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ADMISSIBLE FUNCTIONS IN TWO-SCALE CONVERGENCE

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Abstract: The two-scale convergence method was developed by G. Nguetseng, W. E and G. Allaire in connection with homogenization problems. We introduce a rather large class of "admissible" functions containing Carathéodory integrands of both kinds (continuous on Ω or on Y) and prove a useful continuity result (Proposition 5). Through all the paper we use the notion, introduced by W. E, of "two-scale" Young measures. This also provides natural proofs of some quantitative results of Huyghens type.

1 – Introduction

Recall that Young measures were introduced by L.C. Young [Y] in 1937 in order to retain something of the behavior of the gradients of some minimizing sequences in the Calculus of Variations when the gradients oscillate. After and among several others, L. Tartar [T1–2] and E.J. Balder [Bd1–3] bring some more applications and developments (see more references in my courses [V1–2]). The "two-scale" convergence method was developed in papers by G. Nguetseng [N] in 1989, W. E [E] and G. Allaire [A1] in 1992 (see also [A2]) in connection with applications to homogenization problems. Some ideas go back to Bensoussan–Lions–Papanicolaou [BLP]. We will emphasize the connection with Young measures which is shortly mentioned (in one page) by W. E [E, p.312] and develop a new notion of "admissible" functions.

Starting from a sequence $(u_n)_n$ of functions on an open subset Ω of $\mathbb{R}^N$ satisfying $\sup_n \|u_n\|_{L^2(\Omega)} < +\infty$, the two-scale convergence method permits to obtain

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a function $\widehat{u}$ on(1) $\Omega \times Y$ such that, for any sufficiently "smooth" test function ψ,

$$\text{(Ng)} \qquad \int_\Omega u_\varepsilon(x)\,\psi(x,nx)\,dx \;\to\; \iint_{\Omega\times Y} \widehat{u}(x,y)\,\psi(x,y)\,dx\,dy$$

as $n \to +\infty$ (for a subsequence). Secondarily $x \mapsto \widehat{u}(x,nx)$ is as close to u_n as possible. When $u_n(x) := \overline{u}(x,nx)$ and under some mild assumptions on $\overline{u}$, $\widehat{u}$ coincides with $\overline{u}$ (see Theorem 6).

We use, adding some details, the notion introduced by W. E [E] of a "two-scale" Young measure(2) $(\nu_{(x,y)})_{(x,y)\in\Omega\times Y}$. Then $\widehat{u}(x,y)$ is the mean (or barycenter) of $\nu_{(x,y)}$ (formula (12) below) and the weak limit $u_\infty(x)$ of the sequence under consideration is the mean over Y of $\widehat{u}(x,\cdot)$. Our contributions lie in the introduction of a rather large class of "admissible" functions, in a continuity result (Proposition 5) and in the use of the two-scale Young measures to obtain with easy proofs some quantitative results of Huyghens type which are more or less explicit in [N], [E], [A1] (Proposition 9).

A stochastic result of Bourgeat–Mikelic–Wright [BMW] can be proved in the same way: see [V3, Section 7], where the ideas are due to Gérard Michaille.

2 – Classical Young measures

Let Ω be a bounded open subset of $\mathbb{R}^N$. The Lebesgue measure on Ω is denoted by dx. The Lebesgue measure of a measurable set A is denoted by $|A|$, its characteristic function by 1_A. Let T denote a separable metrizable locally compact space. In the sequel T will be $Y \times \mathbb{R}$ where Y is $[0,1[^N$ with an ad hoc compact topology (it is the N-dimensional torus).

For Young measures we will refer to some fundamemental papers of E.J. Balder [Bd1–3] ([Bd1] appeared in 1984) and to the courses I gave in Italy [V1–2].

To any measurable function $v\colon \Omega \to T$ is associated its *Young measure*, the measure on $\Omega \times T$ which is the image of dx by $x \mapsto (x, v(x))$. A general *Young measure* is a positive measure ν on $\Omega \times T$ such that $\forall A \in \mathcal{B}(\Omega)$, $\nu(A \times T) = |A|$ (this means that the projection of ν on Ω is dx). The set of all Young measures on $\Omega \times T$ will be denoted by $\mathcal{Y}(\Omega, dx; T)$. The *disintegration* of ν is a measurable family $(\nu_x)_{x\in\Omega}$ of probabilities on T such that, for any function $\psi\colon \Omega \times T \to \overline{\mathbb{R}}$

(1) In the literature $\widehat{u}$ is denoted by u_0. We denote by Y the unit cell or N dimensional torus.

(2) From a footnote of [E, p.312], this terminology is due to F. Murat.

which is positive measurable or ν-integrable,

$$\int_{\Omega\times T}\psi\,d\nu=\int_{\Omega}\Big[\int_T \psi(x,t)\,d\nu_x(t)\Big]\,dx\ .$$

One can write

$$\nu=\int_{\Omega}\delta_x\otimes\nu_x\,dx\ .$$

A *Carathéodory integrand* is a real function on $\Omega\times T$ which is separately measurable on Ω and continuous on T. The set of all bounded Carathéodory integrand is denoted by $\mathcal{C}th^b(\Omega;T)$. The *narrow topology* on $\mathcal{Y}(\Omega,dx;T)$ is the weakest one making continuous the maps $\nu\mapsto\int_{\Omega\times T}\psi\,d\nu$ where ψ runs through $\mathcal{C}th^b(\Omega;T)$. It is worthwhile to note that this narrow topology coincides with the weak topology defined by test functions belonging to(3) $\mathcal{C}_c(\Omega\times T)$ (see [Bd2], [V2, Th.3]), hence also with the classical narrow topology defined by test functions belonging to $\mathcal{C}^b(\Omega\times T)$. The proof in [V2, p.362] shows that $\mathcal{C}_c(\Omega)\otimes\mathcal{C}_c(T)$ still works; see below in Proposition 2 an application.

3 – Two-scale Young measures

A sequence $(u_\varepsilon)_\varepsilon$ in $L^2_{\mathbb{R}}(\Omega,dx)$ is a family $(u_\varepsilon)_{\varepsilon\in S}$ where S is a subset of $]0,+\infty[$ with $0\in\overline{S}$. In all this paper the sequence $(u_\varepsilon)_\varepsilon$ is assumed to be bounded in $L^2_{\mathbb{R}}(\Omega,dx)$ (that is $\sup_\varepsilon\|u_\varepsilon\|_2<+\infty$). We are interested by the behavior as $\varepsilon\to0$. A subsequence, still denoted $(u_\varepsilon)_\varepsilon$, is a family $(u_\varepsilon)_{\varepsilon\in S'}$ where $S'\subset S$ and $0\in\overline{S'}$. In examples we will use sequences $(u_n)_{n\in\mathbb{N}^*}$. This obviously corresponds to $S=\{1/n\colon n\in\mathbb{N}^*\}$ and u_n in place of $u_{1/n}$ ($=u_\varepsilon$).

