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LOCAL CONNECTEDNESS AND CONNECTED OPEN FUNCTIONS

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Abstract: Localized versions are proved of some results concerning preservation of local connectivity and related concepts under connected open functions satisfying a condition weaker than continuity.

1 – Introduction

Recently a condition (Z) for functions between topological spaces which is strictly weaker than continuity has been introduced and studied by Y. Zhou [9]. As a basic result of [9] it is proved that local connectedness is preserved under a connected open surjection satisfying the condition (Z). The main purpose of the present paper is to generalize and improve the quoted result of Y. Zhou, and to extend these generalizations to other concepts related to local connectivity of the space at a given point, as weak local connectedness and quasilocal connectedness. To do so we localize the condition (Z) as well as we apply this localization to functions which satisfy some openness and connectivity conditions also considered locally, at a given point. In this way we obtain as corollaries a number of known and new results concerning preservation of local connectedness and related properties under various classes of mappings. A final part of the paper is devoted to open problems concerning some other local properties of the space (paddedness) and of the considered functions (almost openness).

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2 – Preliminaries

Throughout this paper *spaces* mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let a point p of a space X be given. The *component* of p in X means a maximal connected subset of X containing p. The *quasicomponent* of p means the intersection of all simultaneously closed and open subsets of X containing p. If a set A is given with $p \in A \subset X$, we denote by $Q_X(p, A)$ the quasicomponent of A in X containing p. See Part 1 of [1], p. 353–354, for various interrelations of components and quasicomponents. A *neighbourhood* of a point p means a set containing p in its interior.

A space X is said to be:

– *locally connected at* p provided that every neighbourhood of p contains an open connected neighbourhood of p;

– *weakly locally connected* (also called connected "im kleinen") *at* p provided that every neighbourhood of p contains a connected neighbourhood of p;

– *quasilocally connected at* p provided that for every neighbourhood U of p the quasicomponent $Q_X(p, U)$ is a neighbourhood of p ([8], p. 40).

The space X is said to have any of the properties defined above if it has that property at each of its points.

Note that a space is weakly locally connected at p if and only if for each neighbourhood U of p the point p belongs to the interior of the component of U. Observe that some authors (e.g. K. Kuratowski [5], p. 227, and G.T. Whyburn [6], p. 18) use the name of "locally connected at a point" in the sense of "weakly locally connected at a point".

The following implications are known and none of them can be reversed.

2.1 Theorem.

i) *If a space is locally connected at p, then it is weakly locally connected at p (the converse is false – [3], p. 113).*

ii) *If a space is weakly locally connected at p, then it is quasilocally connected at p (the converse is false – [1], Example 5.4, p. 363).*

For an arbitrary space X let $LC(X)$, $WLC(X)$ and $QLC(X)$ denote the sets of points of X at which X is locally connected, weakly locally connected and quasilocally connected correspondingly. Then Theorem 2.1 can be reformulated as follows:

$$LC(X) \subset WLC(X) \subset QLC(X) \ . \tag{2.2}$$

The next theorem is also a well known result (see e.g. [1], Proposition 2.4, p. 355).

2.3 Theorem. *The following conditions on a space X are equivalent:*

i) *X is locally connected;*

ii) *X is weakly locally connected;*

iii) *X is quasilocally connected;*

iv) *(quasi)components of every open subset of X are open.*

Let X and Y be spaces. A function $f\colon X \to Y$ is said to be:

– *a mapping* provided that it is continuous;

– *open (closed, connected)* provided that for each open (closed, connected) set $A \subset X$ its image $f(A)$ is an open (closed, connected) subset of Y;

– *open (connected, connected-open) at a point* $p \in X$ provided that there exists an open set $U \subset X$ containing p such that for each open (connected, connected and open) subset A of U containing p its image $f(A)$ is an open (connected, connected and open) subset of Y;

– *interior at a point* $p \in X$ provided that for each open set $U \subset X$ containing p we have $f(p) \in \operatorname{Int} f(U)$.

We denote by $\operatorname{Op} f$, $\operatorname{Con} f$, $\operatorname{Con-op} f$ and $\operatorname{Int} f$ the sets of all points $p \in X$ at which the function $f\colon X \to Y$ is open, connected, connected-open and interior, respectively. Further, for a given function $f\colon X \to Y$ we denote by $Q(f)$ the set of all points $p \in X$ for which there exists an open set $U \subset X$ containing p such that for each subset A of U containing p we have

$$f(Q_X(p, A)) \subset Q_Y(f(p), f(A)) \ . \tag{2.4}$$

The propositions below are easy consequences of the definitions.

