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EXISTENCE OF PERIODIC ORBITS, SET OF GLOBAL SOLUTIONS AND BEHAVIOR NEAR EQUILIBRIUM FOR VOLTERRA EQUATIONS OF RETARDED TYPE

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Ao nosso querido amigo Nelson Onuchic pelo exemplo de vida

Abstract: A "cut-off" technique for retarded functional differential equations is applied to Volterra models of retarded type, combined with a Kurzweil's theorem, in order to describe global bounded solutions as solutions of an ordinary differential equation. The existence of periodic orbits and a description of the behavior near the equilibrium are also obtained.

1 – Introduction

In his famous book *Leçons sur la théorie mathématique de la lutte pour la vie* [6] Volterra considered a system of retarded functional differential equations (RFDE) in order to describe the hereditary interaction of n species; hereditary here means, for instance, time of incubation or time of gestation of the female predator. When two individuals meet and one eats the other, the population of preys decreases immediately; on the other hand the population of predators takes a while to increase. This delay is interpreted as a constant lag $r > 0$ that appears in the system of RFDE for n species (see equations (4.1)) or the special case of two species (equations (4.14)). In this last case, if we make $\delta_1 = 0$ and $\delta_2 > 0$ one intends to give a meaning for the instantaneous effect of predation on the population of preys and the time lag affects only the predators' population. Of course $\delta_1 > 0$ and $\delta_2 > 0$ also has to be considered; these two cases were studied with some detail along the paper.

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In Section 2 we gave emphasis to a smooth "cut-off" technique for RFDE's. In Section 3, for completeness, we stated Theorem 3.1, a theorem by Kurzweil that describes, under certain hypotheses, the global solutions of a RFDE as solutions of an ODE. In Section 4 we combined the appropriated smooth "cut-off" technique to adapt the Volterra system (4.1) to the hypotheses of the Kurzweil's result and obtained Theorem 4.1; the RFDE to be considered corresponds to a small perturbation of an ODE system. The use of Theorem 4.1 depends strongly on the existence of a compact set Γ, invariant under the flow of the unperturbed ODE system. Theorem 4.2 is applied to the case of two species and proves the existence of r-periodic orbits for system (4.14), under suitable general conditions on the parameters of the system. Remarks 4.3, 4.4 as well as Theorem 4.5 are corollaries of Theorem 4.2. Remark 4.6 treats system (4.14) for the case $\delta_1 = 0$, $\delta_2 > 0$ and Theorems 4.7, 4.8 show that the behavior of the flow can be better understood looking to the other possible periodic orbits with minimum period r/k, $k > 1$ integer. We finish Section 4 with an example of a system with four species obtained as the product of two systems with two species; this procedure leads to a RFDE on a torus T^2.

Section 5 studies the behavior of the solutions in a neighbohood of the equilibrium in the case of two species. Theorem 5.1 shows that the equilibrium is generically hyperbolic, Corollary 5.3 and Theorem 5.4 study the special cases $\delta_1 = 0$, $\delta_2 > 0$ and $\delta_1 = \delta_2 = \delta > 0$, respectively. Corollary 5.5 shows that Hopf bifurcation occurs in this last case.

2 – "cut-off" functions

Let B_1 and B_2 be the closures of two bounded convex open sets in $\mathbb{R}^n$ with smooth boundaries, $\partial B_1 \subset \operatorname{int}(B_2)$. It is well known that there exists a C^∞ function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\Phi(x) = \begin{cases} 1 & \text{on } B_1, \\ 0 & \text{on } \mathbb{R}^n \backslash B_2\,, \end{cases} \qquad 0 \le \Phi(x) \le 1, \text{ otherwise.} \tag{2.1}$$

We easily construct Φ if B_1 and B_2 are choosen as closed balls centered at the origin, with radii r_1 and r_2, respectively, $r_1 < r_2$. From now on B_1 and B_2 will be such balls. One starts with a C^∞ function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(s) = 0$, for $s \ge r_2 - r_1$, $\alpha(s) = 1$, for $s \le 0$ and $0 \le \alpha(s) \le 1$, for $0 < s < r_2 - r_1$; then we define

$$\Phi(x) = \alpha\Big(|x| - r_1\Big), \qquad \text{for all } \; x \in \mathbb{R}^n \;. \tag{2.2}$$

The smoothness of Φ follows from the smoothness of $|x|$ for $x \neq 0$. This kind of "cut-off" function is oftenly used in order to properly modify vector-fields defined in finite dimensional manifolds.