Let $Y=[0,1[^N$ be the N-dimensional unit cube (in homogenization Y or the open cube $]0,1[^N$ is called the *unit cell*(4)) and $\mathbb{T}^N$ the N-dimensional torus, that is the compact topological space quotient of $\mathbb{R}^N$ by its subgroup $\mathbb{Z}^N$. The Lebesgue measure on Y is denoted by dy.

The compact topology of $\mathbb{T}^N$ on $Y=[0,1[^N$ is very useful(5). Firstly, in order to apply general results of Young measure theory, we need the inf-compactness over $Y\times\mathbb{R}^N$ of $(y,\lambda)\mapsto\lambda^2$. Secondly, any continuous function w on Y extends continuously to $\mathbb{R}^N$ in a Y-periodic function (or $\mathbb{Z}^N$-periodic). Then w denotes as well a function on Y, as its Y-periodic extension to $\mathbb{R}^N$. And, for $x\in\mathbb{R}^N$

(3) The subscript c means "with compact supports".

(4) For the measures we will consider, $[0,1[^N\setminus]0,1[^N$ will always be negligible, but see the arguments below...

(5) For $N=1$, any point has the same neighborhoods as in the usual topology induced by $\mathbb{R}$ except for 0: a basis of neighborhoods of 0 is formed by the sets $[0,\delta[\,\cup\,]1-\delta,1[$ ($\delta>0$).

and $\varepsilon > 0$, the element $\mathrm{frac}(\frac{x}{\varepsilon}) \in Y$ (frac is the canonical map from $\mathbb{R}^N$ onto $\mathbb{T}^N$) obtained by taking coordinates modulo 1 of $\frac{x}{\varepsilon} \in \mathbb{R}^N$, will be simply denoted by $\frac{x}{\varepsilon}$ as soon as it is clear that it is an element of Y and we will write $w(\frac{x}{\varepsilon})$. Finally $\mathcal{C}(Y)$ denotes the Banach space of real continuous functions on Y, Y being equipped with its compact topology.

For any sequence of functions $u_\varepsilon : \Omega \to \mathbb{R}$ we consider the Young measures ν_ε associated to the maps $x \mapsto (\frac{x}{\varepsilon}, u_\varepsilon(x))$ from Ω to $Y \times \mathbb{R}$, that is the measures on $\Omega \times Y \times \mathbb{R}$ which are the images of dx by $x \mapsto (x, \frac{x}{\varepsilon}, u_\varepsilon(x))$. We call ν_ε the *two-scale Young measure associated to* u_ε (note that it depends on the value ε of the index). The set of all these Young measures is *tight* in $\mathcal{Y}(\Omega, dx; Y \times \mathbb{R})$ (for this notion see [Bd1], [Bd3], [V1, Prop.8]). Indeed $(y, \lambda) \mapsto |\lambda|^2$ is inf-compact over $Y \times \mathbb{R}$ and

$$\int_{\Omega\times Y\times\mathbb{R}} |\lambda|^2 \, d\nu_\varepsilon(x,y,\lambda) = \int_\Omega |u_\varepsilon(x)|^2 \, dx \le \sup_\gamma (\|u_\gamma\|_2)^2 < +\infty \ .$$

Thus there exists a subsequence, still denoted $(\nu_\varepsilon)_\varepsilon$, convergent to a Young measure ν on $\Omega \times (Y \times \mathbb{R})$. An integrand $\Psi \colon \Omega \times Y \times \mathbb{R} \to \mathbb{R}$ has *linear growth* if there exists $C \in [0, +\infty[$ such that $\forall (x, y, \lambda) \in \Omega \times Y \times \mathbb{R}$, $|\Psi(x, y, \lambda)| \le C(1 + |\lambda|)$. We will denote the set of all Carathéodory integrand on $\Omega \times (Y \times \mathbb{R})$ with linear growth by $\mathcal{C}th^1(\Omega; Y \times \mathbb{R})$. When the functions $x \mapsto \Psi(x, \frac{x}{\varepsilon}, u_\varepsilon(x))$ are uniformly integrable (thanks to the hypothesis $\sup \|u_\varepsilon\|_2 < +\infty$, this holds if $\Psi \in \mathcal{C}th^1(\Omega; Y \times \mathbb{R})$), one has (cf. [Bd1], [Bd3], [V, Th.17], [V2, Th.6]):

$$(1) \qquad \int_\Omega \Psi\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big) \, dx = \int_{\Omega\times Y\times\mathbb{R}} \Psi \, d\nu_\varepsilon \ \to \ \int_{\Omega\times Y\times\mathbb{R}} \Psi \, d\nu \ .$$

It is important to extend (1) to some integrands Ψ which loose some continuity on Y. This will be achieved in Proposition 5.

With classical Young measure, without the factor Y, we would introduce the disintegration on $\mathbb{R}$ with respect to Ω as first factor. Here we disintegrate taking as first factor $\Omega \times Y$. This is possible because the projection of ν on $\Omega \times Y$ is known. It is the limit of the projection θ_ε of ν_ε, that is of the image of dx by $x \mapsto (x, \frac{x}{\varepsilon})$. But these measures converge to $dx \otimes dy$ (see Proposition 2 below). The disintegration is denoted by $(\nu_{(x,y)})_{(x,y)\in\Omega\times Y}$. Thus, for any ad hoc integrand Ψ,

$$(2) \qquad \int_\Omega \Psi\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big) \, dx \ \to \ \int_{\Omega\times Y} \Big[\int_{\mathbb{R}} \Psi(x, y, \lambda) \, d\nu_{(x,y)}(\lambda)\Big] \, dx \, dy \ .$$

Another property of convergence in Young measure Theory is: if a measurable integrand $\Psi \colon \Omega \times Y \times \mathbb{R} \to [0, +\infty]$ is l.s.c. in (y, λ), then

$$(3) \qquad \int_{\Omega\times Y\times\mathbb{R}} \Psi \, d\nu \le \liminf_{\varepsilon\to 0} \int_{\Omega\times Y\times\mathbb{R}} \Psi \, d\nu_\varepsilon \ .$$

4 – Generation of weakly convergent sequences. Admissible functions

The following result is basic in homogenization and has several consequences.

Proposition 1. *Let $\psi \in L^p(Y)$ ($p \in [1,+\infty]$), $\varepsilon \in]0,\varepsilon_0]$ where $0 < \varepsilon_0 < +\infty$. Then the sequence $(\psi(\frac{\cdot}{\varepsilon}))_\varepsilon$ is bounded in $L^p(\Omega)$ and $\psi(\frac{\cdot}{\varepsilon})$ converges weakly in L^p (that is for $\sigma(L^p(\Omega), L^q(\Omega))$ — weakly* if $p = \infty$), as $\varepsilon \to 0$, to $(\int_Y \psi(y)\,dy)\,1_\Omega$.* ∎

Remark. For the boundedness property the condition $\varepsilon \leq \varepsilon_0$ cannot, for $p < +\infty$, be removed: for $N = 1$ and $\psi(y) = y^{-1/2}$ on $]0,1[$, if $\Omega =]0,1[$ and $\varepsilon \geq 1$, $\|(\psi(\frac{\cdot}{\varepsilon}))\|_1 = 2\sqrt{\varepsilon} \to +\infty$ as $\varepsilon \to +\infty$.

