $\mathcal{F}$ on X has θ-adherence points (that is, $\bigcap\{\theta F;\ F\in\mathcal{F}\}\neq\emptyset$).*

Proof: i)$\Rightarrow$ii) Let $X, Y \in \mathcal{T}\mathrm{op}$, with X H-closed, and let $M \subseteq X\times Y$ and $y \in Y\backslash p(\theta M)$. Then, for each $x \in X$, $(x,y)\notin \theta M$, which means that there are open neighbourhoods U_x of x and $V_y(x)$ of y in X such that $(\overline{U_x}\times\overline{V_y(x)})\cap M=\emptyset$. The family $(U_x)_{x\in X}$ is an open cover of X; consequently, there are $U_{x_1}, U_{x_2},\dots,U_{x_n}$ such that $\bigcup\{\overline{U_{x_k}};\ k=1,\dots,n\}=X$. Now, $W_y=\bigcap\{\overline{V_y(x_k)};\ k=1,\dots,n\}$ is a closed neighbourhood of y in Y, and, by assumption, $((\bigcup_{k=1}^n\overline{U_{x_k}})\times W_y)\cap M=(X\times W_y)\cap M=\emptyset$. The last equality implies $W_y\cap p(M)=\emptyset$, hence $y\notin\theta(p(M))$.

ii)$\Rightarrow$iii) follows from 5.2(6).

iii)$\Rightarrow$iv) Assume that $\mathcal{F}=\{F_i : i\in I\}$ is a filter on X with no θ-adherence points. Consider the space $Y=X\cup\{\infty\}$, $\infty\notin X$, where every $x\in X$ is discrete and the basic neighbourhood of ∞ is $\{\infty\}\cup(\bigcap_{k=1}^n F_{i_k})$. Then $X\times X$ is θ-closed in $X\times Y$, while $p(X\times X)=X$ has ∞ as θ-adherence point in Y.

iv)$\Rightarrow$i) Assume that X is not H-closed, and let $\{A_i;\ i\in I\}$ be an open cover of X with no finite subcollection whose closures cover X. Then the family $\mathcal{F}=\{X\backslash\overline{A_i};\ i\in I\}$ is an (open) filter. For each $x\in X$, let A_{i_x} be such that $x\in A_{i_x}$. Then $\overline{A_{i_x}}\cap(X\backslash\overline{A_{i_x}})=\emptyset$, hence x is not θ-adherent to $\mathcal{F}$. ∎

Consequently, Theorem 6.4 gives the Chevalley–Frink product theorem [5] for H-closed spaces, again without any separation condition.

(3) The sequential closure σ defined by

$$\sigma_X(M) = \Big\{x\colon\ x \text{ is a convergence point of a sequence in } M\Big\}$$

is hereditary and satisfies FSPP for *countable* indexing sets I.

Theorem. *For any space X, the following conditions are equivalent:*

(i) *X is sequentially compact (i.e., every sequence in X has a convergent subsequence),*

(ii) *X is σ-compact,*

(iii) *X is $\widehat{\sigma}$-compact,*

(iv) *the projection $X \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ is σ-closed (with $\mathbb{N}_\infty$ the Alexandroff compactification of the discrete $\mathbb{N}$).*

Proof: **i**)$\Rightarrow$**ii**) is well known, and the equivalence of **ii**)$\Rightarrow$**iv**) can be derived with the help of 5.11. ∎

Since in Theorem 6.4 one may restrict the cardinality of the indexing set I throughout, the validity of FSPP for *countable* indexing sets gives the countable productivity of sequentially compact spaces. Moreover, since σ does not satisfy FSPP in general and since sequentially compact spaces are not productive, this example shows that FSPP is essential for the validity of 6.4. (Also the condition that $\mathcal{E}$ be a surjectivity class is essential for 6.4; see 9.7 below.)

9.2. *Birkhoff closure spaces*

Finite additivity of the Kuratowski closure in $\mathcal{T}$op ($\overline{\emptyset} = \emptyset$, $\overline{M \cup N} = \overline{M} \cup \overline{N}$) is certainly a fundamental property when defining topological spaces. It is therefore surprising that this property seems to play no role in the categorical treatment of separation and compactness. However, if one drops the finite additivity requirement from the definition of "space", the notions of Hausdorff separation and compactness become trivial for objects and are still easy for morphisms. More precisely, a *closure space* $(X, \mathcal{F})$ is a set X with a family $\mathcal{F}$ of ("closed") subsets of X which is closed under arbitrary intersections; a map of closure spaces is continuous if inverse images of closed sets are closed. This defines the category $\mathcal{B}$CS which, exactly like $\mathcal{T}$op, has a (surjective, embedding)-factorization structure and a natural closure operator c that is idempotent, hereditary and satisfies FSPP. But (already finite) products in $\mathcal{B}$CS behave totally different from $\mathcal{T}$op: $M \subseteq X \times Y$ is c-closed iff $M = p_X(M) \times p_Y(M)$ with c-closed factors.

Hence every object X is c-compact, but X is c-Hausdorff only if card $X \leq 1$. The c-compact morphisms in $\mathcal{B}$CS are exactly the c-preserving maps, while c-Hausdorff morphisms are simply injective maps. The Stone–Čech compactification of an injection is given by the c-closure of its image.