2.5 Proposition. *For every function $f\colon X \to Y$ the following conditions are equivalent:*

i) *f is open;*

ii) *f is interior at each point of its domain, i.e.,* $\operatorname{Int} f = X$*;*

iii) *f is open at each point of its domain, i.e.,* $\operatorname{Op} f = X$.

2.6 Proposition. *If a function is open at a point, then it is interior at this point, i.e.,* $\operatorname{Op} f \subset \operatorname{Int} f$.

2.7 Remark. The converse to Proposition 2.6 is not true because the well known Cantor step mapping from the Cantor ternary set $C \subset [0,1]$ onto $[0,1]$ is interior at $p = \frac{1}{4}$ while not open at this point.

2.8 Proposition. *If a function* $f\colon X \to Y$ *is connected at a point* $p \in X$ *and open at this point, then it is connected-open at* p*, but not conversely.*

2.9 Proposition. *If a function* $f\colon X \to Y$ *is connected, then it is connected at each point of its domain, i.e.,* $\operatorname{Con} f = X$.

To see that the converse implication does not hold, consider the following example.

2.10 Example: *There exist a connected space* B *and a function* $f\colon B \to [0,1]$ *of* B *into* $[0,1]$ *such that* f *is not connected, while it is connected at each point of* B.

Proof: Denote by C the Cantor ternary set in $[0,1]$, consider the well known Knaster–Kuratowski biconnected set B contained in the Cantor fan $F = C \times [0,1]/C \times \{0\}$ (see e.g. [5], §46, II, Remark, p. 135) and denote by v the vertex of F. Thus B is connected, and the components of $B\backslash\{v\}$ are singletons. Put $B_0 = \{(x,y) \in B : y \in [0,\frac{1}{2}]\}$ and $B_1 = B\backslash B_0$. Let $\pi : B_0 \to [0,\frac{1}{2}]$ be the projection defined by $\pi(x,y) = y$. Denote by $\mathbb{Q}$ the set of all rational numbers and note that the sets B_1 and $(\frac{1}{2},1]\backslash\mathbb{Q}$ have the same cardinality. Thus there exists a one-to-one correspondence $\gamma\colon B_1 \to (\frac{1}{2},1]\backslash\mathbb{Q}$. Put

$$f(p) = \pi(p) \ \text{ if } p \in B_0 \qquad \text{and} \qquad f(p) = \gamma(p) \ \text{ if } p \in B_1 .$$

Thus $f(B) = [0,\frac{1}{2}] \cup ((\frac{1}{2},1]\backslash\mathbb{Q}) \subset [0,1]$. Taking B_0 as a neighbourhood U of v and $B\backslash\{v\}$ as a neighbourhood U of each point $p \neq v$ we see that all the conditions of connectedness at a point are satisfied for f, while f is not connected because the image $f(B)$ of the connected set B is not connected. ∎

2.11 Remark. Obviously for an arbitrary function continuity implies connectedness. However, the localized version of this implication is not true, i.e., a function which is continuous at a point of its domain need not be connected at this point. In fact, define $f : [0,1] \to \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ by $f(0) = 0$ and $f(x) = [x^{-1}]^{-1}$ for $x \neq 0$, where $[r]$ denotes the whole part of a real number r.

2.12 Remark. Note that for every mapping $f\colon X \to Y$ inclusion (2.4) holds true.

A function $f\colon X \to Y$ is said to satisfy

- the *condition* (Z) *with respect to a point* $y \in Y$ provided that for every open subset V of Y containing y we have $f^{-1}(y) \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$;
- the *condition* (Z) provided that it satisfies the condition (Z) with respect to every point $y \in Y$. That is, provided that for every open subset V of Y we have $f(\operatorname{Int} f^{-1}(V)) = V$.

Observe that if a function f satisfies the condition (Z) with respect to a point $y \in Y$, then $y \in f(X)$; therefore, if f satisfies the condition (Z), then f is a surjection. Further, if the considered function f is continuous at a point $x \in X$, then it obviously satisfies the condition (Z) with respect to the point $f(x)$. The converse implication does not hold, even for open connected functions, see [9], Example, p. 65.