Except for the Lipschitz case, we cannot extend the procedure above when B_1 and B_2 are subsets of a general Banach manifold $\mathcal{B}$ because, besides the fact that Φ is globally Lipschitz, its smoothness depends on the differentiability of the norm.

For our purpose, $\mathcal{B}$ will be the Banach space $C = C^0(I, \mathbb{R}^n)$, $I = [-r, 0]$, $r > 0$, of all continuous functions from I to $\mathbb{R}^n$, with the sup norm, viewed as the phase space of a RFDE in $\mathbb{R}^n$. Since the sup norm is not differentiable, we have to change the way of construction of smooth "cut-off" functions.

Let be given

$$F\colon C \to \mathbb{R}^n \tag{2.3}$$

a smooth (at least C^2) RFDE on $\mathbb{R}^n$ (see [2] or [3], for standard definitions, notations and fundamental results).

Let $\tilde{\Phi}\colon C \to C$ defined by

$$(\tilde{\Phi}(\varphi))(\theta) = (\Phi(\varphi(\theta)))\,\varphi(\theta)\,, \qquad \varphi \in C, \quad \theta \in [-r, 0]\ , \tag{2.4}$$

where $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is the C^∞ function (2.1). Remark that $\tilde{\Phi}$ is C^2 (indeed it is C^∞) and that

$$\tilde{\Phi}(C) \subset C^0(I, B_2)\ .$$

Finally we define the smooth RFDE $h\colon C \to \mathbb{R}^n$ by

$$h = F \circ \tilde{\Phi} \quad \text{that is} \quad h(\varphi) = F(\tilde{\Phi}(\varphi))\,, \qquad \varphi \in C\ . \tag{2.5}$$

The function h is smooth (at least C^2) and, moreover, h and F coincide on $C^0(I, B_1)$.

A proof that $\tilde{\Phi}$ is a C^2 function can be done argüing as in [1], pp. 755-756, using the expressions of the first and second derivatives of $\tilde{\Phi}$:

$$\Big((d\tilde{\Phi}(\varphi))(\psi)\Big)(\theta) = \Big[\Phi'((\varphi(\theta))\,\psi(\theta)\Big]\,\varphi(\theta) + \Big[\Phi(\varphi(\theta))\Big]\,\psi(\theta)\ , \tag{2.6}$$

$$\begin{aligned}\Big(d^2\tilde{\Phi}(\varphi)(\psi, \bar{\psi})\Big)(\theta) = {} & \Big[\Big(\Phi''(\varphi(\theta))\,\bar{\psi}(\theta)\Big)\,\psi(\theta)\Big]\,\varphi(\theta) \\ & + \Big[\Phi'(\varphi(\theta))\,\psi(\theta)\Big]\,\bar{\psi}(\theta) + \Big[\Phi'(\varphi(\theta))\,\bar{\psi}(\theta)\Big]\,\psi(\theta)\ ,\end{aligned} \tag{2.7}$$

where $\theta \in [-r, 0]$, $\varphi, \psi, \bar{\psi} \in C$.

3 – A Kurzweil's theorem

In this section, the Theorem 3.1, due to Kurzweil [5] (see also [3]), presented here for sake of completeness, states that all global solutions (i.e. defined on $\mathbb{R}$) of a RFDE, which is sufficiently close to some ordinary differential equation (ODE), are at the same time solutions of a suitable ODE, and vice versa.

Here Y denotes a Banach space and $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function, $\omega(0) = 0$. Let $K > 0$, $k > 0$. Let $g: Y \to Y$ and $h: C \to Y$, $C = C^0(I, Y)$, fulfill the following conditions

$$|g(x)| \le K \quad \text{for } x \in Y\,, \qquad |h(y)| \le k \quad \text{for } y \in C\,, \tag{3.1}$$

there exist derivatives Dg, Dh and

$$\begin{aligned} &|Dg(u) - Dg(v)| \le \omega(|u - v|) \quad \text{for } u, v \in Y\,, \\ &|Dh(\varphi) - Dh(\psi)| \le \omega(|\varphi - \psi|) \quad \text{for } \varphi, \psi \in C\,. \end{aligned} \tag{3.2}$$

Remark that if h is given by (2.5) with F completely continuous together with its derivatives up to the order two, then one can find $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ as above (indeed linear) such that the second inequality (3.2) holds.