References. This is clearly stated (and proved) in P. Suquet's thesis [Su, Lemme 3, p.308]. See also Ball–Murat [BM, Lemma A.1, p.249], Dacorogna [Da, Th. 1.5, p.21].

Comments.

1) This gives a process to generate weakly convergent (usually not strongly convergent) sequences. First let $w \in L^2(Y)$. If w is not constant, $u_n(x) := w(nx)$ weakly converges to the mean of w without converging strongly.

2) Then we turn to less elementary operations. We begin by the simplest procedures, in accordance to the idea that among functions of two variables the simplest ones are those where the variables are independent. An amplitude modulation in x is possible: let $v \in L^2(\Omega)$ and $u_n(x) := v(x)\,w(nx)$. Now, if $p \in L^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega p(x)\,u_n(x)\,dx &= \int_\Omega \Big[p(x)\,v(x)\Big]\,w(nx)\,dx \\ &\to \int_\Omega \Big[p(x)\,v(x)\Big]\Big(\int_Y w(y)\,dy\Big)dx \\ &= \Big\langle p, \Big(\int_Y w(y)\,dy\Big)\,v\Big\rangle\,.\end{aligned}$$

Then one can takes k couples (v_i, w_i) and set $u_n(x) := \sum_{i=1}^k v_i(x)\,w_i(nx)$.

3) Now the problem is: what are the measurable functions $\overline{u}$ on $\Omega \times Y$ such that, if one sets $u_n(x) := \overline{u}(x, nx)$, the following weak convergence

$$u_n = \overline{u}(\cdot, n) \rightharpoonup u_\infty := \Big[x \mapsto \int_Y \overline{u}(x,y)\,dy\Big]$$

still holds? Surely there is a difficulty with taking $\overline{u}$ in $L^2(\Omega \times Y)$: the values of the u_n depends only on the values of $\overline{u}$ on $\Delta := \bigcup_{n\in\mathbb{N}^*} \mathrm{gr}(x \mapsto \mathrm{frac}(nx))$ which is negligible in $\Omega \times Y$. So appear the need for a good subclass of $L^2(\Omega \times Y)$: see Proposition 3 below.

Proposition 2. *The measure θ_ε on $\Omega \times Y$, which is the image of dx by $x \mapsto (x, \frac{x}{\varepsilon})$, converges narrowly to $dx \otimes dy$.*

Proof: By standard facts on narrow convergence (see for example [V2, proof of Theorem 3, p.362]) it is sufficient to show that, for any ψ, $\psi(x,y) = \psi_1(x)\,\psi_2(y)$ where $\psi_1 \in \mathcal{C}_c(\Omega)$ and $\psi_2 \in \mathcal{C}(Y)$, one has

$$\int_{\Omega\times Y} \psi\, d\theta_\varepsilon \;\to\; \int_{\Omega\times Y} \psi\, dx\, dy\ .$$

This holds thanks to Proposition 1:

$$\int_{\Omega\times Y} \psi\, d\theta_\varepsilon = \int_\Omega \psi_1(x)\,\psi_2\Big(\frac{x}{\varepsilon}\Big)\, dx \;\to\; \Big[\int_\Omega \psi_1(x)\, dx\Big]\Big[\int_Y \psi_2(y)\, dy\Big]\ . \blacksquare$$

Now we introduce a class of functions which have nice properties. The property below is inspired by Boccardo–Buttazzo [BB, (2.3), p.23].

Definition. A Borel function $\overline{u}\colon\ \Omega \times Y \to \mathbb{R}$ is *admissible* or belongs to **Adm** if: $\forall\,\delta > 0$, there exists a compact set $Q_\delta \subset \Omega$ and a compact set $K_\delta \subset Y$ satisfying $|\Omega\backslash Q_\delta| \le \delta$, $|Y\backslash K_\delta| \le \delta$ and $\overline{u}|_{Q_\delta\times K_\delta}$ is continuous.

Remark. By the Scorza Dragoni theorem [SD], [ET, VIII.1.3, p.218], this definition is equivalent to: $\forall\,\delta > 0$, there exists a Borel set $\Omega_\delta \subset \Omega$ and a compact set $K_\delta \subset Y$ satisfying $|\Omega\backslash\Omega_\delta| \le \delta$, $|Y\backslash K_\delta| \le \delta$ and $\forall\, x \in \Omega_\delta$, $\overline{u}(x,\cdot)$ is continuous on K_δ. Note that any measurable function $\overline{u}\colon\ \Omega \times Y \to \mathbb{R}$ satisfies the weaker Lusin property: $\forall\,\delta > 0$, there exists a compact subset H of $\Omega \times Y$ such that $|(\Omega \times Y)\backslash H| < \delta$ and $\overline{u}|_H$ is continuous.

Examples. The function $\overline{u}$ is admissible in the following cases:

1) $\overline{u}$ is Carathéodory continuous on Y. Indeed by the Scorza Dragoni theorem, there exists a compact Q_δ satisfying $|\Omega\backslash Q_\delta| \le \delta$ and $\overline{u}|_{Q_\delta\times Y}$ is continuous.

2) $\overline{u}$ is Carathéodory continuous on Ω (still Scorza Dragoni).

3) $\overline{u}$ has the form $\overline{u}(x,y) = v(x)\,w(y)$ where v and w are Borel functions respectively defined on Ω and Y (this follows from the Lusin property applied to w. One can choose $\Omega_\delta = \Omega$). This still holds for a finite sum: $\overline{u}(x,y) = \sum_{i=1}^k v_i(x)\,w_i(y)$.

Remark. Some properties of Carathéodory integrands go back to 1972 or earlier (see Warga's book, specially [W, Th. I.5.25, p.135]): there is a one-to-one correspondence between the set of all Carathéodory integrands $\overline{u}$ such that $[x \mapsto \|\overline{u}(x,\cdot)\|_\infty$ is integrable$]$ and $\mathcal{L}^1(\Omega, dx; \mathcal{C}(Y))$. If $U \in \mathcal{L}^1(\Omega, dx; \mathcal{C}(Y))$, $\overline{u}(x,y) := [U(x)](y)$ defines a Carathéodory integrand.

Notation. Let $\mathbf{Adm}^2(\Omega, dx; Y)$ denote the set of admissible functions $\overline{u}$ satisfying $\exists\, \alpha \in L^2_+(\Omega)$ such that $\forall\, (x,y) \in \Omega \times Y$, $|\overline{u}(x,y)| \le \alpha(x)$.

Thanks to the following, one difficulty when working with $L^2(\Omega \times Y, dx \otimes dy)$, which was noticed before Proposition 2, disappears with admissible functions.

