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ENTROPY DIMENSION OF DYNAMICAL SYSTEMS

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Abstract: The key idea here is borrowed from dimension theory. The starting point is a new concept which behaves like a dimension and is devoted to distinguish zero topological entropy systems. It is a dynamical invariant but also reflects geometrical features.

1 – Introduction

Among all labels used in dynamical systems, topological entropy, Hausdorff dimension and Lyapounov exponents seem to gather the majority of the preferences. Complex systems or complicated geometrical structures may be guessed through positive entropy, nonzero Lyapounov exponents or big Hausdorff dimension of invariant subsets. Each of these methods is linked to a specific approach and depends, in general, on hard calculations. The alternative is to look for sharp estimates of them and, for that purpose, one appeals to connections among topological, metrical and geometrical information. These allow us to overcome inadequacies of each one as a good label; for example, neither the definition of Hausdorff dimension is aware of the dynamics nor the topological entropy provides by itself a geometrical inkling.

Simple systems with respect to these devices need deeper analysis and an acute differentiation between them is expectedly difficult. The notion we will discuss here, in spite of not being a complete invariant, may turn into a useful and suggestive tool to distinguish simple systems — the ones with zero topological entropy — which however are likely to have inner complexity which traditional procedures do not spot.

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Let X be a compact metric space and $f\colon X \to X$ a continuous endomorphism of X. Zero topological entropy essentially means a small growth rate of the number of elements of coverings of X when submitted to the effect of the dynamics. That is, a slow increase with n of the sequence

$$\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) ,$$

where

$$N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = \text{smallest cardinal of the finite subcoverings of } \bigvee_0^{n-1} f^{-i}\alpha$$

and

$$\bigvee_0^{n-1} f^{-i}\alpha = \Big\{A_0 \cap f^{-1}A_1 \cap \ldots \cap f^{-n+1}A_{n-1} \mid A_i \text{ is an element of the covering } \alpha \text{ of } X\Big\}.$$

The concept we will study intends to estimate this speed more accurately, comparing the sequence above not only with n, as usually to calculate the topological entropy, but also with other powers of n. Roughly speaking, it corresponds to topological entropies at different speeds, taking advantage from the canonical relationship between the functions $\log(x)$ and x^s, $s > 0$. More precisely, we will consider open finite coverings of X, take into account, for each $s > 0$, the sequence

$$\frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) ,$$

calculate its upper limit

$$\lim_{n\to+\infty}\sup \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) ,$$

evaluate the least upper bound of these limits when the covering α varies

$$\sup_{\alpha}\lim_{n\to+\infty}\sup \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) ,$$

and finally find the greatest lower bound of the set

$$\Big\{s > 0\colon \sup_{\alpha}\lim_{n\to+\infty}\sup \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = 0\Big\} .$$

The resemblance with the calculation of the Hausdorff dimension and the engagement of the dynamics justify the choice of "entropy dimension" to nominate this number.

Naturally we could extend these comparisons to other choices of test functions instead of x^s, but we might lose contact with the entropy. For example, if we consider the family of quadratic maps given, for each parameter λ in $]0,1]$, by

$$x \in [0,1] \mapsto f_\lambda(x) = 4\lambda x(1-x)$$

when restricted to the parameters such that $h_{\mathrm{top}}(f_\lambda) = 0$ (which correspond to $]0, \lambda_F]$, where λ_F is the first accumulation point of a cascade of period doubling), then

$$\inf\left\{ s>0\colon \lim_{n\to+\infty} \frac{\log\theta(n)}{(\log(n))^s} = 0 \right\} = 1 \iff \lambda = \lambda_F \ ,$$

where $\theta(n)$ denotes the fixed points of $(f_\lambda)^n$; this seems to suggest that $\log(n)$ and $\theta(n)$ are more suitable selections to distinguish the elements of this family. Best approximations however may be, in general, goals beyond reach: indeed, they would enhance a complete knowledge of the topological entropy and this is ultimately not manageable.

As we will prove, and examples will show, this dimension, say D, ranges in the interval $[0,1]$ and positive topological entropy yields $D=1$. So, as we wished, the remaining interval is wholly devoted to characterize the misterious domain of zero entropy dynamics.

A different approach to this subject was given by Katok in [K].

2 – Main definitions

Let X be a compact metric space and $f\colon X \to X$ a continuous endomorphism of X.

Definition 1. Denoting by α any open finite covering of X, the entropy dimension of f in X is given by

$$d_f(X) = \inf\left\{ s>0\colon \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha \Big) = 0 \right\} .$$

It is worthwile pointing out a simple property of this concept which we shall use several times.

Proposition 1.

(**a**) ([W]) *The sequence* $a_n(\alpha) = \log N(\bigvee_0^{n-1} f^{-i}\,\alpha)$ *satisfies the recurrence relation*

$$a_{n+k}(\alpha) \le a_n(\alpha) + a_k(\alpha)\,, \qquad \forall\, k, n \in \mathbb{N}\ .$$

(**b**) ([W]) *The* $\lim_{n\to+\infty} \frac{1}{n}\, a_n(\alpha)$ *exists and is the greatest lower bound of the set* $\{\frac{1}{n}\, a_n(\alpha)\}_{n\in\mathbb{N}}$.

(**c**) *Denote by* $d_f(s, X)$ *the number* $\sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N(\bigvee_0^{n-1} f^{-i}\alpha)$. *If* $s = 1$, *then*

$$d_f(s, X) = h_{\text{top}}(f) = \sup_\alpha \lim_{n\to+\infty} \frac{1}{n}\, a_n(\alpha)\ . \ \blacksquare$$

Definition 2. If Y is an f-invariant subset of X (this means $f(Y) = Y$) and Y is closed and α denotes any open finite covering of X, then the entropy dimension of f restricted to Y is given by

$$d_f(Y) = \inf\Big\{ s > 0\colon\ \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}(\alpha\cap Y)\Big) = 0 \Big\}\ .$$

Remark. Notice that if α is an open covering of X, then $\alpha \cap Y = \{A\cap Y \mid A \in \alpha\}$ is an open covering of Y; reciprocally, if β is an open covering of Y (with the induced topology), then $\alpha = \beta \cup \{X - Y\}$ is an open covering of X.

The next definition is a probabilistic version of former one.