3 – Results

It is well known that local connectedness is preserved under continuous surjective functions which are open or closed ([3], Lemma 3-21, p. 125), almost open ([4], Theorem 3.2, p. 396), quasicompact ([7], Theorem 2, p. 445), or which are arbitrary even, provided that the domain space is pseudocompact ([1], Theorem 4.12, p. 361). Invariance of some other properties that are close to local connectedness has also been studied (see e.g. [4], Theorem 3.4, and Corollaries 3.5, 3.6 and 3.7, p. 396 and 397, and some other results mentioned in the introduction of that paper, p. 393). Recently it was shown that local connectedness is preserved under open connected functions which are strictly weaker than continuous ones, namely which satisfy the condition (Z) ([9], Theorem 4, p. 66).

However, all these results concern preservation of local connectedness when this property is assumed to be satisfied globally, i.e., at every point of the domain space. We intend to prove similar implications, but when the property of local connectedness and related properties are considered locally, i.e., at particular points of the domain and the range spaces, and when the function satisfies the assumed conditions of connectedness and interiority or openness also at the considered points.

The following theorem arose from Theorem 4 of [9], p. 66 (see below, Corollary 3.15).

3.1 Theorem. *Let a surjective function $f\colon X \to Y$ between spaces X and Y, and a point $y \in Y$ be given. Assume that for every open subset V of Y containing y there is a point $x \in X$ such that*

$$x \in f^{-1}(y) \cap \operatorname{Int} f^{-1}(V) \ . \tag{3.2}$$

The following three implications are true:

a) *if*

$$x \in LC(X) \cap \text{Con-op} f \ , \tag{3.3}$$

then $y \in LC(Y)$;

b) *if*

$$x \in WLC(X) \cap \operatorname{Con} f \cap \operatorname{Int} f \ , \tag{3.4}$$

then $y \in WLC(Y)$;

c) *if*

$$x \in QLC(X) \cap Q(f) \cap \operatorname{Int} f \ , \tag{3.5}$$

then $y \in QLC(Y)$.

Proof: Let an open subset V of Y contain the point y, and let a point $x \in X$ satisfy (3.2).

a) By (3.3) the function f is connected-open at x, i.e., there exists an open subset U of X containing x such that if $A \subset U$ is both connected and open, then $f(A) \subset Y$ also is connected and open. Since the space X is locally connected at x and since $U \cap \operatorname{Int} f^{-1}(V)$ is an open subset of X containing x by (3.2), there exists a connected and open set $W \subset X$ such that $x \in W \subset U \cap \operatorname{Int} f^{-1}(V) \subset f^{-1}(V)$. Therefore $y = f(x) \in f(W) \subset f(f^{-1}(V)) \subset V$. Since $f(W)$ is a connected and open subset of V, the space Y is locally connected at y, i.e., $y \in LC(Y)$ as needed.

b) By (3.4) the function f is connected at x, i.e., there exists an open subset U of X containing x such that if $A \subset U$ is connected and contains x, then $f(A) \subset Y$ is connected. Since the space X is weakly locally connected at x by (3.4) and since $U \cap \operatorname{Int} f^{-1}(V)$ is an open subset of X containing x by (3.2), there exists a connected set W in X such that $x \in \operatorname{Int} W \subset W \subset U \cap \operatorname{Int} f^{-1}(V) \subset f^{-1}(V)$. Therefore, the function f being interior at x by (3.4), we have $y = f(x) \in \operatorname{Int} f(\operatorname{Int} W) \subset \operatorname{Int} f(W) \subset f(f^{-1}(V)) \subset V$. Since $W \subset U$ is connected, we conclude $f(W)$ is connected, hence $f(W)$ is a connected neighbourhood of y contained in V, and thus $y \in WLC(Y)$ as needed.

c) By (3.5) we have $x \in Q(f)$, i.e., there exists an open subset U of X containing x such that if $x \in A \subset U$ for a subset A, then

$$f(Q_X(x, A)) \subset Q_Y(y, f(A)) \ . \tag{3.6}$$

Since the space X is quasilocally connected at x by (3.5) and since $U \cap \operatorname{Int} f^{-1}(V)$ is an open subset of X containing x by (3.2), we conclude

$$x \in \operatorname{Int} Q_X(x, U \cap \operatorname{Int} f^{-1}(V)) \ . \tag{3.7}$$