Proposition 3. *If $\overline{u}$ and $\widetilde{u}$ are admissible and $\overline{u}(x,y) = \widetilde{u}(x,y)$ $dx \otimes dy$-a.e., there exist negligible sets $N \subset \Omega$ and $Z \subset Y$ such that $\overline{u}$ and $\widetilde{u}$ coincide on $(\Omega \backslash N) \times (Y \backslash Z)$. Then for any $\varepsilon > 0$, $\overline{u}(x, \frac{x}{\varepsilon}) = \widetilde{u}(x, \frac{x}{\varepsilon})$ dx-a.e.*

When $Z = \emptyset$, this property is called by probabilists (see Dellacherie–Meyer [DM, ch. IV, Definition 7, p.136]) *undistinguishability*.

Proof: **1**) Let, for $\delta > 0$, Q_δ and K_δ be compact sets satisfying the definition of admissibility for $\overline{u}$ and for $\widetilde{u}$. Denoting $Q^n = \bigcap_{m \ge n} Q_{2^{-m}}$ and $K^n = \bigcap_{m \ge n} K_{2^{-m}}$ one gets increasing sequences $(Q^n)_n$ and $(K^n)_n$ such that $N_{-1} := \Omega \backslash (\bigcup_n Q^n)$ and $Z := Y \backslash (\bigcup_n K^n)$ are negligible. Replacing if necessary K^n by the support of the measure $1_{K^n}\, dy$, one may assume that any dy-negligible open subset of K^n is empty. For any n, $\{(x,y) \in Q^n \times K^n \colon \overline{u}(x,y) \ne \widetilde{u}(x,y)\}$ has open cuts and, thanks to Fubini's theorem, they are negligible for dx-almost all x hence empty. Hence there exists a negligible set $N_n \subset Q^n$ such that $\overline{u}(x,\cdot)|_{K^n}$ and $\widetilde{u}(x,\cdot)|_{K^n}$ coincide if $x \notin Q^n \backslash N_n$. It remains to set $N = \bigcup_{n=-1}^{\infty} N_n$.

2) The set $\{x \in \Omega \colon \frac{x}{\varepsilon} \in Z\}$ is dx-negligible. The last assertion follows from the inclusion $\{x \in \Omega \colon \overline{u}(x, \frac{x}{\varepsilon}) \ne \widetilde{u}(x, \frac{x}{\varepsilon})\} \subset N \cup \{x \in \Omega \colon \frac{x}{\varepsilon} \in Z\}$. ∎

5 – Nguetseng's result

The following result is the basic idea of the two-scale convergence method. We follow the proof of G. Allaire, but taking a smaller class of test functions, we get a slightly shorter proof.

Theorem 4 (Nguetseng, 1989). *Let $(u_\varepsilon)_\varepsilon$ be a bounded sequence in $L^2(\Omega)$. There exist a subsequence and a function $\widehat{u} \in L^2(\Omega \times Y)$ such that for any $\psi \in \mathcal{C}_c(\Omega \times Y)$,*

$$\text{(Ng)} \qquad \ell_\varepsilon(\psi) := \int_\Omega u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \;\to\; \int\!\!\int_{\Omega \times Y} \widehat{u}(x,y)\, \psi(x,y)\, dx\, dy\ .$$

Remark. It is interesting to get (maybe for a further subsequence) the above convergence for a larger class than $\mathcal{C}_c(\Omega \times Y)$. We will get it for admissible functions in Theorem 8 formula (11). Note that in [A1, Remark 1.11, pp.1489–1490] G. Allaire obtains (Ng) for the class he defines.

Proof: The ℓ_ε are linear forms on $\mathcal{C}_c(\Omega \times Y)$. Set $C = \sup_\varepsilon \|u_\varepsilon\|_2$. By Cauchy–Schwartz $|\ell_\varepsilon(\psi)| \le C\|\psi\|_\infty\, |\Omega|^{1/2}$. Thus ℓ_ε is a Radon–Bourbaki measure belonging to $\mathcal{M}^b(\Omega \times Y) = (\mathcal{C}_c(\Omega \times Y), \|\cdot\|_\infty)'$ and the sequence is bounded. There exists a weak* convergent subsequence $\ell_\epsilon \to \ell$. But ℓ is continuous with respect to the $L^2(\Omega \times Y)$-norm, denoted $\|\cdot\|_2$ (despite the fact that the ℓ_ϵ are not $\|\cdot\|_2$-continuous: they are singular measures). Indeed

$$|\ell(\psi)| \le C \limsup_{\varepsilon \to 0} \Big(\int_\Omega \psi\Big(x, \frac{x}{\varepsilon}\Big)^2 dx\Big)^{1/2}\ .$$

Since ψ belongs to $\mathcal{C}_c(\Omega \times Y)$, the right-hand member converges to $C\|\psi\|_{L^2(\Omega \times Y)}$ (invoke for example Proposition 2). So the measure ℓ is defined by a density $\widehat{u} \in L^2(\Omega \times Y)$. ∎

6 – A continuity result

We recall that the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $L^2_{\mathbb{R}}(\Omega, dx)$ and that $(\nu_\varepsilon)_\varepsilon$ denotes a narrow convergent subsequence with limit ν. The following result will be useful in cases when $(\beta, p) = (0,1)$, $(\beta, p) = (1,2)$ and $(\beta, p) = (2, +\infty)$ ($a(x,y)$ being constant when $\beta = 2$) which cover the case when Ψ is a trinomial in λ with suitable coefficients.

Proposition 5. *Let $\beta \in [0,2]$ and $p := \frac{2}{2-\beta}$. Let Ψ denote the integrand on $\Omega \times Y \times \mathbb{R}$, $\Psi(x,y,\lambda) := a(x,y)\, \lambda^\beta$, where $a \in$ **Adm** and $\exists\, \alpha \in L^p_+(\Omega)$, such that $\forall\, (x,y)$, $|a(x,y)| \le \alpha(x)$. Suppose that, when $\beta = 2$, the $|u_\varepsilon|^2$ are uniformly integrable. Then*

$$\text{(4)} \qquad \int_{\Omega \times Y \times \mathbb{R}} \Psi\, d\nu_\varepsilon \;\to\; \int_{\Omega \times Y \times \mathbb{R}} \Psi\, d\nu\ .$$

Proof: Let $\delta > 0$ and Q_δ and K_δ be given by $a \in \mathbf{Adm}$ and such that $\alpha|_{Q_\delta}$ is continuous, hence bounded by $M < +\infty$. By Tietze–Urysohn's theorem, $a|_{Q_\delta \times K_\delta}$ has a continuous extension $\overline{a}$ to $Q_\delta \times Y$ with $\forall\,(x,y) \in Q_\delta \times Y$, $|\overline{a}(x,y)| \leq M$. For $x \in Q_\delta$, set $\overline{\Psi}(x,y,\lambda) = \overline{a}(x,y)\,\lambda^\beta$. The functions $\overline{\Psi}(x, \frac{x}{\varepsilon}, u_\varepsilon(x))$ are uniformly integrable on Q_δ because $|\overline{\Psi}(x, \frac{x}{\varepsilon}, u_\varepsilon(x))| \leq M|u_\varepsilon(x)|^\beta$ and, if $\beta < 2$, UI follows from Hölder's inequality (or from La Vallée Poussin criterion), if $\beta = 2$, UI has been assumed in the statement. Moreover, for $x \in Q_\delta$, $(y,\lambda) \to \overline{\Psi}(x,y,\lambda)$ is continuous. Thus, by (1)

$$\int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu_\varepsilon = \int_{Q_\delta} \overline{\Psi}\Big(x, \frac{x}{\varepsilon}, u_\varepsilon(x)\Big)\, dx \;\to\; \int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu\ .$$