Definition 3. Given an f-invariant probability μ, a finite measurable partition α of X and the sequence

$$H_\mu\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) = -\sum_{A\in\bigvee_0^{n-1} f^{-i}\alpha} \mu(A)\log\mu(A)$$

the metric entropy dimension of f in X is given by

$$d_f(\mu, X) = \inf\Big\{ s > 0\colon\ \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log H_\mu\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = 0 \Big\}\ .$$

3 – Examples

One proceeds checking the technical preliminaries above on some examples.

I. Let X be any compact space and f the identity map; as $N(\bigvee_0^{n-1} f^{-i}\alpha) = N(\alpha)$ for all α, we easily conclude that

$$\frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) = \frac{1}{n^s}\log N(\alpha)$$

and therefore $d_f(X) = 0$.

II. Let X be the interval $[0,1]$ and $f(x) = 2x$ (modulo 1); as $h_{\text{top}}(f) = \log 2$,

$$\begin{aligned}\sup_\alpha \lim_{n\to+\infty}\sup \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) &= \sup_\alpha \lim_{n\to+\infty}\sup \Big(\frac{1}{n}\log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big)\Big)(n^{1-s}) \\ &= \begin{cases} +\infty & \text{if } s<1, \\ \log 2 & \text{if } s=1, \\ 0 & \text{if } s>1\;. \end{cases}\end{aligned}$$

Therefore $d_f(X) = 1$.

III. Let X be the interval $[0,1]$, $f(x) = \frac{1}{2}\,x$ and α an open finite covering of $[0,1]$ with $\delta > 0$ as its Lebesgue number. Then

$$N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) \le r_n\Big(\frac{\delta}{2},[0,1]\Big) = \text{minimum cardinal of } (n,\varepsilon)-\text{spanning subsets of } X.$$

As f diminishes distances, we have $r_n(\varepsilon, X) \le r_{n-1}(\varepsilon, X) \le ... \le r_1(\varepsilon, X)$ and so

$$0 \le \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) \le \frac{1}{n^s}\log r_1\Big(\frac{\delta}{2}, X\Big)$$

which approaches zero as n goes to $+\infty$. Thus $d_f(X) = 0$.

IV. Generalizing last example, all isometries or contractions of X have entropy dimension zero.

V. [Sch] Consider a positive integer d, a finite set (alphabet) $A = \{1,2,...,k\}$, where $k \ge 2$, and

$$A^{\mathbb{Z}^d} = \Big\{a\colon\, \mathbb{Z}^d \to A\Big\}$$

which is compact with the product topology. Given a subset F of $\mathbb{Z}^d$, define

$$\begin{aligned} \pi_F : A^{\mathbb{Z}^d} &\to A^F \\ x &\mapsto \pi_F(x) = x_{|_F} \end{aligned}$$

and take, for each n in $\mathbb{Z}^d$ the transformations $\sigma_n :\ A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ given by

$$\sigma_n\Big((x_m)_{m\in\mathbb{Z}^d}\Big) = (\sigma_n(x))_m = (x_{n+m})_m \ .$$

A subset X of $A^{\mathbb{Z}^d}$ is a subshift of finite type if it is closed, shift-invariant (i.e. $\sigma_n(X) = X$ for all n in $\mathbb{Z}^d$) and there exists a finite subset F of $\mathbb{Z}^d$ such that

$$X = \Big\{x \in A^{\mathbb{Z}^d} :\ \pi_F(\sigma_n(x)) \in \pi_F(X) \ \text{ for all } n \text{ in } \mathbb{Z}^d\Big\} \ .$$

When $d = 1$, $A^{\mathbb{Z}^d} = \{1, 2, ..., k\}^{\mathbb{Z}}$ is the full shift of k symbols, that is the set of all doubly infinite sequences of symbols taken from $\{1, 2, ..., k\}$, together with the shift map which moves each sequence one step to the left

$$\sigma((x_n)_n) = (x_{n+1})_n \ .$$

This space has a natural product topology using the discrete metrics on $\{1, 2, ..., k\}$. If we let $\mathcal{M}$ be a matrix with entries $(a_{i,j})_{i,j=1,...,k}$ of zeros and ones such that the entry $a_{i,j}$ is zero precisely when we prohibite "ij" as a word of lenght two, then a subshift of finite type is given by

$$X_{\mathcal{M}} = \Big\{(x_n)_{n\in\mathbb{Z}} :\ x_n \in \{1, 2, ..., k\} \text{ and } a_{x_n, x_{n+1}} = 1 \ \text{ for all } n \text{ in } \mathbb{Z}\Big\} \ .$$

$X_{\mathcal{M}}$ is a compact and σ-invariant subset of the full shift and its metrical and dynamical properties depend essencially on the matrix $\mathcal{M}$. This concept corresponds to the above definition when $d = 1$ and $F = \{0, 1\}$.

Claim [Sch]: *For each positive integer N, denote by $\mathcal{Q}(N)$ the subset of $\mathbb{Z}^d$ given by $\{-N, ..., N\}^d$, by $\partial\mathcal{Q}(N)$ the difference $\mathcal{Q}(N) - \mathcal{Q}(N-1)$ and by $|S|$ the cardinal of S. Then we have*

$$|\pi_{\mathcal{Q}(N)}(X)| \leq |A|^{|\mathcal{Q}(N)|}$$

and

$$h_{\text{top}}(\sigma_{|_X}) = \lim_{N\to+\infty} \frac{1}{|\mathcal{Q}(N)|} \log |\pi_{\mathcal{Q}(N)}(X)|$$

where $\sigma = (\sigma_n)$. Therefore, as when $d = 1$, $h_{\text{top}}(\sigma_{|_X}) \leq \log(|A|)$.