Since f is interior at x by (3.5), we infer from (3.7) and (3.6) that

$$\begin{aligned} y = f(x) \in \operatorname{Int} f(\operatorname{Int} Q_X(x, U \cap \operatorname{Int} f^{-1}(V))) \subset \\ \subset \operatorname{Int} f(Q_X(x, U \cap \operatorname{Int} f^{-1}(V))) \subset \operatorname{Int} Q_Y(y, f(U \cap \operatorname{Int} f^{-1}(V))) \ . \end{aligned}$$

Finally, since for every G, H the inclusions $y \in G \subset H$ imply $Q_Y(y, G) \subset Q_Y(y, H)$ (see [1], Proposition 1.5, p. 354), we see that $Q_Y(y, f(U \cap \operatorname{Int} f^{-1}(V))) \subset Q_Y(y, V)$, and therefore we get $y \in \operatorname{Int} Q_Y(y, V)$. Thus the space Y is quasilocally connected at y, i.e., $y \in QLC(Y)$. The proof is complete. ∎

3.8 Corollary. *Let a function $f\colon X \to Y$ be surjective and let $\mathcal{B}(y)$ be a local base at a point y in Y. If there is a point $x \in X$ such that*

$$x \in f^{-1}(y) \cap \bigcap \left\{ \operatorname{Int} f^{-1}(V) \colon V \in \mathcal{B}(y) \right\} , \tag{3.9}$$

then implications a), b) and c) of Theorem 3.1 hold true.

3.10 Remark. By Proposition 2.8 condition (3.3) of implication a) in Theorem 3.1 can be replaced by

$$x \in LC(X) \cap \operatorname{Con} f \cap \operatorname{Op} f \ . \tag{3.11}$$

3.12 Remark. Observe that if for every open subset V of Y containing the point $y \in Y$ there is a point $x \in X$ such that (3.2) holds, or if (3.9) is satisfied, then the function f satisfies the condition (Z) with respect to y.

The following corollaries are immediate consequences of implications a), b) and c) of Theorem 3.1.

3.13 Corollary. *Let a function $f\colon X \to Y$ satisfying the condition (Z) be open and connected, and let a point $x \in X$ be given. If for every open subset V of Y containing the point $f(x)$ there is a point $x_0 \in f^{-1}(f(x)) \cap \operatorname{Int} f^{-1}(V)$ at which the domain X is* **a**) *locally connected or* **b**) *weakly locally connected, then the range Y at the point $f(x)$ is* **a**) *locally connected or* **b**) *weakly locally connected, respectively.*

3.14 Corollary. *Let a function $f\colon X \to Y$ satisfying the condition (Z) be open, and let a point $x \in X$ be given. If for every open subset V of Y containing the point $f(x)$ there is a point*

$$x_0 \in f^{-1}(f(x)) \cap \operatorname{Int} f^{-1}(V) \cap Q(f)$$

at which the domain X is quasilocally connected, then the range Y is quasilocally connected at the point $f(x)$.

3.15 Corollary ([9], Theorem 4, p. 66)**.** *Let a connected, open function $f\colon X \to Y$ satisfy the condition (Z). If X is locally connected, then so is Y.*

3.16 Corollary. *Let a surjective mapping $f: X \to Y$ between spaces X and Y be open (interior, interior) at a point $x \in X$. If the domain X is locally connected (weakly locally connected, quasilocally connected) at x, then the range Y is locally connected (weakly locally connected, quasilocally connected) at $f(x)$.*

As the next corollary we have the following known result (compare e.g. [2], Chapter VI, 3.5, p. 125, and 1.4, p. 121).

3.17 Corollary. *Let a surjective mapping $f: X \to Y$ be open. If X is locally connected, then so is Y.*

The following example (which is due to the referee) shows that the assumption of openness of the function f at the point x in the implication a) of Theorem 3.1 (see condition (3.3) and compare condition (3.11) of Remark 3.10) cannot be weakened to the assumption of interiority of f at x (Proposition 2.6). Recall that a compact connected Hausdorff space is called a *continuum*. A space is said to be *arcwise* connected provided that any two of its points can be joined by an arc contained in the space.