Now

$$\begin{aligned}
&(5)\quad \Big|\int_{\Omega \times Y \times \mathbb{R}} \Psi\, d\nu_\varepsilon - \int_{\Omega \times Y \times \mathbb{R}} \Psi\, d\nu\Big| \leq \Big|\int_{(\Omega \setminus Q_\delta) \times Y \times \mathbb{R}} \Psi\, d\nu_\varepsilon\Big| \\
&(6)\qquad\qquad + \Big|\int_{Q_\delta \times Y \times \mathbb{R}} \Psi\, d\nu_\varepsilon - \int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu_\varepsilon\Big| \\
&\qquad\qquad\quad + \Big|\int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu_\varepsilon - \int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu\Big| \\
&(7)\qquad\qquad + \Big|\int_{Q_\delta \times Y \times \mathbb{R}} \overline{\Psi}\, d\nu - \int_{Q_\delta \times Y \times \mathbb{R}} \Psi\, d\nu\Big| \\
&(8)\qquad\qquad + \Big|\int_{(\Omega \setminus Q_\delta) \times Y \times \mathbb{R}} \Psi\, d\nu\Big|\ .
\end{aligned}$$

Let us check that the first two and the last two terms are "small". Observe that, using (3), we get

$$\begin{aligned}
\int_\Omega \alpha(x) \Big[\int_Y \Big[\int_{\mathbb{R}} |\lambda|^\beta\, d\nu_{(x,y)}(\lambda)\Big]\, dy\Big] dx &= \int_{\Omega \times Y \times \mathbb{R}} \alpha(x)\, |\lambda|^\beta\, d\nu(x,y,\lambda) \\
&\leq \liminf_{\varepsilon \to 0} \int_{\Omega \times Y \times \mathbb{R}} \alpha(x)\, |\lambda|^\beta\, d\nu_\varepsilon(x,y,\lambda) \\
&= \liminf_{\varepsilon \to 0} \int_\Omega \alpha(x)\, |u_\varepsilon(x)|^\beta\, dx \\
&\leq \sup_\gamma \Big\| |u_\gamma|^\beta \Big\|_{L^{\frac{2}{\beta}}(\Omega)} \Big\|\alpha\Big\|_{L^{\frac{2}{2-\beta}}(\Omega)} \\
&= \Big(\sup_\gamma \|u_\gamma\|_{L^2(\Omega)}\Big)^\beta\, \|\alpha\|_{L^p(\Omega)} \\
&< +\infty\ .
\end{aligned}$$

So firstly, for the term (8),

$$\Big|\int_{(\Omega \setminus Q_\delta) \times Y \times \mathbb{R}} \Psi\, d\nu\Big| \leq \int_{\Omega \setminus Q_\delta} \alpha(x) \Big[\int_Y \Big[\int_{\mathbb{R}} |\lambda|^\beta\, d\nu_{(x,y)}(\lambda)\Big]\, dy\Big]\, dx \xrightarrow{\text{as } \delta \to 0} 0\ .$$

The same calculus with $\alpha \equiv 1$ gives $\int_{\Omega\times Y\times\mathbb{R}} |\lambda|^\beta \, d\nu(x,y,\lambda) < +\infty$ hence, for (7),

$$\begin{aligned}\Big|\int_{Q_\delta\times Y\times\mathbb{R}} \overline{\Psi}\, d\nu - \int_{Q_\delta\times Y\times\mathbb{R}} \Psi\, d\nu\Big| &\le \int_{Q_\delta\times(Y\setminus K_\delta)\times\mathbb{R}} 2M\,|\lambda|^\beta\, d\nu(x,y,\lambda)\\ &\le 2M \int_{Y\setminus K_\delta}\Big[\int_\Omega\Big[\int_{\mathbb{R}} |\lambda|^\beta\, d\nu_{(x,y)}(\lambda)\Big]\,dx\Big]\,dy\\ &\xrightarrow{\text{as } \delta\to 0} 0\ .\end{aligned}$$

Thirdly, for (5)

$$\begin{aligned}\Big|\int_{(\Omega\setminus Q_\delta)\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon\Big| &\le \int_{\Omega\setminus Q_\delta} \alpha(x)\,|u_\varepsilon(x)|^\beta\,dx\\ &\le \sup_\gamma \Big\| 1_{\Omega\setminus Q_\delta}\,|u_\gamma|^\beta\Big\|_{L^{\frac{2}{\beta}}(\Omega)} \Big\|1_{\Omega\setminus Q_\delta}\,\alpha\Big\|_{L^{\frac{2}{2-\beta}}(\Omega)}\\ &= \Big(\sup_\gamma \Big\|1_{\Omega\setminus Q_\delta}\,u_\gamma\Big\|_2\Big)^\beta \Big\|1_{\Omega\setminus Q_\delta}\,\alpha\Big\|_{L^p(\Omega)}\ .\end{aligned}$$

The last line tends to 0 as $\delta\to 0$ because, if $\beta < 2$, p is $< +\infty$ and the second factor tends to 0 and, if $\beta = 2$, the first factor tends to 0 since the $|u_\varepsilon|^2$ are uniformly integrable.

Finally for (6) assume that, for $x\in Q_\delta$, $a(x,\cdot)|_{K_\delta}$ has been extended to Y using Borsuk's theorem(6) [Bo], [Dg, Th. 5.1, p.360], so that $\sup_y |\overline{a}(x,y)|$ remains $\le \alpha(x)$. Then

$$\begin{aligned}\Big|\int_{Q_\delta\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon - \int_{Q_\delta\times Y\times\mathbb{R}} \overline{\Psi}\, d\nu_\varepsilon\Big| &\le \int_{Q_\delta\times(Y\setminus K_\delta)\times\mathbb{R}} 2\,\alpha(x)\,|\lambda|^\beta\, d\nu_\varepsilon(x,y,\lambda)\\ &= 2\int_{Q_\delta} \alpha(x)\,|u_\varepsilon(x)|^\beta\, 1_{Y\setminus K_\delta}\Big(\frac{x}{\varepsilon}\Big)\,dx\ .\end{aligned}$$

We distinguish $\beta < 2$ and $\beta = 2$. In the first case we continue the majoration:

$$\begin{aligned}&\le 2\sup_\gamma \Big\||u_\gamma|^\beta\Big\|_{L^{\frac{2}{\beta}}(\Omega)} \Big\|\alpha(\cdot)\,1_{Y\setminus K_\delta}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^{\frac{2}{2-\beta}}(\Omega)}\\ &= 2\Big(\sup_\gamma \|u_\gamma\|_2\Big)^\beta \Big\|\alpha(\cdot)\,1_{Y\setminus K_\delta}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^p(\Omega)}\ .\end{aligned} \tag{9}$$

Let $\eta > 0$. Here we have to explain a bit how to choose firstly δ such that for ε small enough the second term (6) is less than $\eta/5$. Let $C = 2(\sup_\gamma \|u_\gamma\|_2)^\beta$.