Assume now that $F = \{0,1\}^d$, $|\pi_{\mathcal{Q}(N)}(X)| = |A|^{|\partial \mathcal{Q}(N)|}$ for all N and that $\pi_{\partial\mathcal{Q}(N)}(X)$ determines $\pi_{\mathcal{Q}(N)}(X)$, which means that each x in X depends only on its coordinates along $\partial\mathcal{Q}(N)$. This property ensures that

$$|\pi_{\mathcal{Q}(N)}(X)| = |\pi_{\partial\mathcal{Q}(N)}(X)| \ .$$

Since $|\mathcal{Q}(N)| = (2N+1)^d$, $|\partial\mathcal{Q}(N)| = \frac{d!}{2!\,(d-2)!}.4.2.N^{d-1}$ if $d>1$ and $|\partial\mathcal{Q}(N)| = 2$ if $d=1$, we have

(i) $$\lim_{N\to+\infty} \frac{1}{|\mathcal{Q}(N)|} \log |\pi_{\mathcal{Q}(N)}(X)| = \lim_{N\to+\infty} \frac{1}{|\mathcal{Q}(N)|} \log |A|^{|\partial\mathcal{Q}(N)|} = \\ = \lim_{N\to+\infty} \frac{|\partial\mathcal{Q}(N)|}{|\mathcal{Q}(N)|} \log |A| = 0 \ ;$$

(ii) $$\lim_{N\to+\infty} \frac{1}{(|\mathcal{Q}(N)|)^s} \log |\pi_{\mathcal{Q}(N)}(X)| = \lim_{N\to+\infty} \frac{1}{(2N+1)^{ds}} \log |\pi_{\mathcal{Q}(N)}(X)| = \\ = \lim_{N\to+\infty} \frac{1}{(2N+1)^{ds}} \log |A|^{|\partial\mathcal{Q}(N)|} = \lim_{N\to+\infty} \frac{1}{(2N+1)^{ds}} \log |A|^{\gamma} \ ,$$

where

$$\gamma = \begin{cases} \text{constant} * N^{d-1} & \text{if } d > 1, \\ 2 & \text{if } d = 1 \ . \end{cases}$$

Therefore

$$\lim_{N\to+\infty} \frac{1}{(2N+1)^{ds}} \log |A|^{\gamma} = \begin{cases} +\infty & \text{if } s < 1 - \frac{1}{d}, \\ \text{finite} & \text{if } s = 1 - \frac{1}{d}, \\ 0 & \text{if } s > 1 - \frac{1}{d} \end{cases}$$

and so $d_\sigma(X) = 1 - \frac{1}{d}$ for all $d \geq 1$.

Notice that, in this example, d_σ is ultimately only topological; the dynamics is essentially the same while d varies, but acts on increasing spaces with d and this is enough to alter the entropy dimension.

VI. Consider $X = [0,1]$, $f(x) = 3x$ (modulo 1) and $K_\alpha = \bigvee_0^{+\infty} f^{-i}(\alpha)$, where α is any choice of compact intervals of $[0,1]$. K_α is closed, f-invariant and

- if $\alpha = \{[0, \frac{1}{3}]\}$, then $K_\alpha = \{0\}$ and $d_f(K_\alpha) = 0$;
- in case $\alpha = \{[0,\frac{1}{3}], [\frac{2}{3}, 1]\}$, then $N(\bigvee_0^{n-1} f^{-i}\alpha) = 2^n$ so $\frac{1}{n^s} \log N(\bigvee_0^{n-1} f^{-i}\alpha) = \frac{n}{n^s} \log 2$ which approaches

$$\begin{cases} +\infty & \text{if } s < 1, \\ \log 2 & \text{if } s = 1, \\ 0 & \text{if } s > 1 \ . \end{cases}$$

Thus $d_f(K_\alpha) = 1$ and $d_f(1, K_\alpha) = h_{\mathrm{top}}(f_{|_{K_\alpha}}) = \log 2$ (α is a generator of the entropy of f restricted to K_α).

- if $\alpha = \{[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]\}$ (a Markov partition for f), then K_α is the canonical Cantor set and $N(\bigvee_0^{n-1} f^{-i}\alpha) = 3^n$, so $d_f(K_\alpha) = 1$ and $d_f(1, K_\alpha) = \log 3 = h_{\mathrm{top}}(f)$.

VII. We will use the standard notation $X = \{1, ..., k\}^{\mathbb{Z}}$, σ, $(p_{i,j})_{i,j=1,...,k}$, $(p_i)_{i=1,...,k}$ for the space, map, stochastic matrix and corresponding eigenvector of eigenvalue one of a Markov shift of finite type. Easy calculations lead, if α is the generator covering made up by cilinders, to

$$\frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\,\alpha\Big) = -\frac{n}{n^s} \sum_{i,j} p_i\, p_{i,j} \log(p_{i,j})$$

which yields for $s = 1$, $h_{\mathrm{top}}(\sigma) = -\sum_{i,j} p_i\, p_{i,j} \log(p_{i,j})$. Therefore $d_\sigma(X) = 1$, unless

$$\sum_{i,j} p_i\, p_{i,j} \log(p_{i,j}) = 0 \ ,$$

in which case $d_\sigma(X) = 0$. Meanwhile the only Markov shifts with zero topological entropy are the ones with finite support.

To state explicitely examples of smooth dynamical systems with $0 < d_f < 1$ is an arduous task, as was most likely anticipated from the equally odd difficulties this number inherits from zero topological entropy systems.

4 – Preliminaries

We start with a brief account on expected properties of the entropy dimension, in the following precise sense.

Proposition 2. *Fix $s > 0$ and consider the set $\mathcal{S} = \{$closed f-invariant subsets$\}$. Then*

(a.1) *$d_f(s, Y) = 0$ if Y is empty or finite;*

(a.2) *If Y and Z are in $\mathcal{S}$ and $Y \subseteq Z$, then $d_f(s, Y) \le d_f(s, Z)$;*

(a.3) *If $Y = \bigcup_k Y_k$, where Y, Y_k belong to $\mathcal{S}$, then $d_f(s, Y) \ge \sup_k d_f(s, Y_k)$;*

(a.4) *If $Y = \bigcup_{k=1}^N Y_k$, where Y, Y_k belong to $\mathcal{S}$, then $d_f(s, Y) = \max_k d_f(s, Y_k)$.*

Proof:

(**a.1**) This is immediate from

$$N\Big(\bigvee_0^{n-1} f_{|Y}{}^{-i}\,\alpha\Big) = \begin{cases} 1, & \text{if } Y \text{ is empty,} \\ \text{cardinal of } Y, & \text{otherwise,} \end{cases} \qquad \text{for all } \ \alpha \ .$$

(**a.2**) Given β an open covering of Y, $\alpha_\beta = \beta \cup \{\mathcal{C}Y\}$ is an open covering of Z and

$$N\Big(\bigvee_0^{n-1} f_{|Z}{}^{-i}\,\alpha_\beta\Big) = N\Big(\bigvee_0^{n-1} f_{|Y}{}^{-i}\,\beta\Big) + 1 \ ,$$

hence

$$d_f(s,Y) \le \sup_\beta \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|Z}{}^{-i}\,\alpha_\beta\Big) \le d_f(s,Z) \ .$$

(**a.3**) This results from the application of (a.2) to the inclusion $Y_k \subseteq Y$ for each k.

