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THE CONCEPT OF STRUCTURAL REGULARITY

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Abstract: We introduce the class of structurally regular semigroups. Examples of such semigroups are presented, and relationships with other known generalisations of the class of regular semigroups are explored. Some fundamental results and concepts about regular semigroups are generalised to this new class. In particular, a version of Lallement's Lemma is proved.

1 – Preliminaries and introduction

An element x of a semigroup is said to be (von Neumann) regular if there exists an (inverse) element y such that $xyx = x$ and $yxy = y$; and semigroups consisting entirely of such elements are called *regular*. Regular semigroups have received wide attention (see for example, [16], [17], and [30]). In the literature, the set of all inverses of a regular element x is denoted by $V(x)$. An element x is said to be an *idempotent* if $x^2 = x$; and semigroups consisting entirely of idempotent elements are called *bands*. *Inverse semigroups* are just the regular semigroups with commuting idempotents, or equivalently, they are regular semigroups with unique inverses. Regular semigroups with a unique idempotent element are easily seen to be *groups*; semigroups that are unions of groups are called *completely regular*; and regular semigroups whose idempotent elements form a subsemigroup are called *orthodox*.

The very first class of semigroups to be studied was the class of groups, and some of the important results in semigroup theory came about as a result of attempting to generalise results from group theory. For example, the Vagner–Preston Representation Theorem for inverse semigroups was influenced by Cay-

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ley's Theorem for groups. In the quest to generalise group-theoretic results, inverse semigroups quickly emerged as the most natural class to study, and even today inverse semigroups continue to receive what is arguably more than their fair share of attention. From about 1970 onwards, T.E. Hall and others began the attempt to generalise results on inverse semigroups to orthodox and regular semigroups. They characterised the least inverse semigroup congruence on orthodox semigroups, and hence were able to prove a generalisation of the Cayley's Theorem for orthodox and regular semigroups. The trend towards greater generality has in turn led to the study of various generalisations of regular semigroups, and, in keeping with this trend, we here introduce a new class of semigroups, much larger than the class of regular semigroups, and different from any of the known generalisations. In fact it is shown that the class of all structurally regular semigroups (defined below) is different from each of the following: eventually regular semigroups, locally regular semigroups, nilpotent extensions of regular semigroups, and weakly regular semigroups.

The following countable family of congruences on a semigroup S was introduced by the author in [21]. For each ordered pair of non-negative integers (n, m),

$$\theta(n,m) = \Big\{(a,b)\colon\ uav = ubv, \quad \text{for all } u \in S^n \text{ and } v \in S^m\Big\}\ , \tag{1.1}$$

and we make the convention that $S^1 = S$, and S^0 denotes the set containing the empty word. In particular,

$$\theta(0,m) = \Big\{(a,b)\colon\ av = bv, \quad \text{for all } v \in S^m\Big\}\ ,$$

while $\theta(0,0)$ is the identity relation on S. Many interesting properties of this family of congruences are presented in [21], and a theory which resembles the theory of subnormal series in groups is presented there. It was proved there also that for any semigroup *species* — a class of semigroups closed under homomorphic images say $\mathcal{C}$, the class $\mathcal{C}^{(n,m)}$ of all semigroups S such that $S/\theta(n,m)$ belongs to $\mathcal{C}$, also forms a species. A semigroup S is said to be *structurally regular* if there exists some ordered pair of non-negative integers (n,m) such that $S/\theta(n,m)$ is regular. For any class $\mathcal{C}$ of regular semigroups, we say that a semigroup S is a *structurally* (n,m)-$\mathcal{C}$ semigroup if $S/\theta(n,m)$ belongs to $\mathcal{C}$, and more generally, semigroups in the class $\mathcal{C}^{(\infty,\infty)} = \{S : S/\theta(n,m) \in \mathcal{C}, \text{ for some } (n,m)\}$ will be called *structurally-$\mathcal{C}$ semigroups*. In this paper we lay the foundations for a unified approach to the study of structurally regular semigroups, as a natural generalisation from the concept of regularity. We therefore establish notations and concepts, with the aim of placing this new class of semigroups within the framework of classical semigroup theory. In particular we will be concerned with the classes of semigroups consisting of the following types. A semigroup S is said to be *structurally [orthodox, band, completely regular, inverse]* if and only if $S/\theta(n,m)$ is [orthodox, band, completely regular, inverse] for some (n,m).

After providing many examples and methods of constructing structurally regular semigroups in Section 2, we present in Section 3 a generalisation of the Lallement Lemma. In Section 4 we summarise the relationships that exist between the different classes of semigroups that generalise the concept of regularity.

In the subsequent papers [22] and [23], the author describes the lattices of some semigroup varieties consisting entirely of structurally regular semigroups. We point out that examples of structurally regular semigroups have appeared in the literature under different names. For example: Gerhard has in [11] and [12] studied the lattices of certain structurally band varieties; Bogdanovic and Stamenkovic [5] studied nilpotent extensions of semilattices of right groups; Higgins in [14] determines identities of certain structurally regular semigroups; and inflations of completely regular semigroups were studied by Clarke in [6], where he provides an alternative set of identities that also determine such semigroups. Petrich [27] determined the lattices formed by varieties consisting entirely of 2-nilpotent extensions of orthodox normal bands of groups.

2 – Some examples of structurally regular semigroups

We first give a more useful characterisation of structurally regular semigroups.

Theorem 2.1. *Let (n,m) be an ordered pair of non-negative integers. For any semigroup S, $S/\theta(n,m)$ is regular (and hence, S is structurally regular) if and only if for each element a in S there exists a' such that*

$$zaa'aw = zaw \quad \text{and} \quad za'aa'w = za'w \qquad \text{for all} \ \ z \in S^n \ \ \text{and} \ \ w \in S^m \ .$$

Proof: For each element a of a semigroup S, denote $a\theta(n,m)$ by α. Then $S/\theta(n,m)$ is regular if and only if for every a there exists b such that $b\theta(n,m) = \beta$, $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$, that is, if and only if for every a in S there exists b such that $(aba, a) \in \theta(n,m)$ and $(bab, b) \in \theta(n,m)$, that is, if and only if for every a in S there exist a' in S such that for all z in S^n and w in S^m, $zaa'aw = zaw$ and $za'aa'w = za'w$. ∎

Example 2.2: Take any nontrivial k-nilpotent semigroup N, any regular semigroup R, and consider the direct product $S = N \times R$. Then for any element

$s = (n,r) \in N \times R$, define $s' = (0, r')$, where 0 is the zero element of N, and r' is an inverse of r in the regular semigroup R. Then for all $z = (0,y) \in S^k = \{(0,y)\colon\ y \in R^k\}$, we have

$$zs = (0,y)\,(n,r) = (0,yr) = (0n0n, yrr'r) = (0,y)\,(n,r)\,(0,r')\,(n,r) = zss's$$

and

$$zs' = (0,y)\,(0,r') = (0,yr') = (00n0, yr'rr') = (0,y)\,(0,r')\,(n,r)\,(0,r') = zs'ss'\ ,$$

which proves that $S/\theta(0,k)$ is regular. Hence, by Theorem 2.1, $S = N \times R$ is structurally regular.