Define $q: Y \to C$ as follows: for $u \in Y$, let z be the solution of

$$\frac{dx}{dt} = g(x) \tag{3.3}$$

fulfilling $z(0) = u$. Put $q(u) = z|_{[-r,0]}$ (z restricted to $[-r, 0]$).

Theorem 3.1. *There exists an $\varepsilon > 0$ depending on K and ω only, such that if $k \le \varepsilon$, then there exists a C^1 map $p: Y \to C$ and the following properties are fulfilled:*

(i) *every solution of*

$$\frac{dx}{dt} = g(x) + h(p(x)) \tag{3.4}$$

is at the same time a solution of

$$\frac{dx}{dt} = g(x(t)) + h(x_t)\,; \tag{3.5}$$

(ii) *every global solution of (3.5) is at the same time a solution of (3.4);*

(iii) *every solution of (3.5) (which is of course defined on some positive half line) approaches exponentially some solution of (3.4) as $t \to \infty$;*

(**iv**) *the C^1-norm of $p-q$ is small (there exists an estimate depending on k, K and ω and tending to zero as $k \to 0$).* ∎

4 – The Volterra equations of retarded type

In [6], chapter IV, Volterra introduced the following system of RFDE's in $\mathbb{R}^n$ to describe the hereditary interaction of n species:

$$\frac{dN_i}{dt} = \left[\varepsilon_i + \sum_{s=1}^{n} p_{is}\, N_s(t) + \int_{-r}^{0} \sum_{s=1}^{n} F_{is}(-\theta)\, N_s(t+\theta)\, d\theta\right] N_i(t), \quad i=1,...,n\ , \tag{4.1}$$

where ε_i, p_{is} are constants depending on the species and the "memory" functions

$$F_{is}\colon\ [0,r] \to \mathbb{R} \tag{4.2}$$

are continuous functions. As an example of memory functions it is usual to consider $F_{is}(\tau) = c_{is} \exp\left(-k_{is}\tau\right)$, where $c_{is} \in \mathbb{R}$, $k_{is} \geq 0$ and $\tau \in [0,r]$. It is not clear if (4.1), with some $F_{is} \not\equiv 0$, may have nontrivial periodic solutions or even nontrivial global solutions. Later on we will analyse a special case that presents nontrivial periodic solutions.

The system (4.1) is defined by the function $G = (G_1, ..., G_n)\colon\ C \to \mathbb{R}^n$ given by

$$G_i(\varphi) = \left[\varepsilon_i + \sum_{s=1}^{n} p_{is}\, \varphi_s(0) + \int_{-r}^{0} \sum_{s=1}^{n} F_{is}(-\theta)\, \varphi_s(\theta)\, d\theta\right] \varphi_i(0), \quad i=1,...,n\ , \tag{4.3}$$

where $\varphi = (\varphi_1, ..., \varphi_n) \in C$.

We will also consider the vector-field X on $\mathbb{R}^n$ defined by the system of ODE's associated to (4.1) (see [6], p. 97)

$$\frac{dN_i}{dt} = \left[\varepsilon_i + \sum_{s=1}^{n} p_{is}\, N_s(t)\right] N_i(t)\,, \quad i = 1, ..., n\ . \tag{4.4}$$

The system (4.4) generalizes the classical prey-predator Lotka–Volterra model (see again [6], p. 14). The autonomous RFDE defined by $G = (G_1, ..., G_n)$ in (4.3) is of class C^∞, so it holds for it the classical results on local existence and uniqueness of solutions of initial condition problems (see [2]). On the other hand, if an initial condition $(\varphi; t_0) \in C \times \mathbb{R}$, where $\varphi = (\varphi_1, ..., \varphi_n)$, is such that $\varphi_i(\theta) > 0$ for all $\theta \in [-r, 0]$, then the solution $N_i(t; \varphi, t_0)$, $i = 1, ..., n$, of initial condition $(\varphi; t_0)$, is defined for all $t \geq t_0$ and, moreover, we have $N_i(t; \varphi, t_0) > 0$, $i = 1, ..., n$, for all $t \geq t_0$. This means that the set

$$C_{++} = \Big\{ \varphi = (\varphi_1, ..., \varphi_n) \in C \mid \varphi_i(\theta) > 0, \ \theta \in [-r, 0], \ i = 1, ..., n \Big\}$$

is positively invariant under the semi-flow of (4.1).