9.3. *Uniform spaces*

In the category $\mathcal{U}$nif of uniform spaces and uniformly continuous maps, consider the usual topological closure. The functor "induced topology" $\mathcal{U}\mathrm{nif} \to \mathcal{T}\mathrm{op}$ preserves direct products, hence topologically compact spaces are compact objects in $\mathcal{U}$nif. Conversely, observe that in the Kuratowski–Mrówka Theorem, one can restrict oneself to zerodimensional Hausdorff spaces as "test spaces", and such spaces are in particular uniformizable. Hence compact objects in $\mathcal{U}$nif are topologically compact. Also Hausdorffness takes on the usual topological meaning.

Likewise, perfect morphisms in $\mathcal{U}$nif are exactly the topologically perfect maps, since the "test spaces" Z in 5.8 may be restricted to compact Hausdorff spaces (cf. [13], 3.7.15).

9.4. *Projection spaces*

We present a category with an idempotent closure operator which, like the σ-closure in $\mathcal{T}$op, is finitely additive and hereditary, but for which even the *countable* Tychonoff Theorem fails, because of failure of the countable FSPP. The objects of $\mathcal{P}$ro are unary algebras $(X, (\alpha_n)_{n\in\mathbb{N}})$ whose operations satisfy $\alpha_n \cdot \alpha_m = \alpha_{\min\{n,m\}}$ for all $n, m \in \mathbb{N}$; morphisms are homomorphisms. With respect to the (surjective, injective)-factorization system, one considers the closure operator c given by

$$c_X(M) = \Big\{ x \in X\colon\ (\forall\, n \in \mathbb{N})\ \alpha_n(x) \in M \Big\}$$

(cf. [18]). Calling an object discrete if all operations are identity morphisms, one easily shows that the discrete 2-point object D_2 is c-compact, but that no infinite discrete object is c-compact. Consequently, the discrete object D_2^ω is not c-compact. It is interesting to observe that, as in the previous example, one has a concrete functor $\mathcal{P}\mathrm{ro} \to \mathcal{T}\mathrm{op}$ (since the closure operator c may be extended from subalgebras to all subsets and is then finitely additive, hence it defines a topology). But unlike $\mathcal{U}\mathrm{nif} \to \mathcal{T}\mathrm{op}$, this functor does not preserve products.

9.5. *Topological groups*

In the category $\mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rp}$ of topological groups and continuous homomorphisms with the usual topological closure, topologically compact groups are certainly categorically compact in the sense of 5.1, but it is not known whether the converse proposition holds. Nevertheless, Theorem 6.4 is applicable and gives that the product of categorically compact groups is again categorically compact — a result that has been established independently and with topological methods by Dikranjan and Uspenskij [12].

Compact morphisms must be closed and must have a compact kernel, but again it is not known whether the converse proposition holds true.

9.6. *Locales*

That the Kuratowski–Mrówka characterization of compact objects remains valid in the category $\mathcal{L}\mathrm{oc}$ of locales (= the dual of the category $\mathcal{F}\mathrm{rm}$ of frames, i.e., of complete lattices with $x \wedge \bigvee y_i = \bigvee x \wedge y_i$) was shown in [34] (using choice) and in [38] (without choice), while compact morphisms in $\mathcal{L}\mathrm{oc}$ have been fully characterized in [38] and [39]. As mentioned previously, the usual closure operator satisfies FSPP and ILSPP, hence 6.2 gives the Tychonoff Theorem in $\mathcal{L}\mathrm{oc}$.

9.7. [J. Adámek, private communication]

The following example shows that even in the presence of FSPP the Tychonoff Theorem may fail when $\mathcal{E}$ fails to be a surjectivity class.

$$\begin{tikzcd}
 & D_0 \arrow[r, hook, "d_0"] & Z_0 \arrow[r, hook, "z_0"] & X_0 \\
 & D_1 \arrow[u] \arrow[r, hook, "d_1"] & Z_1 \arrow[u] \arrow[r, hook, "z_1"] & X_1 \arrow[u] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
M \arrow[uuur] \arrow[uur] \arrow[r, hook, "m"] & D_\omega \arrow[r, hook, "d_\omega"] & Z_\omega \arrow[r, hook, "z_\omega"] & X_\omega
\end{tikzcd}$$

Let the non-trivial $\mathcal{E}$-morphisms be those denoted by $\twoheadrightarrow$ and the non-trivial $\mathcal{M}$-morphisms be those denoted by $\hookrightarrow$. Consider the idempotent and hereditary closure operator c defined by

$$c(z_n \cdot d_n) = z_n \ ,$$
$$c(z_\omega) = z_\omega \ ,$$
$$c(z_\omega \cdot d_\omega \cdot m) = z_\omega \cdot d_\omega \ .$$

To see that the productivity of c-compactness fails, consider the product X_ω of $(X_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, X_n is c-compact, but X_ω is not c-compact, since $p\colon X_\omega \times Z_0 = Z_\omega \to Z_0$ is not c-preserving:

$$p(c(d_\omega \cdot m)) = p(d_\omega) = d_0 \neq 1_{Z_0} = c(d_0) = c(p(d_\omega \cdot m)) \ .$$

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