(**a.4**) Given a covering α of Y, $\alpha_i = \alpha \cap Y_i$ is a covering of Y_i and

$$\begin{aligned} N\Big(\bigvee_0^{n-1} f_{|Y}{}^{-i}\,\alpha\Big) &\le N\Big(\bigvee_0^{n-1} f_{|Y_1}{}^{-i}\,\alpha_1\Big) + N\Big(\bigvee_0^{n-1} f_{|Y_2}{}^{-i}\,\alpha_2\Big) + \dots + N\Big(\bigvee_0^{n-1} f_{|Y_k}{}^{-i}\,\alpha_k\Big) \\ &\le k \max_{1\le j\le k} \Big\{ N\Big(\bigvee_0^{n-1} f_{|Y_j}{}^{-i}\,\alpha_j\Big)\Big\} \ , \end{aligned}$$

so

$$\begin{aligned} \log N\Big(\bigvee_0^{n-1} f_{|Y}{}^{-i}\,\alpha\Big) &\le \log k + \log \max_{1\le j\le k} \Big\{ N\Big(\bigvee_0^{n-1} f_{|Y_j}{}^{-i}\,\alpha_j\Big)\Big\} \\ &\le \log k + \max_{1\le j\le k} \Big\{ \log N\Big(\bigvee_0^{n-1} f_{|Y_j}{}^{-i}\,\alpha_j\Big)\Big\} \end{aligned}$$

since log is an increasing function; hence

$$\begin{aligned} \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|Y}{}^{-i}\,\alpha\Big) &\le \lim_{n\to+\infty} \sup \max_{1\le j\le k} \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|Y_j}{}^{-i}\,\alpha_j\Big) \\ &= \max_{1\le j\le k} \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|Y_j}{}^{-i}\,\alpha_j\Big) \end{aligned}$$

and

$$\sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|_Y}{}^{-i}\,\alpha\Big) \leq$$

$$\leq \max_{1\leq j\leq k} \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f_{|_{Y_j}}{}^{-i}\,\alpha_j\Big) \ .$$

Therefore

$$d_f(s,Y) \leq \max_{1\leq j\leq k} d_f(s,Y_j) \ .$$

With (a.3) we complete the other inequality. ∎

These properties induce on d_f similar ones:

Proposition 3.

(**b.1**) *If Y and Z are in $\mathcal{S}$ and $Y \subseteq Z$, then $d_f(Y) \leq d_f(Z)$;*

(**b.2**) *If $Y = \bigcup_1^k Y_i$, where Y_i is an element of $\mathcal{S}$, then $d_f(Y) = \max_{1\leq j\leq k} d_f(Y_j)$;*

(**b.3**) *If $Y = \bigcup_1^{+\infty} Y_i$, where (Y_i) is an increasing union of elements of $\mathcal{S}$, then $d_f(Y) = \sup_j d_f(Y_j)$.*

Proof:

(**b.1**) If s is bigger than $d_f(Z)$, then $d_f(s,Z) = 0$ and so $d_f(s,Y) = 0$, which implies that $d_f(Y) \leq s$. Since this holds for all $s > d_f(Z)$, we must have $d_f(Y) \leq d_f(Z)$.

(**b.2**) For all $s > 0$, $d_f(s,Y) = \max_j d_f(s,Y_j)$. If we take s bigger than $\max_j d_f(s,Y_j)$, then $d_f(s,Y_j)$ vanishes for all j and so $d_f(s,Y) = 0$. This implies that $d_f(Y) \leq s$ for all such s and therefore

$$d_f(Y) \leq \max_j d_f(s,Y_j) \ .$$

Reciprocally, if we pick s bigger than $d_f(Y)$, then $d_f(s,Y) = 0$ and $d_f(s,Y_j) = 0$ for all j, and therefore $d_f(Y_j) \leq s$ for such s, which yields

$$\max_{1\leq j\leq k} d_f(Y_j) \leq s \ ,$$

$$\max_{1\leq j\leq k} d_f(Y_j) \leq d_f(Y) \ .$$

(**b.3**) We already know that $(d_f(Y_j))_j$ forms an increasing sequence whose limit, $\sup_j d_f(Y_j)$, is less or equal to $d_f(Y)$. If $\sup_j d_f(Y_j)$ were strictly smaller than $d_f(Y)$, then we could take s in the interval $]\sup_j d_f(Y_j), d_f(Y)[$ for which $d_f(s,Y) = +\infty$ but $d_f(s,Y_j) = 0$ for all j. However, since Y_j is approaching Y as j increases, given a covering α of Y,

$$\lim_j N\Big(\bigvee_0^{n-1} {f_{|_{Y_j}}}^{-i}\,\alpha\Big) = N\Big(\bigvee_0^{n-1} {f_{|_Y}}^{-i}\,\alpha\Big)$$

hence

$$\lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} {f_{|_Y}}^{-i}\,\alpha\Big)$$

is close to the corresponding limit of $\frac{1}{n^s} \log N(\bigvee_0^{n-1} {f_{|_{Y_j}}}^{-i}\,\alpha)$, for j big enough, and so

$$d_f(s,Y_j) = d_f(s,Y)\ . \ \blacksquare$$

Proposition 4.

(**a**) *X finite $\Rightarrow d_f(X) = 0$.*

(**b**) *If $\Omega_f(X)$ denotes the nonwandering set of f in X, then, for all closed subset Y of X,*

$$d_f(1,Y) \le d_f(1,X) = d_f(1,\Omega_f(X))\ .$$

Proof:

(**a**) If X is finite, $N(\bigvee_0^{n-1} f^{-i}\alpha) \le$ cardinal of X for all n in $\mathbb{N}$ and so $d_f(X) = 0$. Besides, if X is countable and may be written as an increasing union of finite f-invariant subsets $(X_i)_i$, then, by Proposition 3 (b.3), we get

$$d_f(X) = \sup_i d_f(X_i) = 0\ .$$

(**b**) $d_f(1,X) = h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f_{|_{\Omega_f(X)}})$. $\blacksquare$

As previously mentioned, the variable s on the definition of $d_f(X)$ makes this invariant remindful of a fractional dimension.