The condition that for each element a there exists b such that $zaw = zabaw$ for all z in S^n and w in S^m implies that there exists an element, namely $a^* = bab$, such that $zaw = zaa^*aw$ and $za^*w = za^*aa^*w$. Other examples of structurally regular semigroups are presented in Example 2.6, 2.7 and 3.9 of [21]. In fact, the method of construction described in Example 3.10 of that same paper can be used to construct more such examples. As pointed out in [15], P.M. Edwards defined a semigroup S to be *eventually regular* if for each x in S there exists some positive integer n such that x^n is regular. In [26] Munn termed the inverses of the regular element x^n the *pseudoinverses* of x.

Example 2.3: Take any nontrivial k-nilpotent semigroup N and consider the semigroup $S = N^{(1)}$, the semigroup obtained from N by adjoining an identity element. Clearly, for each element x in S, the k-th power x^k is either the zero element of the nilpotent semigroup or the adjoined identity element. Thus, S is eventually regular. However since S is a monoid, it is reductive and so $S/\theta(i,j) = S$ for every ordered pair (i,j). Hence, eventual regularity does not imply structural regularity.

A semigroup S is called *reductive* if both the congruences $\theta(1,0)$ and $\theta(0,1)$ reduce to the identity relation on S. It is shown in Example 4.1 that the class of all structurally regular semigroups is not contained in the class of all eventually regular semigroups. However, for the cases considered in Lemma 2.4 below, every structurally regular semigroup is necessarily eventually regular.

A semigroup is said to be *completely regular* if it is a union of groups.

Denote the set of all regular elements of S by $\mathrm{Reg}(S) = \{x \in S\colon\ xx'x = x$ for some $x' \in S\}$, and the union of all its idempotent $\theta(n,m)$-classes as follows:

$$E_{(n,m)}(S) = \Big\{x\colon\ (x,x^2) \in \theta(n,m),\ x \in S\Big\}\ .$$

We shall say an element x is *(n,m)-idempotent* if it is $\theta(n,m)$ related to some idempotent element, that is, if $ux^2v = uxv$ for all u and v in S^n and S^m, respectively. We will demonstrate in this paper that (n,m)-idempotents play an analogous role to that played by idempotent elements in regular semigroups. In fact, as shown in Theorem 2.12 if $S/\theta(n,m)$ is orthodox, then $E_{(n,m)}$ forms a subsemigroup of S. We shall simply denote by $E(S)$ the set of all idempotent elements of S, and in the above notation it would be $E_{(0,0)}(S)$.

Lemma 2.4. *Let S be a semigroup. If $S/\theta(n,m)$ satisfies $x = x^{k+1}$ for some positive integer k, then S satisfies $x^{(n+1+m)} = x^{(n+1+m)(k+1)}$, $k \geq 1$. Hence, if $\mathcal{V}$ is a variety consisting entirely of completely regular semigroups, then $\mathcal{V}^{(n,m)}$ consists entirely of eventually regular semigroups.*

Proof: Suppose that $S/\theta(n,m)$ satisfies an identity of the form $x^{k+1} = x$, for some $k \geq 1$. Then for each element a of S, $(a, a^{k+1}) \in \theta(n,m)$. This implies that for all $u \in S^n$ and $v \in S^m$, $uav = ua^{k+1}v$. In particular, $a^{n+1+m} = a^{n+k+1+m}$. Now, putting $b = a^{n+1+m}$, we see that

$$
\begin{aligned}
b^{k+1} = (a^{n+1+m})^{k+1} &= a^{(n+1+m)(k+1)} \\
&= a^{(n+1+m)}\, a^{k(n+1+m)} \\
&= a^{(n+k+1+m)}\, a^{k(n+m)} \\
&= a^{(n+1+m)}\, a^{k(n+m)} \\
&= a^{(n+k+1+m)}\, a^{k(n+m-1)} \\
&\;\vdots \\
&= a^{(n+k+1+m)} \\
&= b\ .
\end{aligned}
$$

In the case $k > 1$ the element $b = a^{n+1+m}$ is regular since $b(b^{k-1})b = b$; and in the case $k = 1$, b is also regular since $b(b)b = b$. In any case S is eventually regular. Now, if $\mathcal{V}$ is a variety consisting entirely of completely regular semigroups, then as shown in Corollary 14 of [14], every semigroup in $\mathcal{V}$ satisfies an identity of the form $x^{k+1} = x$, for some $k \geq 1$. Then from what we have just proved, the class $\mathcal{V}^{(n,m)}$ consists of eventually regular semigroups. ∎

An element x is said to be a *weak inverse* (see [31]; Page 537) of y if $xyx = x$. This does not, in general, imply that $yxy = y$ but of course x is a regular element. We dub the semigroups consisting entirely of such elements as *weakly regular* semigroups, and we point out that the semigroup in Example 2.3 above is one such example. For, if we put $x' = 1$ when x is the identity element, and put $x' = 0$ otherwise, then, it is easy to verify that $x'xx' = x'$. This then establishes the fact that the class of all structurally regular semigroups does not even contain the class of all weakly regular semigroups. In fact any semigroup with a zero element is weakly regular.

A semigroup S is said to be an *inflation* (see Clifford and Preston [7]) of a regular subsemigroup if there exists a homomorphism ϕ from S into itself such that $S^2 \subseteq S\phi$, $S\phi$ is a regular subsemigroup, and $x\phi = x$ for every x in $S\phi$. This implies that for any elements a, b of S, the product $ab = (a\phi)\,(b\phi)$. As before, $\mathrm{Reg}(S)$ denotes the set of all regular elements of S in the following lemma.

Lemma 2.5. *A semigroup S is an inflation of a regular subsemigroup if and only if $\mathrm{Reg}(S)$ forms a subsemigroup and for each $a \in S$ there exists a^* such that for all $x \in S$*

$$(\ddagger) \qquad xaa^*a = xa \quad \text{and} \quad aa^*ax = ax\ .$$

Hence such semigroups are structurally regular.

Proof: Suppose that S is an inflation of a regular semigroup. Then by definition there exists a homomorphism ϕ from S into itself such that for any elements a, b of S, the product $ab = (a\phi)\,(b\phi)$. We note that if $a \in \mathrm{Reg}(S)$ then $a \in S^2 \subseteq S\phi$. Indeed, since $S\phi$ is regular we deduce that $\mathrm{Reg}(S) = S\phi$. For each element a of S let $a^* = (a\phi)'$ denote an inverse of the regular element $a\phi$. Then for all element x in S we have

$$xaa^*a = (x\phi)\,(a\phi)\,(a\phi)'\,(a\phi) = (x\phi)\,(a\phi) = xa\ ;$$

and by symmetry, we also have $aa^*ax = ax$.