(6) This gives a measurable integrand because, for $x\in Q_\delta$, the $\mathcal{C}(Y)$-valued map $x\mapsto \overline{a}(x,\cdot)$ inherits the measurability of $x\mapsto a(x,\cdot)|_{K_\delta}$.

Since p is $< +\infty$, by Proposition 1,

$$\tag{10}\begin{aligned}\Big\|\alpha(\cdot)\,1_{Y\setminus K_\delta}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^p(\Omega)} &= \Big(\int_\Omega \alpha(x)^p\,1_{Y\setminus K_\delta}\Big(\frac{x}{\varepsilon}\Big)\,dx\Big)^{\frac1p}\\ &\xrightarrow{\text{as }\delta\to 0} \Big(|Y\setminus K_\delta|\int_\Omega \alpha(x)^p\,dx\Big)^{\frac1p}\\ &\le \Big(\delta\int_\Omega \alpha(x)^p\,dx\Big)^{\frac1p}\ .\end{aligned}$$

One can choose δ small enough in order that $C(\delta\int_\Omega \alpha(x)^p\,dx)^{\frac1p} \le \eta/10$. Then, thanks to (10), for ε small enough, one gets $\le \eta/5$ in (9).

Now consider the second case when $(\beta,p)=(2,+\infty)$. We continue the majoration:

$$\le 2\int_{\{x\in\Omega:\ \frac{x}{\varepsilon}\notin K_\delta\}} \alpha(t)\,|u_\varepsilon(t)|^2\,dt\ .$$

By Proposition 1, $|\{x\in\Omega:\ \frac{x}{\varepsilon}\notin K_\delta\}| \to |\Omega|\,\delta$. So the result follows from the uniform integrability of the $|u_\varepsilon|^2$ and $\alpha\in L^\infty$. ∎

7 – Some properties of admissible functions

Theorem 6. *Let $\overline{u}\in \mathbf{Adm}^2(\Omega,dx;Y)$(7). Let $u_\varepsilon(x)=\overline{u}(x,\frac{x}{\varepsilon})$ and ν_ε be the two-scale Young measure associated to u_ε. Then ν_ε converges narrowly to the image ν of $dx\otimes dy$ by $(x,y)\mapsto(x,y,\overline{u}(x,y))$ which writes $\nu=\int_{\Omega\times Y}\delta_{(x,y,\overline{u}(x,y))}\,dx\,dy$ (or $\nu_{(x,y)}=\delta_{\overline{u}(x,y)}$). Moreover u_ε converges weakly in $L^2(\Omega)$ to u_∞ where $u_\infty(x)=\int_Y \overline{u}(x,y)\,dy$.*

Remark. So, in this case, the function $\widehat{u}$ defined later (formula (12)) as the barycenter of $\nu_{(x,y)}$ coincides with $\overline{u}$. Specially we get the periodic case $u_n(x)=u_1(nx)$ when $\overline{u}$ does not depend on x. Then the classical Young measure theory cannot distinguish some different behaviors. For example, if $N\ge 2$, the sequences $u_n(x)=\sin(nx_1)$ and $u'_n(x)=\sin(nx_2)$ give the same classical Young measure(8). For example, if $N=1$ and $u_1(x)=x$ on $Y=[0,1[$ and $u'_1(x)=2|x-1/2|$ on Y, the limit Young measure ν_x is the same: the Lebesgue measure on $[0,1]$. In these examples the classical Young measures cannot recover the original pattern or the directions of oscillations, but $\widehat{u}$ recovers exactly the generator function of the sequence.

(7) With a less short proof, one can treat the case when $\overline{u}\in\mathbf{Adm}$ and $\exists\,\beta\in\mathcal{L}^2_+(Y)$ such that $\forall\,(x,y)$, $|\overline{u}(x,y)|\le\beta(y)$.

(8) Its disintegration ν_x $(x\in\Omega)$ is (see for $N=1$, [V2, Th. 4]) the probability on $\mathbb{R}$ with density $\lambda\mapsto(\pi\sqrt{1-\lambda^2})^{-1}$ on $]-1,1[$.

Now we must say that a different notion was introduced by L. Tartar in [T3]: the H-measures. They also permit to distinguish different oscillation directions.

Proof: 1) Since the set $\mathcal{H}$ of all measures ν_ε is tight, it is relatively compact in $\mathcal{Y}(\Omega, dx; Y \times \mathbb{R})$. So it suffices to prove that $(\nu_\varepsilon)_\varepsilon$ has ν as a unique limit point. Moreover, since the space of continuous functions with compact supports $\mathcal{C}_c(\Omega \times Y \times \mathbb{R})$ is separable, $\overline{\mathcal{H}}$ is metrizable for the narrow topology of Young measures.

Suppose that $\varepsilon_k \to 0$ and that $\nu_k := \nu_{\varepsilon_k}$ converges narrowly to τ. We are going to prove that τ is carried by $\mathrm{gr}(\overline{u})$. Since the projection of τ on $\Omega \times Y$ is $dx \otimes dy$ (cf. Prop. 2), this will prove $\tau_{(x,y)} = \delta_{\overline{u}(x,y)} = \nu_{(x,y)}$ hence $\tau = \nu$. Set $\Psi(x,y,\lambda) = |\lambda - \overline{u}(x,y)|^2 = \lambda^2 - 2\,\overline{u}(x,y)\,\lambda + |\overline{u}(x,y)|^2$. Each term satisfies the hypotheses of Proposition 5 (the $|u_\varepsilon|^2$ are UI since $|u_\varepsilon(x)| \le \alpha(x)$ with $\alpha \in L^2(\Omega)$). Hence

$$\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon \;\to\; \int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu\ .$$

But $\forall\,\varepsilon > 0$, $\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon = \int_\Omega \Psi(x, \frac{x}{\varepsilon}, \overline{u}(x, \frac{x}{\varepsilon}))\, dx = 0$. This implies $\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu = 0$, hence $\Psi(x,y,\lambda) = 0$ ν-a.e. This means that ν is carried by $\{(x,y,\lambda)\colon |\lambda - \overline{u}(x,y)|^2 = 0\}$, that is by $\mathrm{gr}(\overline{u})$.