Proposition 5.

[**1**] *The map $s > 0 \mapsto d_f(s,X)$ is positive and decreasing with s.*

[**2**] *There exists $s_0 \in [0,+\infty]$ such that*

$$d_f(s,X) = \begin{cases} +\infty & \text{if } 0 < s < s_0, \\ 0 & \text{if } s > s_0\ . \end{cases}$$

Proof:

I. For all covering α,

$$\left.\begin{array}{l} 0 < s \leq t \\ n \in \mathbb{N} \end{array}\right\} \Rightarrow n^s \geq n^t \Rightarrow \frac{1}{n^s} \leq \frac{1}{n^t}$$

$$\Rightarrow \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \leq \frac{1}{n^t} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \ .$$

This inequality is preserved by the action of $\lim\limits_{n\to+\infty}\sup$ and $\sup\limits_{\alpha}$.

II.

(**1i**) If for all positive s we have $d_f(s,X) = +\infty$, then $d_f(X) = +\infty$;

(**2i**) If for all positive s we have $d_f(s,X) = 0$, then $d_f(X) = 0$;

(**3i**) If there exists a positive s such that $0 \neq d_f(s,X) \leq +\infty$, let s_0 be the biggest one of them (s_0 belongs to $]0,+\infty]$); then

(**3i.1**) in case $0 \neq d_f(s,X) < +\infty$ and s_0 is in $]0,+\infty[$, we have

$$\lim_{n\to+\infty}\sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = \lim_{n\to+\infty}\sup \frac{1}{n^{s-s_0}}\,\frac{1}{n^{s_0}} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$
$$= \begin{cases} +\infty & \text{if } s < s_0, \\ d_f(s_0,X) & \text{if } s = s_0, \\ 0 & \text{if } s > s_0 \ . \end{cases}$$

(**3i.2**) in case $d_f(s,X) = +\infty$ and s_0 is in $]0,+\infty[$, we have

$$\lim_{n\to+\infty}\sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = \lim_{n\to+\infty}\sup \frac{1}{n^{s-s_0}}\,\frac{1}{n^{s_0}} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$
$$= \begin{cases} +\infty & \text{if } s \leq s_0, \\ 0 & \text{if } s > s_0 \ . \end{cases}$$

In fact, if $s > s_0$ and $d_f(s,X) > 0$, then $d_f(t,X) = +\infty$ for all $s_0 < t < s$ which contradicts the definition of s_0.

(**3i.3**) the case $s_0 = +\infty$ was already considered in (1i). ∎

The metric on X suggests an alternative method to estimate $d_f(X)$, using the special coverings with balls. The notion of (n,ε)-spanning subset we shall evoke can be interpreted as the number of orbits of length n up to an error ε.

Proposition 6. *Denote by K any closed subset of X, by $r_n(\varepsilon,K)$ the minimum cardinal among all (n,ε)-spanning subsets of K and by $s_n(\varepsilon,K)$ the maximum cardinal among all (n,ε)-separated subsets of K. Then*

(**a**) $d_f(s,X) = \sup\limits_K \lim\limits_{\varepsilon\to 0} \lim\limits_{n\to+\infty} \sup \dfrac{1}{n^s} \log r_n(\varepsilon,K)$;

(**b**) $d_f(s,X) = \lim\limits_{\varepsilon\to 0} \lim\limits_{n\to+\infty} \sup \dfrac{1}{n^s} \log r_n(\varepsilon,X)$;

(**c**) $d_f(s,X) = \sup\limits_K \lim\limits_{\varepsilon\to 0} \lim\limits_{n\to+\infty} \sup \dfrac{1}{n^s} \log s_n(\varepsilon,K)$;

(**d**) $d_f(s,X) = \lim\limits_{\varepsilon\to 0} \lim\limits_{n\to+\infty} \sup \dfrac{1}{n^s} \log s_n(\varepsilon,X)$.

Proof: This is immediate from

Lemma. ([W])

(**1**) *If α is an open covering of X with Lebesgue number δ, then*

$$N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \le r_n\Big(\frac{\delta}{2},X\Big) \le s_n\Big(\frac{\delta}{2},X\Big) \ ;$$

(**2**) *Given $\varepsilon>0$ and an open covering α of X with diameter less or equal to ε, then*

$$r_n(\varepsilon,X) \le s_n(\varepsilon,X) \le N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \ ;$$

(**3**) *If α_ε is an open covering of X made up by balls of radius ε, then*

$$N\Big(\bigvee_0^{n-1} f^{-i}\alpha_\varepsilon\Big) \le r_n(\varepsilon,X) \le s_n(\varepsilon,X) \le N\Big(\bigvee_0^{n-1} f^{-i}\alpha_{\frac{\varepsilon}{2}}\Big) \ . \ \blacksquare$$

5 – Main results

The above examples suggest that the entropy dimension is a device shaped to distinguish zero topological entropy systems. The next theorem states more precisely that the whole interval $]0,1[$ is assigned by d_f to these systems, being $d_f(X)$ an unnecessary label where the topological entropy already provides a good catalogue.

Theorem 1.

(**a**) $h_{\mathrm{top}}(f) < +\infty \Rightarrow d_f(X) \leq 1$.

(**b**) $h_{\mathrm{top}}(f) = +\infty \Rightarrow d_f(X) \geq 1$.

(**c**) $0 < h_{\mathrm{top}}(f) < +\infty \Rightarrow d_f(X) = 1$.

Proof:

(**a**) If $h_{\mathrm{top}}(f) < +\infty$, then

$$\lim_{n\to+\infty} \sup \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = \lim_{n\to+\infty} \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) < +\infty \ ,$$

therefore, for all s bigger than 1 and all covering α, we have

$$\begin{aligned}\lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) &= \lim_{n\to+\infty} \sup \frac{1}{n^{s-1}} \cdot \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \\ &= \lim_{n\to+\infty} \frac{1}{n^{s-1}} \lim_{n\to+\infty} \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \\ &= 0.h_{\mathrm{top}}(f) = 0 \ .\end{aligned}$$

So $d_f(X) \leq 1$.