Conversely, suppose that $\mathrm{Reg}(S)$ is a subsemigroup and that for each a in S there exists a^* such that $(\ddagger)$ holds. Consider the congruence

$$\delta_1 = \theta(1,0) \cap \theta(0,1) = \Big\{(a,b)\colon\ xa = xb,\ ax = bx \ \text{ for all } x \text{ in } S\Big\}\ .$$

Then

$$(2.6) \qquad \delta_1 \text{ separates the regular elements of } S.$$

To see this, let $a, b \in \mathrm{Reg}(S)$ be such that $(a,b) \in \delta_1$. Then for any $a' \in V(a)$ and $b' \in V(b)$, we have $a = aa'a = (aa')b$ and $b = bb'b = (bb')a$. These, together, imply that $(a,b) \in \mathcal{L}$ (Green's relation). Then it follows from (Howie [17] that there exists $a'' \in V(a)$ and $b'' \in V(b)$ such that $a''a = b''b$, and that implies that $a = aa''a = ba''a = bb''b = b$. Next,

every δ_1-class contains a regular element.

To see this, consider an arbitrary a in S. By assumption, there exists a^* such that (‡) holds. Then by (‡)

$$(aa^*a)\, a^*\, (aa^*a) = \Big[(aa^*)\, (aa^*a)\Big] a^*a = \Big[(aa^*)a\Big] a^*a = (aa^*a)a^*a = aa^*a \ ,$$

and so aa^*a is regular. Also, directly from (‡) we have $(a, aa^*a) \in \delta_1$.

We deduce from (2.6) that every δ_1-class $a\delta_1$ contains a unique regular element aa^*a. If we define $\phi\colon S \to \mathrm{Reg}(S)$ by $a\phi = aa^*a$ $(a \in S)$then certainly $\phi^2 = \phi$. Clearly ϕ is onto. Also, by (‡), for all a, b, in S, $ab = aa^*ab = aa^*abb^*b \in \mathrm{Reg}(S)$. Hence $S^2 \subseteq S\phi$. This also proves that ϕ is a homomorphism, and so S is, as required, an inflation of a regular semigroup. It follows from Theorem 2.1 that S is structurally regular. ∎

A subset I of a semigroup S is said to form an *ideal* if both $IS \subseteq I$ and $SI \subseteq I$ in which case I forms a subsemigroup and we say that S is an *ideal extension* of I by S/I, where S/I is the quotient taken under the Rees congruence: $\{(a,b)\colon\ a,b \in I\} \cup \{(a,a)\colon\ a \in S \backslash I\}$. If there exists a homomorphism ϕ from S onto I such that $a\phi = a$ for every a in I, then such an ideal extension is called a *retract extension* (See Petrich [28]). A retract extension by an n-nilpotent semigroup is called an *n-inflation*. We point out that the semigroups given in Lemma 2.5 are precisely the retract extensions of regular semigroups by null semigroups. A semigroup S is said to be an *n-nilpotent extension* of a regular semigroup if S^n is regular for some $n \geq 1$.

Theorem 2.8. *The following statements concerning a semigroup S are equivalent:*

- **i)** *S is a $(n+1)$-nilpotent extension of a regular semigroup, and there exists a regular-element-separating congruence γ on S with the property that every γ-class contains a regular element.*
- **ii)** *S is an $(n+1)$-inflation of a regular semigroup.*
- **iii)** Reg(S) *forms a subsemigroup and for each element a of S there exists a^* in S such that for all elements x in S^n, we have*

$$xa = xaa^*a \quad \text{and} \quad aa^*ax = ax \ .$$

Proof: i)⇒ii) Suppose that i) holds, and define $\phi\colon\ S \to S$ to be the map which sends each element x to the unique regular element contained in the γ-class that contains x. Then ii) holds.

ii)⇒iii) We are supposing that there exists a retract homomorphism $\phi\colon S \to R$, where R is a regular ideal of S and where S/R is a $(n+1)$-nilpotent semigroup. If $x \in S^n$ and $a \in S$ then $xa \in S^{n+1} \subseteq R$, and so $xa = (xa)\,\phi$. If $a^* \in V(a\phi)$ in R then $a^*\phi = a^*$, and so

$$xa = (xa)\phi = (x\phi)\,(a\phi) = (x\phi)\,(a\phi)\,(a^*\phi)\,(a\phi) = (xa)\phi\,(a^*a)\phi = xaa^*a\ .$$

Similarly, $ax = aa^*ax$ for all x in S^n. It is clear that $\mathrm{Reg}(S) = R$, a subsemigroup.

iii)⇒i) Suppose that iii) holds in S and consider the congruence $\delta_n = \theta(n,0) \cap \theta(0,n)$. It is clear by the assumption that for each element a there exists a^* such that $(a,(aa^*)^n a), (a^*,(a^*a)^n a^*) \in \delta_n$. In S the element $(aa^*)^n a$ is regular since

$$\Big[(aa^*)^n\, a\Big] a^* \Big[(aa^*)^n\, a\Big] = (aa^*)^n\, (aa^*)^{n+1}\, a = (aa^*)^n\, a\ ,$$

by repeated use of the equality $xaa^*a = xa$. Thus every δ_n-class contains a regular element. One can show that δ_n is regular element separating, by the same proof used in Lemma 2.5 to prove that δ_1 has this same property. Hence the map $\phi\colon\ S \to S$, $a \mapsto (aa^*)^n a$ is well defined. If a is regular, then $a = (aa')^n a = a\phi$ for any $a' \in V(a)$; and it follows that $\mathrm{Reg}(S) = S\phi$. The regular elements form a subsemigroup, by assumption, and so ϕ is a homomorphism. Moreover, since $s\phi$ is regular for all $s \in S$, it follows that $(s\phi)\phi = s\phi$. If s is a regular element, then it can also be expressed in the form $s = s(s's)^{n+1}$ and is therefore contained in S^{n+1}. Now, for any elements $a_1, a_2, a_3, ..., a_n, a_{n+1}$ of S,

$$\begin{aligned} a_1a_2a_3\cdots a_na_{n+1} &= a_1a_2a_3\cdots a_n(a_{n+1}\phi) \qquad (\text{since } (a_{n+1}, a_{n+1}\phi) \in \delta_n) \\ &= (a_1\phi)\,(a_2\phi)\,(a_3\phi)\cdots(a_n\phi)\,(a_{n+1}\phi)\ . \end{aligned}$$

We have the last equality since $a_{n+1}\phi$ is contained in $S^{n+1} \subseteq S^n$. Thus we have proved that $S^{n+1} = \mathrm{Reg}(S)$. Hence S is an $(n+1)$-nilpotent extension of the subsemigroup $\mathrm{Reg}(S) = S\phi$, proving that i) holds. ∎

We point out that, as a consequence of Theorem 2.12 below, if $S/\theta(n,m)$ is orthodox then $\mathrm{Reg}(S)$ forms a subsemigroup, and so in that case the requirement of $\mathrm{Reg}(S)$ to form a subsemigroup in the above result would not be necessary. Also, the congruence δ_n appearing in the above proof is regular-element-separating even for structurally regular semigroups in general, and not just for n-inflations.