The continuation property and the positive invariance mentioned above were proved by Volterra for two species, but the generalization for $n > 2$ species is quite obvious (see [6], p. 191).

In our setting the RFDE (4.1) can be written as

$$G(\varphi) = X(\varphi(0)) + F(\varphi) \tag{4.5}$$

so $F = (F_1, ..., F_n)\colon C \to \mathbb{R}^n$ is given by

$$F_i(\varphi) = \varphi_i(0) \int_{-r}^{0} \sum_{s=1}^{n} F_{is}(-\theta)\, \varphi_s(\theta)\, d\theta \ . \tag{4.6}$$

It is easy to see that the function F is completely continuous. The "cut-off" procedure of Section 2 will be used for the vector-field X and for the RFDE F in order to adjust them to the result of Kurzweil of Section 3.

So, in the notation of Theorem 3.1, let $Y = \mathbb{R}^n$ and define $g\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon C \to \mathbb{R}$ in the following way:

$$g(x) = \Phi(x)\, X(x) \quad \text{ for } \ x \in \mathbb{R}^n \ , \tag{4.7}$$

$$h(\varphi) = F(\tilde{\Phi}(\varphi)) \quad \text{ for } \ \varphi \in C \ , \tag{4.8}$$

where $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is given by (2.1) and $\tilde{\Phi}\colon C \to C$ is defined in (2.4). Let $K > 0$ be a bound for X on the compact set B_2 and $k > 0$ be a bound of F on $C^0(I, B_2)$; the existence of k follows from the fact that F is completely continuous and k can be made arbitrarily small by taking the memory functions (4.2) sufficiently small. The existence of the function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ and the inequalities (3.2) follow from the fact that the functions F_i in (4.6) are completely continuous together with their derivatives up to the order 2.

Define $q\colon \mathbb{R}^n \to C$ as in Theorem 3.1, that is, for $u \in \mathbb{R}^n$ let z be the solution of

$$\frac{dx}{dt} = \Phi(x)\, X(x)$$

of initial condition $z(0) = u$. Put

$$q(u) = z|_{[-r,0]} \ . \tag{4.9}$$

As an application of Theorem 3.1 we have the following result:

Theorem 4.1. *Let X be the vector-field on $\mathbb{R}^n$ defined by the ODE system (4.4):*

$$\frac{dN_i}{dt} = \Big[\varepsilon_i + \sum_{s=1}^{n} p_{is}\, N_s(t)\Big] N_i(t)\,, \qquad i = 1, ..., n\ ,$$

and let $G\colon C \to \mathbb{R}^n$ be the RFDE defined by the Volterra equations of retarded type (4.1):

$$\frac{dN_i}{dt} = \bigg[\varepsilon_i + \sum_{s=1}^{n} p_{is}\, N_s(t) + \int_{-r}^{0} \sum_{s=1}^{n} F_{is}(-\theta)\, N_s(t+\theta)\, d\theta\bigg] N_i(t), \quad i = 1, ..., n\ ,$$

that is, $G(\varphi) = X(\varphi(0)) + F(\varphi)$ where the function $F = (F_1, ..., F_n)$ is given by $F_i(\varphi) = \varphi_i(0) \int_{-r}^{0} \sum_{s=1}^{n} F_{is}(-\theta)\, \varphi_s(\theta)\, d\theta$, $\varphi = (\varphi_1, ..., \varphi_n) \in C$. Let B_1 and B_2 be the closures of two bounded convex open sets in $\mathbb{R}^n$ with smooth boundaries, $\partial B_1 \subset \operatorname{int}(B_2)$, and consider the "cut-off" functions $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ and $\tilde{\Phi}\colon C \to C$ constructed in Section 2. Assume now the existence of a compact set Γ, invariant under the flow of X, such that $\partial\Gamma \subset \operatorname{int}(B_1)$. Under these hypotheses, there exists an $\bar{\varepsilon} > 0$ such that if $|F(\varphi)| \le \bar{\varepsilon}$ for all $\varphi \in C^0(I, B_2)$, then there exists a C^1 map $p\colon \mathbb{R}^n \to C$ such that $p(\Gamma) \subset C^0(I, B_1)$ and the following hold:

a) *Every solution $x = x(t)$ of the ODE $\dfrac{dx}{dt} = X(x) + F(p(x))$ such that $x(0) \in \Gamma$ is defined for $t \in [-r, 0]$ and $x|_{[-r,0]} \in C^0([-r, 0], B_1)$. Moreover, if $x(t) \in \Gamma$ for $t \in [-r, t_1)$, this solution is also a solution of $\dfrac{dx}{dt} = X(x(t)) + F(x_t)$ for $t \in [0, t_1)$ with initial condition $x|_{[-r,0]}$, that is, is a solution of the Volterra equations (4.1). In particular, every global solution of $\dfrac{dx}{dt} = X(x) + F(p(x))$ contained in Γ is a (global) solution of $\dfrac{dx}{dt} = X(x(t)) + F(x_t)$.*

b) *Every global solution of the Volterra equations (4.1) contained in Γ is also a solution of the ODE $\dfrac{dx}{dt} = X(x) + F(p(x))$.*

Proof: Apply Theorem 3.1 to the functions (4.7) and (4.8) and consider the equations:

$$\frac{dx}{dt} = \Phi(x)\, X(x) \tag{4.10}$$

$$\frac{dx}{dt} = \Phi(x(t))\, X(x(t)) + F(\tilde{\Phi}(x_t)) \tag{4.11}$$

corresponding to (3.3) and (3.5), respectively. The compactness of Γ and standard arguments of continuity of solutions of small perturbations of (4.10) imply, by choosing properly $\bar{\varepsilon}$ smaller than the number $\varepsilon > 0$ given by Theorem 3.1, that the C^1 map $p\colon \mathbb{R}^n \to C$ satisfies $p(\Gamma) \subset C^0(I, B_1)$. The conclusions a) and b) follow now from Theorem 3.1 where (3.4) is the equation

$$\frac{dx}{dt} = \Phi(x)\, X(x) + F(\tilde{\Phi}(p(x)))\ . \blacksquare \tag{4.12}$$

A crucial point for the application of Theorem 4.1 is the choice of a compact set Γ, invariant under the flow of the unperturbed system $\dfrac{dx}{dt} = X(x)$, in such a way that one can obtain global solutions of $\dfrac{dx}{dt} = X(x(t)) + F(x_t)$ inside Γ; then, as the theorem asserts, these solutions are global solutions of the ODE $\dfrac{dx}{dt} = X(x) + F(p(x))$.

In the sequel we will analyse the special case of system (4.1) with $n = 2$ species and constant memory functions, describing a classical prey-predator model of retarded type. In this case there are compact sets Γ invariant under the flow of the associated Lotka-Volterra ODE

$$\begin{aligned} \frac{dN_1}{dt} &= N_1[\varepsilon_1 - \gamma_1 N_2]\ , \\ \frac{dN_2}{dt} &= N_2[-\varepsilon_2 + \gamma_2 N_1]\ , \end{aligned} \tag{4.13}$$

where ε_1, ε_2, γ_1, γ_2 are positive fixed numbers. Moreover, Γ can be choosen containing global solutions of the following Volterra system of retarded type:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(t)\Big[\varepsilon_1 - \gamma_1 N_2(t) - \delta_1 \int_{-r}^{0} N_2(t+\theta)\, d\theta\Big]\ , \\ \frac{dN_2}{dt} &= N_2(t)\Big[-\varepsilon_2 + \gamma_2 N_1(t) + \delta_2 \int_{-r}^{0} N_1(t+\theta)\, d\theta\Big]\ , \end{aligned} \tag{4.14}$$

where $r > 0$ is the lag and the parameters $\delta_1, \delta_2 \geq 0$ are such that $\delta_1 + \delta_2 > 0$. Notice that system (4.14) with $\delta_1 = 0$, so $\delta_2 > 0$, means that the effect of predation is instantaneous for preys and carries a time lag for predators.

The next result is a theorem of existence of periodic solution for the Volterra retarded system (4.14).

Theorem 4.2. *Assume that the parameters δ_1, δ_2 and the lag r, in (4.14), satisfy the inequalities*

$$\delta_1\delta_2 < \frac{\varepsilon_1\varepsilon_2\,\gamma_1\gamma_2}{4\pi^2} \qquad \text{and} \qquad r > \frac{1}{2\alpha}\Big[\beta + \sqrt{\beta^2 + 4\alpha\,\gamma_1\gamma_2}\Big]\ ,$$

where α and β are positive numbers given by $\alpha = [\varepsilon_1\varepsilon_2\gamma_1\gamma_2/(4\pi^2)] - \delta_1\delta_2$ and $\beta = \gamma_1\delta_2 + \gamma_2\delta_1$. Then the system (4.14) has a periodic solution with minimum period r.