(**b**) If $h_{\mathrm{top}}(f) = +\infty$, for each s less than 1, we have

$$\lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = \lim_{n\to+\infty} \sup n^{s-1} \cdot \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) = +\infty$$

and so $d_f(X) \geq 1$. Equivalently, $d_f(X) < 1$ yields $h_{\mathrm{top}}(f) < +\infty$.

(**c**) Under the hypothesis $h_{\mathrm{top}}(f) < +\infty$, we get from (a) that $d_f(X) \leq 1$. As $h_{\mathrm{top}}(f) > 0$ it is at s equal to 1 that the map $s \mapsto d_f(s, X)$ changes its value:

$$\begin{aligned} &d_f(s,X) = +\infty \quad \text{ if } \ s < 1 \ ; \\ &d_f(1,X) = h_{\mathrm{top}}(f) \ ; \\ &d_f(s,X) = 0 \quad \text{ if } \ s > 1 \ . \end{aligned}$$

This means that $d_f(X) = 1$. ∎

Theorem 2. *If f is a continuous endomorphism of a compact set X, then* $d_f(X) \leq 1$.

Proof: Let α be an open covering of X, s be a real number greater than one and $a_n(\alpha)$ denote the sequence $(\log N(\bigvee_0^{n-1} f^{-i}\alpha))_n$. By Proposition 1, for all n and k in $\mathbb{N}$,

$$a_{n+k}(\alpha) \leq a_n(\alpha) + a_k(\alpha)$$

and therefore, if n is written as $n = p\,k + r$, p a fixed integer and r the remainder from the integer division by p $(0 \leq r < p)$, we have

$$0 \leq \frac{a_n(\alpha)}{n^s} = \frac{a_{r+kp}(\alpha)}{(r+kp)^s} \leq \frac{a_r(\alpha)}{(r+kp)^s} + \frac{a_{kp}(\alpha)}{(r+kp)^s} \leq \frac{a_r(\alpha)}{(r+kp)^s} + \frac{k\,a_p(\alpha)}{(r+kp)^s} \ .$$

Therefore

$$0 \leq \frac{a_r(\alpha)}{n^s} \leq \frac{\max\Big\{a_i(\alpha) \mid i \in \{0,...,p\}\Big\}}{n^s}$$

and, as n approaches $+\infty$, the sequence $(\frac{a_r(\alpha)}{n^s})$ converges towards zero. Besides, since

$$\frac{k\,a_p(\alpha)}{(r+k\,p)^s} = \frac{a_p(\alpha)}{k^{s-1}(\frac{r}{k}+p)^s}$$

and, as n goes to $+\infty$, k also approaches $+\infty$ and $(p+\frac{r}{k})^s$ converges to p^s, we get

$$\lim_{k\to+\infty} \frac{k\,a_p(\alpha)}{(r+k\,p)^s} = 0 \ ,$$

$$\lim_{k\to+\infty} \frac{1}{k^{s-1}} = 0$$

and

$$\lim_{n\to+\infty} \frac{a_n(\alpha)}{n^s} = 0 \ .$$

Therefore $d_f(X) \leq 1$. In particular, since this limit exists, we may replace, in Definition 1, $d_f(X)$ by

$$\inf\Big\{s > 0\colon \sup_\alpha \lim_{n\to+\infty} \frac{1}{n^s}\, a_n(\alpha) = 0\Big\}_{|\alpha \text{ an open finite covering of } X} \ .$$

If $s = 1$, the inequality

$$\frac{a_n(\alpha)}{n} \leq \frac{a_r(\alpha)}{n} + \frac{a_p(\alpha)}{p+\frac{r}{k}}$$

yields

$$\lim_{n\to+\infty} \sup \frac{a_n(\alpha)}{n} \leq \inf_p \frac{a_p(\alpha)}{p} \leq \lim_{n\to+\infty} \inf \frac{a_n(\alpha)}{n}$$

and

$$\lim_{n\to+\infty} \frac{a_n(\alpha)}{n} = \inf_n \frac{a_n(\alpha)}{n} \ . \ \blacksquare$$

Question: Does $d_f(X) = 1$ imply that $0 < d_f(1, X) = h_{\mathrm{top}}(f)$?

This question has a known answer if we are considering the Hausdorff dimension: if an invariant subset of a manifold X has maximum Hausdorff dimension (whose value equals the topological dimension of X), then the Lebesgue measure of Y must be positive. See [H] for details. Notice however that there are functions $L(n)$ such that

$$\lim_{n\to+\infty} \frac{L(n)}{n} = 0$$

and

$$\forall\, 0 < s < 1 \quad \lim_{n\to+\infty} \frac{L(n)}{n^s} = +\infty \ .$$

For instance, if

$$L(n) = \int_2^n \frac{1}{\log(t)}\, dt \ ,$$

then

$$\lim_{n\to+\infty} \frac{L(n)}{n} = \lim_{n\to+\infty} \frac{1}{\log(n)} = 0$$

but, for all s in $]0, 1[$,

$$\lim_{n\to+\infty} \frac{L(n)}{n^s} = \lim_{n\to+\infty} \frac{1}{s\, n^{s-1} \log(n)} = +\infty \ .$$

Unfortunately, it is less immediate to find a dynamical system f whose sequence $(\log N(\bigvee_0^{n-1} f^{-i}\alpha))_n$ equals $L(n)$.

The next step is to extend to the entropy dimension the basic methods for calculating the topological entropy.

Proposition 7. *If (X, f) has a generator covering α of the entropy, then $d_f(s, X) = d_f(s, \alpha, X)$.*

Proof: We have just to repeat the proof of the analogue property for $s = 1$. See, for instance, [W]. $\blacksquare$