Corollary 2.9. *Any n-inflation of a regular semigroup is structurally regular.* ∎

We point out that for the semigroup S given in Example 3.9 of [21], $S/\theta(n,m)$ is regular but $S/\theta(0,1)$ is not regular. Therefore the regularity of $S/\theta(n,0)$ does not, in general, imply the regularity of $S/\theta(0,n)$. Hence the conditions in the statements of Lemma 2.5 and Theorem 2.8 cannot be weakened. It follows also that the class of all n-inflations of regular semigroups is properly contained in the class of all structurally regular semigroups.

Example 2.10: Consider the two element semilattice $A = \{a, 0\}$ and on the Cartesian product $S = A^{(1)} \times A = \{(x,y) : x \in A^{(1)} \text{ and } y \in A\}$, where $A^{(1)}$ is the semigroup obtained by adjoining an identity element to A, define a binary operation $\otimes$ by $(a,b) \otimes (c,d) = (ad, bd)$. It can be shown that $(S, \otimes)$ is a semigroup and that $S/\theta(1,0)$ is isomorphic to the semilattice A (see [21]; Example 3.10). Therefore, S is structurally regular. However, for each positive integer n, $S^n = \{(a,a), (a,0), (0,0), (0,a)\}$ is not regular, since the element $(a,0)$ is not regular. Thus structural regularity does not imply nilpotent extension.

Lemma 2.11. *If $S/\theta(n,m)$ is regular then every $\theta(n,m)$-class contains a regular element. Moreover, every element x of S^{n+1+m} can be expressed in the form $x = abc$, where $a \in S^n$, $b \in \mathrm{Reg}(S)$ and $c \in S^m$.*

Proof: Suppose that $S/\theta(n,m)$ is regular. Then for each element a in S there exists an element a' such that for all $u \in S^n$ and $v \in S^m$,

$$uav = uaa'av = u(aa')^n\, a(a'a)^m\, v\ ;$$

hence the elements a and $b = (aa')^n\, a(a'a)^m$ are $\theta(n,m)$-related. Since

$$ba'b = (aa')^n\, a(a'a)^m\, a'(aa')^n\, a(a'a)^m = (aa')^n\, a(a'a)^m = b\ ,$$

b is a regular element. Now, take any element x in S^{n+1+m}. Then there exist elements $x_1, x_2, x_3, \ldots, x_{n+1+m}$ in S such that

$$x = (x_1x_2x_3\cdots x_n)\, x_{n+1}\, (x_{n+2}x_{n+3}x_{n+4}\cdots x_{n+1+m}) = abc\ ,$$

where $a = x_1x_2x_3\cdots x_n$, $b = (x_{n+1}x'_{n+1})^n\, x_{n+1}\, (x'_{n+1}x_{n+1})^m$, a regular element $\theta(n,m)$-related to x_{n+1}, and $c = x_{n+2}x_{n+3}x_{n+4}\cdots x_{n+1+m}$. ∎

Theorem 2.12. *If $S/\theta(n,m)$ is orthodox then $E_{(n,m)}$, $E(S)$ and $\mathrm{Reg}(S)$ form subsemigroups of S. In particular, the following equalities hold if $S/\theta(n,m)$*

is an inverse semigroup:

$$\theta^{E_{(n,m)}}(n,m) = \theta^S(n,m) \cap (E_{(n,m)} \times E_{(n,m)}) \ , \tag{2.13}$$

$$\theta^{\mathrm{Reg}(S)}(n,m) = \theta^S(n,m) \cap (\mathrm{Reg}(S) \times \mathrm{Reg}(S)) \ , \tag{2.14}$$

$$\theta^E(n,m) = \theta^S(n,m) \cap (E \times E) \ . \tag{2.15}$$

Proof: Suppose that $S/\theta(n,m)$ is orthodox. Let $x, y \in E_{(n,m)}$. Then $x\theta(n,m)$ and $y\theta(n,m)$ are idempotents of the orthodox semigroup $S/\theta(n,m)$. Hence $(xy)\theta(n,m)$ is also idempotent and so $xy \in E_{(n,m)}$. For any elements e, f of $E(S)$, we have that $(ef)^2 = efef = e^n(ef)^2f^m = e^n(ef)f^m = ef$; and so $E(S)$ also forms a subsemigroup. Now, we see that $\mathrm{Reg}(S)$ also forms a subsemigroup, since for any a, b in $\mathrm{Reg}(S)$ and any a', b' in $V(a)$ and $V(b)$, respectively, we have that $(ab)b'a'(ab) = a(aa'bb')\,(aa'bb')b = ab$ (since $E(S)$ forms a subsemigroup).

Let $S/\theta(n,m)$ be an inverse semigroup. To prove (2.13), take any $(a,b) \in \theta^{E_{(n,m)}}$, and let $u \in S^n$ and $v \in S^m$,

$$\begin{aligned} uav &= ua^naa^mv = ua^nba^mv = uabav = uab^2av = uba^2bv \\ &= ubabv = ub^nab^mv = ub^nbb^mv = ubv \ , \end{aligned}$$

and so $(a,b) \in \theta^S(n,m) \cap (E_{(n,m)} \times E_{(n,m)})$. Since the reverse containment holds trivially, the equality (2.13) follows.

To prove (2.14), take any $(a,b) \in \theta^{\mathrm{Reg}(S)}(n,m)$, and let $d = aa' = (aa')^n$, $e = a'a = (a'a)^m$, $f = bb' = (bb')^n$ and $g = b'b = (b'b)^m$, where $a' \in V(a)$, $b' \in V(b)$. Then $d, e, f, g \in \mathrm{Reg}(S)$. Now, for all u in S^n, and v in S^m,

$$\begin{aligned} uav &= udaev = udbev \qquad (\text{since } (a,b) \in \theta^{\mathrm{Reg}(S)}(n,m)) \\ &= udfbgev = ufdbegv = ufdaegv = ufagv = ufbgv = ubv \ . \end{aligned}$$

Thus $(a,b) \in \theta^S(n,m)$. Since the reverse containment holds trivially, the equality (2.14) follows. One can easily show, in the same way, that (2.15) also holds. ∎

The concepts of *engamorphic products* were first introduced in [19]. Take a semigroup $(S, \circ)$ and any homomorphism ϕ from S into itself with the property that $(x\phi)\phi = x\phi$ for every x. Such a map is called a *retractive endomorphism*. It can be shown that the binary operation $a \oplus b = a \circ (b\phi)$ is associative; and the semigroup $(S, \oplus)$ (alternatively, written as $(S, \circ, \phi; l)$ is called the *left engamorphic product of* $(S, \circ)$ *with respect to* ϕ. The *right engamorphic product* $(S, \circ, \phi; r)$ is defined by duality. Example 2.17 below shows that these concepts are different from taking inflations of semigroups. In fact, if $(S, \circ)$ is a monoid and ϕ is chosen to be the constant map that sends every element of S to the identity element, then $(S, \circ, \phi; l)$ forms a left zero band. Historically speaking, the concept of engamorphic products in [19] led the author to the idea of the family of congruences $\theta(n, m)$.