Proof: Assume that for a global solution $(N_1(t), N_2(t))$ of (4.14) one has

$$\int_{-r}^{0} N_1(t+\theta)\,d\theta = K_1 \quad \text{and} \quad \int_{-r}^{0} N_2(t+\theta)\,d\theta = K_2$$

for all $t \in \mathbb{R}$; then this solution satisfies the ODE system:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1\Big[(\varepsilon_1 - \delta_1 K_2) - \gamma_1 N_2\Big] \,, \\ \frac{dN_2}{dt} &= N_2\Big[-(\varepsilon_2 - \delta_2 K_1) + \gamma_2 N_1\Big] \,. \end{aligned} \tag{4.15}$$

It is well known that if $(\varepsilon_1 - \delta_1 K_2)$, $(\varepsilon_2 - \delta_2 K_1)$ are positive, all solutions of (4.15) on the positive quadrant are periodic with one only equilibrium given by $((\varepsilon_2 - \delta_2 K_1)/\gamma_2, (\varepsilon_1 - \delta_1 K_2)/\gamma_1)$. Moreover, for any T greater than the limit period

$$T_\ell = \frac{2\pi}{\sqrt{(\varepsilon_1 - \delta_1 K_2)\,(\varepsilon_2 - \delta_2 K_1)}} \,, \tag{4.16}$$

through each point on the positive quadrant passes only one periodic trajectory of (4.15) with minimum period T (see [6], p. 19 and [4]). So, if the lag r satisfies $r > T_\ell$ and the global solution $(N_1(t), N_2(t))$ is periodic with minimum period r, we have necessarily, by the averaging conservation law([6], p. 19),

$$\begin{aligned} \frac{1}{r}K_1 &= \frac{1}{r}\int_{-r}^{0} N_1(t+\theta)\,dt = \frac{\varepsilon_2 - \delta_2 K_1}{\gamma_2} \,, \\ \frac{1}{r}K_2 &= \frac{1}{r}\int_{-r}^{0} N_2(t+\theta)\,dt = \frac{\varepsilon_1 - \delta_1 K_2}{\gamma_1} \,. \end{aligned}$$

From this we get

$$K_1 = \frac{r\,\varepsilon_2}{\gamma_2 + r\,\delta_2} \quad \text{and} \quad K_2 = \frac{r\,\varepsilon_1}{\gamma_1 + r\,\delta_1} \,. \tag{4.17}$$

Consider now the ODE system (4.15) with K_1 and K_2 given by (4.17), therefore,

$$\begin{aligned} \varepsilon_1 - \delta_1 K_2 &= \varepsilon_1 - \frac{\delta_1\,r\,\varepsilon_1}{\gamma_1 + r\,\delta_1} = \frac{\varepsilon_1\,\gamma_1}{\gamma_1 + r\,\delta_1} > 0 \,, \\ \varepsilon_2 - \delta_2 K_1 &= \varepsilon_2 - \frac{\delta_2\,r\,\varepsilon_2}{\gamma_2 + r\,\delta_2} = \frac{\varepsilon_2\,\gamma_2}{\gamma_2 + r\,\delta_2} > 0 \,. \end{aligned} \tag{4.18}$$

Then system (4.15) with $\varepsilon_1 - \delta_1 K_2$ and $\varepsilon_2 - \delta_2 K_1$ given by (4.18) has all of its periodic solutions in the positive quadrant with limit period

$$T_\ell = 2\pi \sqrt{\frac{(\gamma_1 + r\,\delta_1)\,(\gamma_2 + r\,\delta_2)}{\varepsilon_1\varepsilon_2\,\gamma_1\gamma_2}}\;. \tag{4.19}$$