3.18 Example: *There exists an arcwise connected continuum Y in the plane, a point $y \in Y$ and a surjective function $f\colon [0,1] \to Y$ such that for every open* *subset V of Y containing y there is a point $x \in [0,1]$ satisfying*

$$x \in f^{-1}(y) \cap \operatorname{Int} f^{-1}(V) \cap LC([0,1]) \cap \operatorname{Con} f \cap \operatorname{Int} f \setminus \operatorname{Op} f \ ,$$

while $y \notin LC(Y)$.

Proof: In the Euclidean plane $\mathbb{R}^2$ let ab denote the straight line segment joining a with b. For every $m \in \mathbb{N}$ and $n \in \{0\} \cup \mathbb{N}$ put

$$y = (0,0)\,, \qquad v_n = (2^{1-n}, 0) = u_{n+1,0} \qquad \text{and} \qquad u_{n,m} = (2^{-n}, 2^{-(m+n)}) \ .$$

Then define

$$B_{n,m} = v_{n+1}\, u_{n+1,m} \ \text{ for } m,n \in \{0\} \cup \mathbb{N}\,, \qquad B_n = \bigcup \Big\{ B_{n,m} \colon\, m \in \{0\} \cup \mathbb{N} \Big\} \ ,$$

and

$$Y = \{y\} \cup \bigcup \Big\{ B_n \colon\, n \in \{0\} \cup \mathbb{N} \Big\} \ .$$

Note that for each $n \in \{0\} \cup \mathbb{N}$ the set B_n is the cone with the vertex v_{n+1} over the (noncompact) set $\{u_{n+1,m} : m \in \mathbb{N}\}$. For a picture of a continuum homeomorphic to Y see [3], Fig. 3–9, p. 113.

Further, for every $m,n \in \{0\} \cup \mathbb{N}$ define subintervals A_n and $A_{n,m}$ of $[0,1]$ as follows:

$$A_n = (2^{-(n+1)}, 2^{-n}] \qquad \text{and} \qquad A_{n,m} = \Big[2^{-(n+1)}(1 + 2^{-(m+1)}), 2^{-(n+1)}(1 + 2^{-m}) \Big] \ .$$

Note that the intervals A_n are mutually disjoint,

$$A_n = \bigcup \{ A_{n,m} \colon\, m \in \{0\} \cup \mathbb{N} \} \qquad \text{for each} \ \ n \in \{0\} \cup \mathbb{N} \ ,$$

and that

$$[0,1] = \{0\} \cup \bigcup \Big\{ A_n \colon\, n \in \{0\} \cup \mathbb{N} \Big\} \ .$$

Now, for each $n \in \{0\} \cup \mathbb{N}$, let $f_n \colon A_n \to B_n$ be defined by the conditions: $f_n | A_{n,0} \colon A_{n,0} \to B_{n,0}$ is a linear, surjective mapping with $f_n(2^{-n}) = v_n = u_{n+1,0}$, and $f_n(2^{-(n+1)}(\frac{3}{2})) = v_{n+1}$; and for each $k \in \mathbb{N}$ the partial functions

$$f_n | A_{n,2k-1} \colon A_{n,2k-1} \to B_{n,k} \qquad \text{and} \qquad f_n | A_{n,2k} \colon A_{n,2k} \to B_{n,k}$$

are linear surjections with

$$f_n\Big(2^{-(n+1)}(1 + 2^{1-2k}) \Big) = v_{n+1} \qquad \text{and} \qquad f_n\Big(2^{-(n+1)}(1 + 2^{-2k}) \Big) = u_{n+1,k}$$

(i.e., the right end point of $A_{n,2k-1}$ which coincides with the left one of $A_{n,2k}$ is mapped onto the end point v_{n+1} of $B_{n,k}$ while the left end point of $A_{n,2k-1}$ which coincides with the right one of $A_{n,2k}$ is mapped onto the other end point $u_{n+1,k}$ of $B_{n,k}$). Thus for each $n \in \{0\} \cup \mathbb{N}$, the function $f_n\colon A_n \to B_n$ is a continuous surjection.