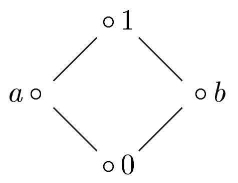
Theorem 2.16. *Every engamorphic product of a regular semigroup is structurally regular.*

Proof: Suppose that $(S, \circ)$ is regular, and take any retractive endomorphism ϕ. Then for each $a \in S$ let a' be an inverse of a. Denote $a'\phi$ by a^*. Then for all s in S, we have by the retractive nature of ϕ that

$$s \oplus a = s \circ (a\phi) = s \circ (a\phi) \circ (a'\phi) \circ (a\phi) = s \oplus a \oplus a^* \oplus a \ ;$$

and $(S, \oplus)/\theta(1,0)$ is regular. Hence $(S, \oplus) = (S, \circ, \phi; l)$ is structurally regular. ∎

Example 2.17: Consider a 4-element diamond semilattice $(S, \circ) = \{1, a, b, 0\}$, and define a map ϕ from S into itself which sends $1 \mapsto 1$, $a \mapsto 1$, $b \mapsto 0$, and $0 \mapsto 0$. Define a binary operation on the set S by $x \oplus y = x \circ (y\phi)$.



$\oplus$	1	a	b	0
1	1	1	0	0
a	a	a	0	0
b	b	b	0	0
0	0	0	0	0

The semigroup $(S, \circ, \phi; l) = (S, \oplus)$ is not regular since the element b is not regular. Moreover, since $S^k = \{1, a, b, 0\} = S$ for all $k \geq 1$, $(S, \oplus)$ is not a nilpotent extension of a regular semigroup. Thus not every engamorphic product is a nilpotent extension.

3 – A general concept of idempotency

If $S/\theta(n, m)$ is regular then for each element x of S one can define the following set:

$$\begin{aligned} V_S(x; n, m) &= \Big\{ y\colon\ uxyxv = uxv \ \text{ and } \ uyxyv = uyv, \ \ u \in S^n \ \text{ and } \ v \in S^m \Big\} \\ &= \Big\{ y\colon\ y\theta(n, m) \in V(x\theta(n, m)) \Big\} \ ; \end{aligned} \tag{3.1}$$

and call each member of the set an (n,m)-*inverse* of x. In particular, if the element x is regular, then the set of all its inverses coincides with the set $V_S(x;0,0)$, and of course $V_S(x;0,0) \subseteq V_S(x;n,m)$. For any semigroup S, any ordered pair (n,m), and for all u in S^n, and v in S^m we have the following concepts. Recall that an element x is called (n,m)-*idempotent* if $ux^2v = uxv$, and that the set of all such elements is denoted by $E_{(n,m)}(S)$. Semigroups that consist entirely of such elements will be called (n,m)-*bands*. The concept of $(0,0)$-band coincides with the usual meaning of the word band. A semigroup will be called (n,m)-*orthodox* if $S/\theta(n,m)$ is orthodox. Equivalently, these are structurally regular semigroups for which the union of idempotent $\theta(n,m)$-classes form a subsemigroup. In this section, we demonstrate that (n,m)-idempotents behave in a way somewhat similar to the way in which idempotent elements do. In fact for any element x' of $V_S(x;n,m)$ both xx' and $x'x$ are (n,m)-idempotents.

We refer the reader to Clifford and Preston [7], Howie [17] or Higgins [16] for the definitions of the five Green's relations $\mathcal{L},\mathcal{R},\mathcal{H},\mathcal{D},\mathcal{J}$. The following five relations, which are in fact generalisations of these Green's relations, will prove quite useful later in the study of structurally regular semigroups. For any Green's relation $\mathcal{X} \in \{\mathcal{R},\mathcal{L},\mathcal{H},\mathcal{D},\mathcal{J}\}$, define a new relation $\mathcal{X}_{(n,m)}$ as follows: for any elements a,b of S, we say $(a,b) \in \mathcal{X}_{(n,m)}$ if and only if the classes $a\theta(n,m)$ and $b\theta(n,m)$ are $\mathcal{X}$-related in $S/\theta(n,m)$. For example, $(a,b) \in \mathcal{R}_{(n,m)}$ in S if and only if there exist $x,y \in S^{(1)}$ such that

$$b\theta(n,m) = a\theta(n,m)\, x\theta(n,m) \quad \text{and} \quad a\theta(n,m) = b\theta(n,m)\, y\theta(n,m)\ .$$

This is equivalent to saying that (b,ax) and (a,by) are $\theta(n,m)$-related pairs in S.

Theorem 3.2. *Take any structurally (n,m)-regular semigroup S, and any elements a and b. Then for any $a' \in V_S(a;n,m)$ and $b' \in V_S(b;n,m)$ the following statements hold:*

- **i)** $(a,b) \in \mathcal{L}_{(n,m)}$ *in S if and only if there exist (n,m)-inverses a' and b' of a and b, respectively, such that $(a'a,b'b)$ are $\theta(n,m)$-related.*
- **ii)** $(a,b) \in \mathcal{R}_{(n,m)}$ *in S if and only if there exists a' and b' such that (aa',bb') are $\theta(n,m)$-related.*
- **iii)** $(a,b) \in \mathcal{H}_{(n,m)}$ *in S if and only if there exists a' and b' such that (aa',bb') and $(a'a,b'b)$ are $\theta(n,m)$-related pairs.*

Proof: We prove only statement i). The remaining statements can be proved similarly. Let $a\theta(n,m) = \alpha$, $b\theta(n,m) = \beta$, and suppose that $(a,b) \in \mathcal{L}_{(n,m)}$ in S. Then in the regular semigroup $S/\theta(n,m)$, $(\alpha,\beta) \in \mathcal{L}$. Hence by Howie [17]

(Lemma II:4.7), there exist $\alpha' \in V(\alpha)$ and $\beta' \in V(\beta)$ such that $\alpha'\alpha = \beta'\beta$. If a' and b' are in S such that $a'\theta(n,m) = \alpha'$, $b'\theta(n,m) = \beta'$, then $(a'a, b'b) \in \theta(n,m)$ as required. ∎

The following theorem is a generalisation of one due to T.E. Hall (see Exercise 14 on Page 55 of [17]).