2) Note that $\overline{u} \in L^2(\Omega \times Y) \subset L^1(\Omega \times Y)$ implies that u_∞ is well defined and that $u_\infty \in L^2(\Omega)$:

$$\int_\Omega |u_\infty(x)|^2\, dx = \int_\Omega \Big|\int_Y \overline{u}(x,y)\, dy\Big|^2\, dx \le \int_{\Omega\times Y} \Big|\overline{u}(x,y)\Big|^2\, dx\, dy < +\infty\ .$$

Let $p \in L^2(\Omega)$ and $\Psi(x,y,\lambda) := p(x)\,\lambda$. By Proposition 5

$$\begin{aligned}\int_\Omega p(x)\, u_\varepsilon(x)\, dx &= \int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon \\ &\to \int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu \\ &= \int_{\Omega\times Y} \Psi\Big(x, y, \overline{u}(x,y)\Big)\, dx\, dy \\ &= \int_{\Omega\times Y} p(x)\, \overline{u}(x,y)\, dx\, dy \\ &= \int_\Omega p(x)\, u_\infty(x)\, dx\ . \blacksquare\end{aligned}$$

Proposition 7. *If $\overline{u} \in \mathbf{Adm}^2(\Omega, dx; Y)$, for $p = 1$ and $p = 2$,*

$$(*) \qquad \int_\Omega \left|\overline{u}\left(x, \frac{x}{\varepsilon}\right)\right|^p dx \;\to\; \int_{\Omega\times Y} \left|\overline{u}(x,y)\right|^p dx\, dy \;.$$

Comments. G. Allaire [A1, (1.3)] calls admissible a function $\overline{u} \in L^2(\Omega \times Y)$ satisfying this property for $p = 2$. When $\overline{u}(x,y) := v(x)\, w(y)$ ($v \in L^2(\Omega)$, $w \in L^2(Y)$) and $p = 1$, $(*)$ is a consequence of Proposition 1. Indeed, with the scalar product of $L^2(\Omega)$,

$$\int_\Omega \left|\overline{u}\left(x, \frac{x}{\varepsilon}\right)\right| dx = \left\langle |v|, \left|w\left(\frac{\cdot}{\varepsilon}\right)\right|\right\rangle \;\to\; \left\langle |v|, \left(\int_Y |w(y)|\, dy\right) 1_\Omega \right\rangle = \int_{\Omega\times Y} |\overline{u}(x,y)|\, dx\, dy.$$

Proof: We use the functions $u_\varepsilon(x) := \overline{u}(x, \frac{x}{\varepsilon})$ and the measures ν_ε and ν of Theorem 6. Let $\Psi(x,y,\lambda) = |\lambda|^p$. Then

$$\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon = \int_\Omega \left|\overline{u}\left(x, \frac{x}{\varepsilon}\right)\right|^p dx$$

and

$$\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu = \int_{\Omega\times Y\times\mathbb{R}} |\lambda|^p\, d\nu(x,y,\lambda) = \int_{\Omega\times Y} |\overline{u}(x,y)|^p\, dx\, dy \;.$$

The result follows from (4). For $p = 2$ the uniform integrability of the $|u_\varepsilon|^2$ comes from $|u_\varepsilon(x)|^2 \le \alpha(x)^2$. ∎

8 – Properties of the limit measure

We recall once more that the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $L^2_{\mathbb{R}}(\Omega, dx)$ and that $(\nu_\varepsilon)_\varepsilon$ denotes a narrow convergent subsequence with limit ν.

Theorem 8. *For the subsequence under consideration,*

1) *the disintegration of the limit ν is for $dx \otimes dy$-almost every (x,y) of first order, that is*

$$\int_{\mathbb{R}} |\lambda|\, d\nu_{(x,y)}(\lambda) < +\infty \;.$$

Setting $\widehat{u}(x,y) = \int_{\mathbb{R}} \lambda\, d\nu_{(x,y)}(\lambda)$, one has $\widehat{u} \in L^2(\Omega \times Y)$. Moreover

$$(11) \qquad \forall\, \psi \in \mathbf{Adm}^2(\Omega, dx; Y), \; \int_\Omega u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx \;\to\; \int_{\Omega\times Y} \widehat{u}(x,y)\, \psi(x,y)\, dx\, dy \;.$$

2) *$u_\infty(x) = \int_Y \widehat{u}(x,y)\,dy$ defines a function belonging to $L^2(\Omega)$ and u_ε converges weakly to u_∞ in L^2.*

Remark. The convergence (11) extends the simplified version of the two-scale convergence method we gave in Section 5. It partially recovers Remark 1.11 of [A1].

Proof: **1**) With the choice $\Psi(x,y,\lambda) = |\lambda|$, (3) gives:

$$\int_{\Omega\times Y} \Big[\int_{\mathbb{R}} |\lambda|\, d\nu_{(x,y)}(\lambda)\Big]\, dx\, dy \leq \sup_\varepsilon \|u_\varepsilon\|_1 < +\infty$$

hence

$$\int_{\mathbb{R}} |\lambda|\, d\nu_{(x,y)}(\lambda) < +\infty \qquad dx \otimes dy\text{-a.e.}$$

So we can set

$$\widehat{u}(x,y) := \int_{\mathbb{R}} \lambda\, d\nu_{(x,y)}(\lambda)\ . \tag{12}$$

We check $\widehat{u} \in L^2(\Omega \times Y)$. We use again (3):

$$\begin{aligned} \int_{\Omega\times Y} |\widehat{u}(x,y)|^2\, dx\, dy &= \int_{\Omega\times Y} \Big|\int_{\mathbb{R}} \lambda\, d\nu_{(x,y)}(\lambda)\Big|^2\, dx\, dy \\ &\leq \int_{\Omega\times Y\times\mathbb{R}} |\lambda|^2\, d\nu(x,y,\lambda) \\ &\leq \liminf \int_{\Omega\times Y\times\mathbb{R}} |\lambda|^2\, d\nu_\varepsilon(x,y,\lambda) \\ &= \liminf \int_\Omega |u_\varepsilon(x)|^2\, dx < +\infty\ . \end{aligned}$$

Thus $\widehat{u} \in L^2(\Omega \times Y)$. Proposition 5 applies to $\Psi(x,y,\lambda) := \psi(x,y)\,\lambda$, and (4) gives

$$\begin{aligned} \int_\Omega u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \ \to\ &\int_{\Omega\times Y} \Big[\int_{\mathbb{R}} \lambda\, d\nu_{(x,y)}(\lambda)\Big]\, \psi(x,y)\, dx\, dy \\ =\ &\int_{\Omega\times Y} \widehat{u}(x,y)\, \psi(x,y)\, dx\, dy\ . \end{aligned}$$

2) As in part 2) of the proof of Theorem 6, $u_\infty \in L^2(\Omega)$ and the weak convergence $u_\varepsilon \to u_\infty$ follows from (4) applied to $\Psi(x,y,\lambda) := p(x)\lambda$ ($p \in L^2(\Omega)$). ∎

Remark. If we set

$$\sigma_x := \int_Y \nu_{(x,y)}\, dy\ , \tag{13}$$

the choice $\Psi(x,y,\lambda):=\Phi(x,\lambda)$ where $\Phi\in\mathcal{C}th^b(\Omega;\mathbb{R})$ and (2) give

$$\begin{aligned}\int_\Omega \Phi(x,u_\varepsilon(x))\,dx \;&\to\; \int_{\Omega\times Y}\Big[\int_{\mathbb{R}}\Phi(x,\lambda)\,d\nu_{(x,y)}(\lambda)\Big]\,dx\,dy\\ &=\int_\Omega\Big[\int_{\mathbb{R}}\Phi(x,\lambda)\,d\sigma_x(\lambda)\Big]\,dx\ .\end{aligned}$$

This proves that, as noticed by W. E [E, (4.6)], the subsequence $(u_\varepsilon)_\varepsilon$ (more precisely the sequence of classical Young measures on $\Omega\times\mathbb{R}$ associated to the functions u_ε) converges to the Young measure on $\Omega\times\mathbb{R}$, σ, whose disintegration is defined in (13).