Finally we define $f:\ [0,1] \to Y$ putting $f(0) = (0,0)$ and $f(t) = f_n(t)$ if $t \in A_n$ for some $n \in \{0\} \cup \mathbb{N}$. Then f is a surjective but not continuous function, and it has the needed properties for $x = 0$. In particular, any neighbourhood U of x contains the interval $A = [0, 2^{-n}]$ for some sufficiently large $n \in \mathbb{N}$ whose image $f(A)$ is not open in Y. ∎

4 – Problems

In this chapter we shall discuss implications similar to a), b) or c) of Theorem 3.1 for two other concepts, one of which is related to the spaces X and Y, and the other is related to the function $f\colon X \to Y$. First, the condition of local connectivity of the space at a point will be replaced by paddedness of the space at the point. Second, an openness or interiority condition of the function at a considered point will be replaced by a weaker one, of almost openness of f at the point.

A space X is said to be *padded at a point* $p \in X$ provided that for every neighbourhood U of p there exist open sets W_1 and W_2 such that $p \in W_1 \subset \overline{W}_1 \subset W_2 \subset U$ and $W_2 \backslash \overline{W}_1$ has only finitely many components. The space X is said to be *padded* provided it is padded at each of its points (see [1], p. 355; see also [6], p. 19, where the name of a semilocally connected space is used for a connected space with the same property). For an arbitrary space X let $P(X)$ denote the set of all points of X at which the space X is padded. Below we recall some known facts about the concept of paddedness.

4.1 Proposition ([1], Proposition 2.3, p. 355). *If a connected space is padded at a point p, then it is locally connected at p, i.e., if X is connected, then $P(X) \subset LC(X)$ (the converse is false – [1], Example 5.1, p. 361).*

A space X is said to be *rim-compact at a point* $p \in X$ provided that every neighbourhood of p contains an open neighbourhood of p with compact boundary. Clearly every locally compact space is rim-compact. We quote the following result.

4.2 Theorem ([1], Theorem 3.3, p. 356). *If X is a continuum, or — more generally — a rim-compact connected Hausdorff space, then the following conditions are equivalent:*

i) *X is locally connected;*

ii) *X is padded;*

iii) *components and quasicomponents of every open subset of X coincide.*

4.3 Problem. Let a surjective function $f\colon X \to Y$ between spaces X and Y, and a point $y \in Y$ be given. Assume that for every open subset V of Y containing y there is a point $x \in X$ such that

$$x \in f^{-1}(y) \cap \operatorname{Int} f^{-1}(V) \ , \tag{3.2}$$

and that the domain X is padded at x, i.e., $x \in P(X)$. Under what conditions concerning the behaviour of f locally at x (i.e., in an open subset U of X containing x) and/or the space X it follows that $y \in P(Y)$?

The above discussed Theorem 4 of [9], p. 66 (quoted here as Corollary 3.15) has even been extended to a wider class of almost open functions in place of open ones — see Theorem 8 of [9], p. 67. In the light of this theorem a natural problem arises concerning a possibility of generalizing the results of the previous chapter — in particular of Theorem 3.1 — to almost open functions if this notion is considered locally, at a given point. Therefore let us accept the following definition.

A function $f\colon X \to Y$ between spaces X and Y is said to be *almost open at a point* $p \in X$ provided that there exists an open set $U \subset X$ containing p such that for each open subset A of U containing p its image $f(A)$ satisfies the inclusion

$$f(A) \subset \operatorname{Int} \overline{f(A)} \ . \tag{4.4}$$

A function $f\colon X \to Y$ is said to be *almost open* provided that (4.4) holds for each open subset A of X (see [4], p. 394, where an equivalent condition is taken as a definition of this concept). Denote by Al-op f the set of all points $p \in X$ at which the function $f\colon X \to Y$ is almost open. Thus the inclusion of Proposition 2.6 can be supplied as follows:

$$\operatorname{Op} f \subset \operatorname{Int} f \subset \text{Al-op}\, f \ . \tag{4.5}$$

4.6 Problems. Can the assumption $x \in \text{Con-op}\, f$ in (3.3) of implication a) of Theorem 3.1 be relaxed to the following:

(4.7) There exists an open subset U of X containing x such that for each connected open subset A of U containing x inclusion (4.4) holds true?

4.8 Problems. Can the assumption $x \in \operatorname{Int} f$ in (3.4) and (3.5) of implications b) and c) of Theorem 3.1 respectively be relaxed to $x \in \text{Al-op}\, f$?

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