Proposition 8. *$d_f(s, X)$ and $d_f(X)$ are invariant under conjugacy.*

Proof: Let X and Y be compact metric spaces and $f\colon X \to X$, $g\colon Y \to Y$ be continuous dynamical systems supported on them and such that there is a homeomorphism h satisfying $h \circ f = g \circ h$. Take α a finite covering of X. Then $h(\alpha)$ is a finite open covering of Y and

$$N\Big(\bigvee_0^{n-1} g^{-i}\, h(\alpha)\Big) = N\Big(\bigvee_0^{n-1} h\, f^{-i}\alpha\Big) = N\Big(h\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)\Big) \leq N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$

since a subcovering of $\bigvee_0^{n-1} f^{-i}\alpha$ is taken by h bijectively onto a subcovering of $\bigvee_0^{n-1} g^{-i}\, h(\alpha)$. Therefore

$$\lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} g^{-i}\, h(\alpha)\Big) \leq \lim_{n\to+\infty} \sup \frac{1}{n^s} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$

and

$$d_g(s,Y) \leq d_f(s,X) \quad \text{for all positive } s \ .$$

Taking the least upper bound over all coverings α and into account that $(h(\alpha))_\alpha$ ranges among all covers of Y, we obtain $d_g(Y) \leq d_f(X)$. Analogously, using h^{-1}, we conclude that $d_g(s,Y) \geq d_f(s,X)$ and $d_g(Y) \geq d_f(X)$. ∎

Proposition 9 ([N], [Ym]). *$d_f(X)$ is upper semicontinuous within families of C^∞ endomorphisms of a compact smooth manifold X. That is, if g converges to f in the C^∞ topology, then*

$$\limsup d_g(X) \leq d_f(X) \ .$$

Proof: This is an interesting property the topological entropy shares. Since the main component in the definition of $d_f(X)$ that depends on f is given by

$$\frac{1}{n^{s-1}}\, \frac{1}{n} \log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$

and its second factor (the unique involved in the entropy) varies nicely with f among C^∞ families, according to [N] and [Ym], the same may be expected for $d_f(X)$. And in fact, the estimates and the arguments of [N] and [Ym] may be pursued in our context with the extra exponent s. ∎

Proposition 10.

(**a**) $\forall\, s > 0$, $\forall\, m \in \mathbb{N}$, $d_{f^m}(s,X) \leq m^s\, d_f(s,X)$;

(**b**) $\forall\, 0 < s \leq 1$, $\forall\, m \in \mathbb{N}$, $d_{f^m}(s,X) \leq m\, d_f(s,X)$;

(**c**) $\forall\, m \in \mathbb{N}$, $d_{f^m}(X) = d_f(X)$.

Proof:

(**a**) Given $s>0$ and $m\in\mathbb{N}$,

$$\frac{1}{n^s}\log r_n(\varepsilon,K,f^m)\le\frac{1}{n^s}\log r_{nm}(\varepsilon,K,f)=\frac{m^s}{(n\,m)^s}\log r_{nm}(\varepsilon,K,f)\ .$$

Hence $d_{f^m}(s,X)\le m^s\,d_f(s,X)$.

(**b**) This is straightforward from the inequality

$$m^s\le m\quad\ \forall\,0<s\le 1\ \ \forall\,m\in\mathbb{N}\ .$$

(**c**) By Theorem 2, to estimate $d_f(X)$ we only need to consider $s\in]0,1[$; (b) then implies that $d_{f^m}(X)\le d_f(X)$.

Denote by $\mathcal{D}$ any metric inducing the topology in X. Taking into account that, fixing m in $\mathbb{N}$, the powers of f

$$f,\ f^2,\ ...,\ f^m$$

are uniformly continuous on X, given $\varepsilon>0$, there exists $\delta>0$ such that

$$\Big[\mathcal{D}(x,y)<\delta\ \Rightarrow\ \sup_{0\le i\le m-1}\mathcal{D}(f^i(x),f^i(y))<\varepsilon\Big]\ .$$

Then we get $[m\,r(\varepsilon,K,f)\le r(\delta,K,f^m)]$ and, using Proposition 6, we conclude that

$$d_f(X)\le d_{f^m}(X)\ .\ \blacksquare$$

6 – Variational principle

Let us now turn to the probabilistic version of the entropy dimension. Throughout this section, we will keep the notation

$f\colon X\to X$ is a continuous endomorphism of the compact metric space X;

μ is an f-invariant measure.

The next theorems assemble properties of the metric entropy dimension similar to the ones formulated in previous sections for $d_f(X)$.

Theorem 3.

(**a**) $\forall\, f, X, \mu,\ \ d_f(X,\mu) \le 1.$

(**b**) $0 < h_{\rm top}(f) < +\infty \underset{I}{\Rightarrow} \exists\, \mu : \ 0 < h_\mu(f) < +\infty \underset{II}{\Rightarrow} d_f(X,\mu) = 1 \underset{III}{\Rightarrow}$ $d_f(X,\mu) = \sup_\nu d_f(X,\nu)$.

(**c**) $\forall\, s>1, \forall\, \mu, d_f(s,X) \ge d_f(s,X,\mu)$ *and so* $\forall\, s>1, d_f(s,X) \ge \sup_\mu d_f(s,X,\mu)$.

(**d**) $\forall\, 0<s<1,\ d_f(s,X) \le \sup_\mu d_f(s,X,\mu)$.

(**e**) *For* $s=1,\ d_f(1,X) = \sup_\mu d_f(1,X,\mu)$.

(**f**) $\forall\, f\colon\ X \to X,\ d_f(X) \le \sup_\mu d_f(X,\mu)$.

Proof:

(**a**) This is the analogue to Theorem 2, using $a_n(\alpha) = \log H_\mu(\bigvee_0^{n-1} f^{-i}\alpha)$.

(**b**) (**I**) This first implication is a consequence of the Variational Principle, see [W].

(**II**) Since

$$d_f(s,X,\mu) = \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^s} \log H_\mu\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$
$$= \sup_\alpha \lim_{n\to+\infty} \sup \frac{1}{n^{s-1}}\,\frac{1}{n} \log H_\mu\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)\ ,$$

we have, taking into account that $h_{\rm top}(f) > 0$,

$$d_f(s,X,\mu) = \begin{cases} +\infty & \text{if } s<1, \\ 0 & \text{if } s>1\ . \end{cases}$$

Hence $d_f(X,\mu) = 1$.

(**III**) Since, by (a), $d_f(X,\nu) \le 1$ for all f-invariant probability ν, if $d_f(X,\mu)$ is equal to 1, then it has reached the maximum.