9 – Huyghens type results

Since, for any $(x,y)\in\Omega\times Y$, $\widehat{u}(x,y)$ is the mean of $\nu_{(x,y)}$, Huyghens' theorem says:

$$\forall r\in\mathbb{R},\ \int_{\mathbb{R}}|\lambda-r|^2\,d\nu_{(x,y)}(\lambda)=\int_{\mathbb{R}}|\lambda-\widehat{u}(x,y)|^2\,d\nu_{(x,y)}(\lambda)+|\widehat{u}(x,y)-r|^2. \tag{14}$$

Proposition 9. *Suppose that the $|u_\varepsilon|^2$ are uniformly integrable and that $\widehat{u}$ given by Theorem 8 belongs to $\mathbf{Adm}^2(\Omega,dx;Y)$. The subsequence under consideration satisfies:*

1) *For any $\widetilde{u}\in\mathbf{Adm}^2(\Omega,dx;Y)$,*

$$\lim_{\varepsilon\to0}\Big\|u_\varepsilon-\widetilde{u}\Big(\cdot,\frac{\cdot}{\varepsilon}\Big)\Big\|_2^2=\int_{\Omega\times Y}\Big[\int_{\mathbb{R}}|\lambda-\widetilde{u}(x,y)|^2\,d\nu_{(x,y)}(\lambda)\Big]\,dx\,dy \tag{15}$$

and the minimum is attained when $\widetilde{u}=\widehat{u}$.

$$\textbf{2})\ \lim_{\varepsilon\to0}\Big\|\widehat{u}\Big(\cdot,\frac{\cdot}{\varepsilon}\Big)-u_\infty\Big\|_2^2=\int_{\Omega\times Y}\Big|\widehat{u}(x,y)-u_\infty(x)\Big|^2\,dx\,dy\ . \tag{16}$$

$$\textbf{3})\ \lim_{\varepsilon\to0}\Big\|u_\varepsilon-u_\infty\Big\|_2^2=\lim_{\varepsilon\to0}\Big\|u_\varepsilon-\widehat{u}\Big(\cdot,\frac{\cdot}{\varepsilon}\Big)\Big\|_2^2+\lim_{\varepsilon\to0}\Big\|\widehat{u}\Big(\cdot,\frac{\cdot}{\varepsilon}\Big)-u_\infty\Big\|_2^2\ . \tag{17}$$

Remarks. Suppose that the $|u_\varepsilon|^2$ are uniformly integrable and let $v\in L^2(\Omega)$. Taking $\widetilde{u}(x,y)=v(x)$ in (15), one gets, with σ defined by (13),

$$\begin{aligned}\lim_{\varepsilon\to0}\|u_\varepsilon-v\|_2^2&=\int_{\Omega\times Y}\Big[\int_{\mathbb{R}}|\lambda-v(x)|^2\,d\nu_{(x,y)}(\lambda)\Big]\,dx\,dy\\ &=\int_{\Omega\times\mathbb{R}}|\lambda-v(x)|^2\,d\sigma(x,\lambda)\ .\end{aligned} \tag{18}$$

1) An analogous formula for the L^1-norm and classical Young measures already exists [V2, Th. 9]. Other quantitative results in L^1 combining Young measures and the biting lemma are given in Saadoune–Valadier [SV].

2) For $\Omega =]0,1[$ and $u_n = n^{1/2}\,1_{]0,\frac{1}{n}[}$, $\nu_{(x,y)} = \delta_0$, $\widehat{u} = 0$ and $u_\infty = 0$. Formula (18) with $v = u_\infty$ does not hold: the left member equals 1, the second 0. The uniform integrability of the $|u_\varepsilon|^2$ is lacking.

Proof: **1**) Let us choose $\Psi(x,y,\lambda) = |\lambda - \widetilde{u}(x,y)|^2$. Proposition 5 applies:

$$\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu = \lim_{\varepsilon\to 0} \int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon\ .$$

So (15) follows from

$$\int_{\Omega\times Y\times\mathbb{R}} \Psi\, d\nu_\varepsilon = \int_\Omega \left| u_\varepsilon(x) - \widetilde{u}\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx\ .$$

Finally by Huyghens' theorem, $\int_{\mathbb{R}} |\lambda - \widetilde{u}(x,y)|^2\, d\nu_{(x,y)}(\lambda)$ is minimal when $\widetilde{u}(x,y)$ is chosen to be $\widehat{u}(x,y)$.

2) The measure θ_ε on $\Omega\times Y$ image of dx by $x \mapsto (x, \frac{x}{\varepsilon})$ converges to $dx\otimes dy$ (cf. Prop. 2). Set $\psi(x,y) = |\widehat{u}(x,y) - u_\infty(x)|^2$. The expected result follows from the uniform integrability of the functions $|\widehat{u}(\cdot, \frac{\cdot}{\varepsilon}) - u_\infty(\cdot)|^2$ and from

$$\int_{\Omega\times Y} \psi\, d\theta_\varepsilon = \int_\Omega \left|\widehat{u}\left(x, \frac{x}{\varepsilon}\right) - u_\infty(x)\right|^2 dx\ .$$

3) Formula (17) follows easily from (18) (with $v = u_\infty$), (14) (with $r = u_\infty(x)$), (15) and (16). ∎

Comment. One has 0 in (15) (with $\widetilde{u} = \widehat{u}$) if and only if $\nu_{(x,y)}$ is the Dirac mass $\delta_{\widehat{u}(x,y)}$. Getting 0 in (15) means that the two-scale method recovers quite well the (sub)sequence. When a strictly positive value is obtained, some oscillations escape to this analysis.

The strong convergence $u_\varepsilon \to u_\infty$ of the subsequence under consideration holds if and only if the right members of (15) (with $\widetilde{u} = \widehat{u}$) and of (16) are 0. Each one can be 0 independently. For example, if $u_n(x) = \sin(nx_1)$, (15) is 0 and (16) is not. But, if $u_n(x) = \sin(n^2 x_1)$, $\widehat{u}(x,y) \equiv 0$, hence (16) is 0 but (15) is not.

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