Theorem 3.3. *Let ϕ be a homomorphism from S onto T. If $S/\theta(n,m)$ is regular, then for any $t \in T$ and any $t' \in V_T(t;n,m)$ there exists $s' \in V_S(s;n,m)$ such that $(s\phi, t)$ and $(s'\phi, t')$ are $\theta(n,m)$-related pairs in T.*

Proof: We have by ([21]; Theorem 2.4) that $T/\theta^T(n,m)$ is a homomorphic image of $S/\theta^S(n,m)$ under the map $\phi_{(n,m)}\colon\ a\theta^S(n,m) \mapsto a\phi\theta^T(n,m)$, for each element a of S. Hence, every $\theta^T(n,m)$-class is an image of some $\theta^S(n,m)$-class under ϕ, and the quotient $T/\theta^T(n,m)$ is a homomorphic image of $S/\theta^S(n,m)$ under $\phi_{(n,m)}$. Denote the $\theta(n,m)$-classes of S and T, respectively, as follows:

$$\left\{S_\alpha\colon\ \alpha \in \Gamma = S/\theta(n,m)\right\} \quad \text{and} \quad \left\{T_\alpha\colon\ \alpha \in \Lambda = T/\theta(n,m)\right\}.$$

Take any $t \in T_\alpha$, $\alpha \in \Lambda$, and any $t' \in T_{\alpha'}$, where α' is a inverse of the regular element α. Then by Hall's generalisation of Lallement's Lemma, and by the commutativity of the diagram in ([21]; Theorem 2.4), there exist elements β and β' in Γ such that $(\beta)\phi_{(n,m)} = \alpha$ and $(\beta')\phi_{(n,m)} = \alpha'$. This means that here exists s and s' in the $\theta^S(n,m)$-classes S_β and $S_{\beta'}$ respectively, such that $s\phi \in T_\alpha$ and $s'\phi \in T_{\alpha'}$. ∎

It is known that Lallement's lemma does not hold true in arbitrary semigroups. In fact, this lemma fails to hold in the semigroup of all positive integers under addition, since it does not have any idempotent element but the entire semigroup can be mapped onto a trivial semigroup, which of course is an idempotent.

Corollary 3.4. *Let ϕ be a homomorphism from S onto T. If $S/\theta(n,m)$ is regular, then for each idempotent f of T, there exists an idempotent element e of S such that $e\phi = f$.*

Proof: Since ϕ is onto, there exists some $a \in S$ such that $a\phi = f$. Take any $x \in V_S(a^2;n,m)$ and consider $e = (axa)^{n+1+m}$. We will show that e is an idempotent of S such that $e\phi = a\phi = f$. It is not difficult to see that (axa) is $\theta(n,m)$-related to $(axa)^i$ in S for every $i \geq 1$.

Now,

$$\begin{aligned}
e^2 &= (axa)^{n+1+m}\,(axa)^{n+1+m}\\
&= (axa)^n\Big[(axa)^{1+m}\,(axa)^{n+1}\Big](axa)^m\\
&= (axa)^n\,[axa]\,(axa)^m \qquad (\text{since } (axa,(axa)^{1+m}\,(axa)^{n+1})\in\theta(n,m))\\
&= (axa)^{n+1+m} = e\ ;
\end{aligned}$$

and

$$\begin{aligned}
e\phi &= ((axa)^{n+1+m})\phi = \Big((axa)\,(axa)^{n+m-1}\,(axa)\Big)\phi\\
&= (a\phi)\Big[xa\,(axa)^{n+m-1}\,ax\Big]\phi(a\phi)\\
&= (a\phi)^{n+2}\Big[xa\,(axa)^{n+m-1}\,ax\Big]\phi(a\phi)^{m+2}\\
&= \Big(a^{n+2}\Big[xa\,(axa)^{n+m-1}\,ax\Big]a^{m+2}\Big)\phi\\
&= (a^{n+2}\,[x]\,a^{m+2})\phi \qquad (\text{since } x\in V_S(a^2;n,m))\\
&= (a^{n+2}\,[x]\,a^{m+2})\phi = (a^n\,a^2xa^2\,a^m)\phi\\
&= (a^{n+2+m})\phi = (a\phi)^{n+1+m} = a\phi = f\ . \ \blacksquare
\end{aligned}$$

4 – Some generalisations of the class of regular semigroups

The following counter example proves that the class of all structurally regular semigroups is not contained in the class of all eventually regular semigroups. Combining that with Example 2.3, we conclude that these two classes are not comparable; that is, neither contains the other.

Example 4.1: Let N denote the set of all positive integers, and consider the semigroup $S = N\times N$ with the multiplication $\square$ given by

$$(n,m)\,\square\,(p,q) = \Big(n-m+\max(m,p),\ q-p+\max(m,p)\Big)\ . \tag{4.2}$$

This is the so-called *bicyclic* semigroup, which plays an important role in the theory of inverse semigroups. Now, consider $T = S^{(1)}\times S$, where $S^{(1)}$ denotes the semigroup obtained by adjoining an identity element $\mathbf{1}$ to S, and define a multiplication $\diamond$ on T as follows:

$$x\diamond y = [a,b]\diamond[c,d] = [a\,\square\,d, b\,\square\,d]\,, \qquad x=[a,b],\ y=[c,d]\ \in\ T = S^{\mathbf{1}}\times S\ . \tag{4.3}$$

More precisely,

$$x \diamond y = \begin{cases} \Big[\Big(r-s+\max(s,k), l-k+\max(k,s)\Big), \Big(t-u+\max(u,k), l-k+\max(k,u)\Big)\Big], \\ \qquad \text{if } \; x = [(r,s),(t,u)] \; \text{ and } \; y = [(i,j),(k,l)] \text{ or } y = [\mathbf{1},(k,l)] \ , \\ \Big[(k,l),\ \Big(t-u+\max(u,k),\ l-k+\max(k,u)\Big)\Big], \\ \qquad \text{if } \; x = [\mathbf{1},(t,u)] \; \text{ and } \; y = [(i,j),(k,l)] \text{ or } y = [\mathbf{1},(k,l)] \ . \end{cases}$$

We have from ([21]; Example 3.10) that $(T,\diamond)$ forms a semigroup, and that $T/\theta(1,0)$ is isomorphic to S. Hence, $(T,\diamond)$ is structurally regular. We will prove that it is not eventually regular.