(**c**),(**d**) These are analogous to the corresponding result for $s=1$ (the Variational Principle). We have only to check a few estimates and how they change with the intervention of the exponent s, which we summarize as follows:

(**c**) Given a covering ξ of X, there is a refinement α of ξ such that, for each n in $\mathbb{N}$,

$$H_\mu\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \le \log\Big(N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)\, 2^n\Big)$$

and this yields

$$\frac{1}{n^s}\log H_\mu\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \le \frac{n}{n^s}\log 2 + \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big)$$

and so

$$\frac{1}{n^s}\log H_\mu\Big(\bigvee_0^{n-1} f^{-1}\xi\Big) \le \frac{n}{n^s}\log 2 + \frac{1}{n^s}\log N\Big(\bigvee_0^{n-1} f^{-i}\alpha\Big) \ ;$$

since $s > 1$, as n goes to $+\infty$, $\frac{n}{n^s}$ converges towards zero, thus

$$d_f(s, X, \mu) \le d_f(s, X) \qquad \forall \mu \ .$$

This yields

$$\sup_\mu d_f(s, X, \mu) \le d_f(s, X) \ .$$

(**d**) Given $\varepsilon > 0$, the argument follows by exhibiting an f-invariant probability μ which is an accumulation point of iterates by f of Dirac measures μ_n supported on (n, ε)-separated sets, satisfying

[**1**] $\forall$ covering ξ $\quad \frac{n}{q^s}\log s_q(\varepsilon, X) \le H_{\mu_q}\Big(\bigvee_0^{n-1} f^{-i}\xi\Big) + \frac{2n^2}{q^s}\log(\text{cardinal of } \xi)$;

[**2**] $\quad n \lim_{q\to+\infty} \frac{1}{q^s}\log s_q(\varepsilon, X) \le H_\mu\Big(\bigvee_0^{n-1} f^{-i}\xi\Big)$.

Therefore

[**3**] $\quad \lim_{q\to+\infty} \frac{1}{q^s}\log s_q(\varepsilon, X) \le \frac{1}{n} H_\mu\Big(\bigvee_0^{n-1} f^{-i}\xi\Big) \le \frac{1}{n^s} H_\mu\Big(\bigvee_0^{n-1} f^{-i}\xi\Big)$

since

$$0 < s < 1 \ \Rightarrow\ n^s \le n \ \Rightarrow\ \frac{1}{n^s} \ge \frac{1}{n} \ .$$

Hence, letting n go to $+\infty$, we obtain

$$d_f(s, X) \le d_f(s, X, \mu) \le \sup_\mu d_f(s, X, \mu) \ .$$

(**e**) This is precisely the contents of the Variational Principle.

(**f**) If $\sup_\mu d_f(s_0, X, \mu)$ vanishes for some s_0 in $]0, 1]$, then $d_f(s_0, X, \mu) = 0$ for all μ; besides, (d) and (e) above yield

$$d_f(s_0, X) = 0 \ .$$

Therefore

$$d_f(X) \leq s_0$$

and so

$$d_f(X) \leq \inf\Big\{s > 0\colon \sup_\mu d_f(s,X,\mu) = 0\Big\}\ .$$

Notice that s bigger than one is irrelevant for $d_f(X)$ and was already discarded. The missing step is the prove that $\inf\{s > 0 : \sup_\mu d_f(s,X,\mu) = 0\} = \sup_\mu\{\inf\{s > 0\colon d_f(s,X,\mu) = 0\}\}$. This is the contents of coming Lemma.

Lemma. *Let S and $\widetilde{S}$ be defined as*

$$S = \inf\Big\{s > 0\colon \sup_\mu d_f(s,X,\mu) = 0\Big\}$$

and

$$\widetilde{S} = \sup_\mu\Big\{\inf\{s > 0\colon\ d_f(s,X,\mu) = 0\}\Big\}\ .$$

Then $S = \widetilde{S}$.

Proof:

I. $S \geq \widetilde{S}$.

If S were less than $\widetilde{S}$, we might take t in $]S,\widetilde{S}[$ and then, as $t > S$,

$$\sup_\mu d_f(t,X,\mu) = 0$$

or, equivalently, for all μ, $d_f(t,X,\mu) = 0$. But since $t < \widetilde{S}$, there would be an f-invariant measure μ_t such that

$$\Big\{\inf\{s > 0\colon\ d_f(s,X,\mu_t) = 0\}\Big\} > t$$

which implies that $d_f(t,X,\mu_t) \neq 0$ and this is a contradiction.

II. $S \leq \widetilde{S}$.

Assume S is bigger than $\widetilde{S}$. As, for all μ,

$$\widetilde{S} \geq \inf\Big\{s > 0\colon\ d_f(s,X,\mu) = 0\Big\}$$

we have, for instance,

$$d_f\left(\widetilde{S} + \frac{S - \widetilde{S}}{3}, X, \mu\right) = 0 \quad \forall\,\mu\ .$$

This implies that

$$[\mathbf{1}] \qquad \sup_\mu d_f\Big(\widetilde{S}+\frac{S-\widetilde{S}}{3},X,\mu\Big)=0$$

and

$$[\mathbf{2}] \qquad S>\inf\Big\{s>0\colon \sup_\mu d_f(s,X,\mu)=0\Big\}$$

which is not consistent with the definition of S since $\widetilde{S}+\frac{S-\widetilde{S}}{3}$ is smaller than S. ∎

To end the proof of the Theorem, notice that

$$\begin{aligned} d_f(X) &\leq \inf\Big\{s>0\colon \sup_\mu d_f(s,X,\mu)=0\Big\} \\ &= \sup_\mu\Big\{\inf\{s>0\colon d_f(s,X,\mu)=0\}\Big\} \\ &= \sup_\mu d_f(X,\mu) \ . \blacksquare \end{aligned}$$

Question: If $0<h_{\mathrm{top}}(f)<+\infty$, then $d_f(X)=1=\sup_\mu d_f(X,\mu)$. Is $d_f(X)=\sup_\mu d_f(X,\mu)$ always valid?

REFERENCES

[**H**] Hurewicz, Wallman – *Dimension theory*, Princeton Univ. Press, 1948.

[**K**] Katok – Monotone equivalence in ergodic theory, *Math. USSR Izvestija,* 1 (1977), 99–146.

[**N**] Newhouse – Entropy and volume, *Ergod. Th. & Dynam. Sys.,* 8 (1988), 283–299.

[**Sch**] Schmidt – *The cohomology of expansive $\mathbb{Z}^d$-actions*, preprint.

[**Ym**] Yomdin – Volume growth and entropy, *Israel J. Math.,* 57(3) (1987), 285–317.

[**W**] Walters – *An introduction to Ergodic Theory*, Springer Verlag, 1982.

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