First we note that

$$\text{Reg}(T) = \Big\{[(a,b),(c,d)] \in T\colon \; b \geq d, \; b,d \in N\Big\} \ . \tag{4.4}$$

First notice that if $x = [1,b]$ then x is not regular. Now, take any regular element, say $x = [(a,b),(c,d)]$ of T. Then by assumption there exists an element, say $x' = [(e,f),(g,h)]$ such that $x \diamond x' \diamond x = x$ and $x' \diamond x \diamond x' = x'$. This implies that the following equalities hold in the bicyclic semigroup:

$$(c,d) \mathbin{\square} (g,h) \mathbin{\square} (c,d) = (c,d) \qquad \text{and} \qquad (g,h) \mathbin{\square} (c,d) \mathbin{\square} (g,h) = (g,h) \ , \tag{4.5}$$

$$(a,b) \mathbin{\square} (g,h) \mathbin{\square} (c,d) = (a,b) \ . \tag{4.6}$$

From (4.5), we have by the uniqueness of inverses in S that $(g,h) = (d,c)$; and by substituting this equality into (4.6) we have

$$(a,b) \mathbin{\square} (d,c) \mathbin{\square} (c,d) = (a,b) \ . \tag{4.7}$$

But since $(d,c) \mathbin{\square} (c,d) = (d,d)$, it follows that $a - b + \max(b,d) = a$, and we have that $b \geq d$.

Conversely, it is straightforward, but tedious, to verify that for any $y = [(n,m),(p,q)]$ with $m \geq q$, the element $y' = [(r,s),(q,p)]$ with $s \geq p$ is an inverse of y. Thus the set in (4.4) gives all the regular elements of T.

To show that T is not eventually regular, consider $x = [\mathbf{1},(1,2)]$, where $\mathbf{1}$ is the adjoined identity element of $S^{(\mathbf{1})}$. Then

$$x^2 = [(1,2),(1,3)]\,, \qquad x^3 = [(1,3),(1,4)]\,, \qquad x^4 = [(1,4),(1,5)]\,, \qquad \ldots$$

In general,

$$x^k = [(1,k),(1,k+1)] \qquad \text{for} \quad k \geq 2$$

and so does not belong to $\mathrm{Reg}(T)$. Hence, x is not eventually regular. Thus T is not eventually regular, but it is structurally regular. ∎

A semigroup S is said to be *locally regular* if for every idempotent e of S, the subsemigroup $eSe = \{exe\colon x \in S\}$ is regular.

Lemma 4.8. *Every structurally regular semigroup is locally regular.*

Proof: Suppose that $S/\theta(n,m)$ is regular, and take any $x \in eSe$ with $e \in E(S)$. Then for any $x' \in V(x;n,m)$, $x = exe = e^n x e^m = e^n(xx'x)e^m = xx'x$. By straightforward verification, one can show that the element $x^* = e(x'xx')e \in eSe$ is indeed an inverse of x, and so x is regular in eSe. Hence eSe is a regular subsemigroup, proving that S is locally regular. ∎

The next example shows that the converse of Lemma 4.8 does not hold.

Example 4.9: Let N be the semigroup of all positive integers under addition, G be a non trivial group, and $\phi\colon\ N \to G$ be the constant map which sends every element of N to the identity element e of G. Denote by $(S,\diamond)$ the ideal retract extension of G by N with respect to the homomorphism ϕ. Then the multiplication $\diamond$ on S is defined as follows:

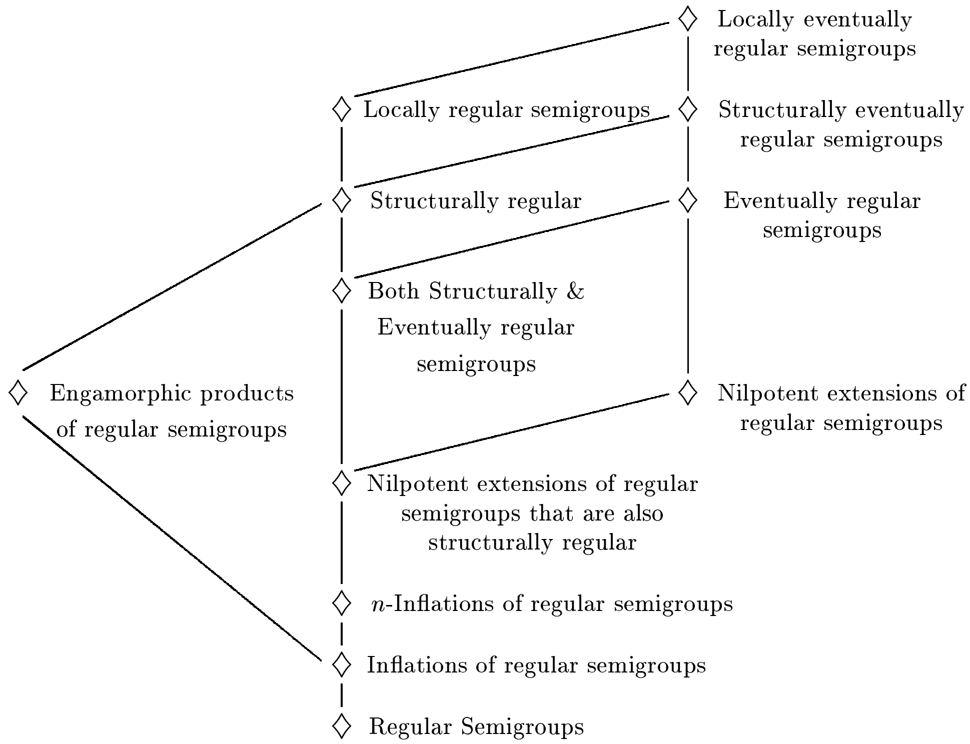
$$x \diamond y = \begin{cases} x, & \text{if } x \in G,\ y \in N, \\ y, & \text{if } x \in N,\ y \in G, \\ xy, & \text{otherwise .} \end{cases}$$

The identity element e of G becomes the unique idempotent element of S. Since $eSe = eGe = G$, the semigroup $(S,\diamond)$ is locally regular. However, we see that $(S,\diamond)$ is not structurally regular since for every (i,j) and any element x of N, $x\theta(i,j)$ forms a singleton set, and that the element x is not regular in S. Thus not every locally regular semigroup is structurally regular.

Lemma 4.10. *The class of all nilpotent extensions of regular semigroups and the class of all structurally regular semigroups are not comparable.*

Proof: Ruskuc produced an example of a nilpotent extension of a regular semigroup which is not structurally regular (see [23]; Example 2.2). The semigroup we encountered earlier in Example 2.10 is structurally regular but is not a nilpotent extension of some regular semigroup. These examples, together, prove that neither of the classes contain the other, and are therefore not comparable in this sense.

Figure 1 – Some species that contain the class of all regular semigroups.



Semigroup *species* [21] are just classes of semigroups that are closed under homomorphic images. In Figure 1, we summarise the containment relationships that exist between some known species containing the class of all regular semigroups. In the diagram, a continuous line indicates a strict containment. A semigroup S is said to be *locally eventually regular* if for every idempotent e of S, the subsemigroup $eSe = \{exe\colon\, x \in S\}$ is eventually regular; and S is *structurally eventually regular* if $S/\theta(n,m)$ is eventually regular for some (n,m). It is clear that both the classes of all structurally regular semigroups and the class of all eventually regular semigroups belong to the class of all structurally eventually regular semigroups. And a semigroup is called *structurally locally eventually regular* if $S/\theta(n,m)$ is locally eventually regular for some (n,m). The semigroup $(N,+)$ does not belong to any of the classes so far considered, although it appears as a subsemigroup of some regular semigroups. Hence the classification presented in Figure 1 does not exhaust the class of all semigroups. However, the class of all finite semigroups is included here since they are eventually regular.

Lemma 4.11. *Every structurally eventually regular semigroup is locally eventually regular.*

Proof: Suppose that S is structurally eventually regular, and consider any idempotent element $e \in E(S)$. Then $S/\theta(n,m)$ is eventually regular for some ordered pair of non negative integers (n,m). For each $x \in eSe$ there exists an element b of S, and a positive integer k such that $(x^k bx^k, x^k) \in \theta(n,m)$. This implies that $u\,x^k bx^k\,v = u\,x^k\,v$ for all $u \in S^n$ and $v \in S^m$.

Now,

$$x^k = e\,x^k\,e = e^n\,x^k\,e^m = e^n\,(x^k bx^k)\,e^m = x^k bx^k\ .$$

One can show that the element $a = e(bx^k b)e \in eSe$ is an inverse of x^k. Hence it follows that S is locally eventually regular. ∎

The previously encountered Example 4.9 serves to show that the converse of Lemma 4.11 does not hold.

A class $\mathcal{C}$ is said to be *structurally closed* if for every $S \in \mathcal{C}$, and any ordered pair (n,m) of non-negative integers, the quotient $S/\theta(n,m)$ belongs to $\mathcal{C}$.

Lemma 4.12. *The class of all locally regular semigroups is structurally closed.*

Proof: Suppose that $S/\theta(n,m)$ is locally regular, and take any $e \in E(S)$. Then for any $x \in eSe$, $x\theta(n,m)$ is regular in $S/\theta(n,m)$ since $x\theta(n,m)$ is contained in the local subsemigroup of $S/\theta(n,m)$ with identity element $e\theta(n,m)$. Therefore, by assumption, there exists $a \in S$ such that (xax, x) and (axa, a) are $\theta(n,m)$-related pairs in S. Hence $uxaxv = uxv$ and $uaxav = uav$ for all u in S^n and v in S^m. Now, in S, $x = exe = e^n xe^m = e^n xaxe^m = xax$. It can be shown that $y = e(axa)e \in eSe$ is an inverse of x, and so S is locally regular. ∎

Lemma 4.13. *The class of all locally eventually regular semigroups is structurally closed.*

Proof: Suppose that $S/\theta(n,m)$ is locally eventually regular, and take any $e \in E(S)$. Then for any $x \in eSe$, there exists a positive integer $k \geq 1$, such that $x^k\theta(n,m)$ is regular in $S/\theta(n,m)$, since $x\theta(n,m)$ is contained in the local subsemigroup of $S/\theta(n,m)$ with identity element $e\theta(n,m)$. Therefore, by assumption, there exists $a \in S$ such that $(x^k ax^k, x^k)$ and $(ax^k a, a)$ are $\theta(n,m)$-related pairs in S. Hence $ux^k ax^k v = ux^k v$ and $uax^k av = uav$ for all u in S^n and v in S^m. Now, in S, $x^k = ex^k e = e^n x^k e^m = e^n x^k ax^k e^m = x^k ax^k$. It can be shown that $y = e(ax^k a)e \in eSe$ is an inverse of x^k, and so S is locally eventually regular. ∎

Finally, we now demonstrate how one can produce concrete examples of semigroups from the types given in Figure 1. Let R_0 be a non trivial regular semigroup, say the bicyclic semigroup, and N be a non trivial nilpotent semigroup.

i) Let $R_1 = R_0 \times N$ be the direct product of R_0 and N. Then, as shown in Example 2.2, R_1 is both a nilpotent extension and a structurally regular semigroup.

ii) Let $R_2 = R_1^{(1)}$ be the semigroup obtained by adjoining an identity element to R_1. And as shown in Example 2.3, R_2 is eventually regular, but is neither structurally regular nor a nilpotent extension.

iii) Let $R_3 = (R_2^{(1)} \times R_2, \Theta)$, where $R_2^{(1)}$ is the semigroup obtained by adjoining an identity element to R_2, $R_2^{(1)} \times R_2$ is the Cartesian product, and the multiplication Θ is defined as follows: $(a, b)\,\Theta\,(c, d) = (ad, bd)$. Then as was the case for the semigroup in Example 4.1, R_3 is a structurally eventually regular semigroup but is not eventually regular.

iv) Let R_4 be the ideal extension of R_3 by the semigroup $(N, +)$ of all positive integers under addition determined by a constant map which sends every element of N to a fixed idempotent element of R_3. Then every local subsemigroup of R_4 turns out to be a local subsemigroup of R_3. As was the case for the semigroup in Example 4.9, R_4 is not structurally eventually regular but is locally eventually regular.

v) From Lemma 4.12 and Lemma 4.13 any structurally locally [eventually] regular semigroup is again locally [eventually] regular.

One can construct structurally regular semigroup using the method described in Example 3.10 of [21]. Example 4.9 gives a locally regular semigroup that is not structurally regular. The construction of engamorphic products on a regular semigroup, or the taking of a nilpotent extension of a regular semigroup are well known procedures. Thus each of the classes given on Figure 1 are distinct and non empty.

We complete this paper with a characterisation of structurally permutative semigroups. A semigroup is said to be *permutative* if it satisfies a permutation identity. In particular a semigroup is said to be *commutative* if it satisfies the permutation identity $xy = yx$.

Theorem 4.12. *A semigroup S is permutative if and only if it is a structurally commutative semigroup.*

Proof: Suppose that $S/\theta(n,m)$ is commutative. Then it follows from ([21]; Theorem 4.7) that for all $x,y \in S$, $u \in S^n$ and $v \in S^m$: $uxyv = uyxv$. Clearly, this is a permutation identity. To prove the converse, we need to show that for every permutative semigroup S, $S/\theta(n,m)$ is commutative for some (n,m). But that follows from the following result of Putcha and Yaqub [32]. ∎

Theorem 4.13 ([32]; Theorem 1). *Let S be a semigroup such that, for all $x_1, x_2, ..., x_n$ in S,*

$$x_1\,x_2 \cdots x_n = x_{\sigma(1)}\,x_{\sigma(2)} \cdots x_{\sigma(n)} \quad (n \geq 2)\ , \tag{4.14}$$

where σ is a fixed permutation of $\{1,2,...,n\}$ distinct from the identity permutation. Then there exists an integer k such that, for all $u,v \in S^k$ and for all $x_1, x_2 \in S$ we have that

$$ux_1x_2v = ux_2x_1v\ .\ ∎$$

It is well know that any commutative regular semigroup is an inverse semigroup. The following analogous result holds for structurally regular semigroups.

Corollary 4.15. *Let S be a structurally regular semigroup. If S is permutative, then it is a structurally inverse semigroup.* ∎

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