If the lag r satisfies $r > T_\ell$, that is,

$$r > 2\pi \sqrt{\frac{(\gamma_1 + r\,\delta_1)\,(\gamma_2 + r\,\delta_2)}{\varepsilon_1\varepsilon_2\,\gamma_1\gamma_2}}\;, \tag{4.20}$$

the system (4.15), together with conditions (4.18) and (4.20), admits, through each point of the positive quadrant, just one periodic trajectory $(\bar N_1(t), \bar N_2(t))$ with minimum period r that satisfies

$$\frac{1}{r}\int_{t-r}^{t} \bar N_1(\tau)\,d\tau = \frac{\varepsilon_2}{\gamma_2 + r\,\delta_2} \quad \text{and} \quad \frac{1}{r}\int_{t-r}^{t} \bar N_2(\tau)\,d\tau = \frac{\varepsilon_1}{\gamma_1 + r\,\delta_1}\;. \tag{4.21}$$

This periodicity and (4.17) and (4.21) imply

$$K_1 = \int_{-r}^{0} \bar N_1(t+\theta)\,d\theta \quad \text{and} \quad K_2 = \int_{-r}^{0} \bar N_2(t+\theta)\,d\theta\;. \tag{4.22}$$

Combining (4.22) and (4.15) one obtains that $(\bar N_1(t), \bar N_2(t))$ satisfies (4.14). Elementary computations show that (4.20) is equivalent to the two inequalities of the hypotheses of Theorem 4.2. ∎

Remark 4.3. If r is greater than the limit period $2\pi/\sqrt{\varepsilon_1\varepsilon_2}$ of the periodic solutions of (4.13), it follows that (4.20) is verified for δ_1, δ_2 sufficiently small and, therefore, one can obtain from Theorem 4.2 a periodic solution of (4.14) with minimum period r.

Remark 4.4. If we have $\delta_1 = 0$ in (4.14), the hypotheses of Theorem 4.2 reduce to the inequality

$$r > \frac{2\pi}{\varepsilon_1\varepsilon_2\,\gamma_2}\left(\pi\delta_2 + \sqrt{\pi^2\,\delta_2^2 + \varepsilon_1\varepsilon_2\,\gamma_2^2}\right) \stackrel{\text{def}}{=} f(\delta_2)\;.$$

It is easy to check that the positive function $f(\delta_2)$ has the following properties:

1) $f'(\delta_2) > 0$ and $f''(\delta_2) > 0$ for all $\delta_2 > 0$;

2) $\lim_{\delta_2\to 0} f(\delta_2) = 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$ and $\lim_{\delta_2\to\infty} f'(\delta_2) = 4\pi^2/(\varepsilon_1\varepsilon_2\,\gamma_2)$.

These properties show that if $r > 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$, system (4.14) with $\delta_1 = 0$ has a periodic solution with minimum period r.

The next result will follow from a suitable combination of theorems 4.1 and 4.2 through the remark 4.3 in order to obtain a compact invariant set for the Volterra system of retarded type (4.14).

Let us denote by γ_0 the orbit of the r-periodic solution of (4.14) obtained according to Remark 4.3. Let us choose the compact set $\Gamma \subset \mathbb{R}^2$, required by the hypotheses of Theorem 4.1, as the closure of the inner points of an orbit (Jordan curve) of (4.13) in such a way that $\gamma_0 \subset \mathrm{int}(\Gamma)$.

Take B_1 and B_2 as closed balls in $\mathbb{R}^2$ centered at the origin with radii r_1, r_2, respectively, $r_1 < r_2$, and such that $\Gamma \subset \mathrm{int}(B_1)$. Then, by Theorem 4.1 applied to systems (4.13) and (4.14), γ_0 is also an orbit of a certain C^1 vector-field $\bar{X}$, provided δ_1, δ_2 are choosen sufficiently small. The vector-field $\bar{X}$ is a small perturbation of the Lotka-Volterra system (4.13); indeed, $\bar{X}$ corresponds precisely to the ODE $\dfrac{dx}{dt} = X(x) + F(p(x))$ introduced in the statement of Theorem 4.1.

Therefore, all solutions of $\bar{X}$ contained in Γ are global solutions of system (4.14); in particular γ_0 and all solutions of $\bar{X}$ encircled by γ_0 define in $C^0(I, \mathbb{R}^2)$ a two-dimensional compact manifold with boundary invariant under the flow of Volterra system (4.14).

This proves the following

Theorem 4.5. *Let $r > 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$. Then for δ_1, δ_2 sufficiently small there exists in $C = C^0([-r, 0], \mathbb{R}^2)$ a two-dimensional compact manifold with boundary, diffeomorphic to a disk D in $\mathbb{R}^2$, which is invariant under the flow of system (4.14). Moreover, the flow of (4.14) restricted to D is the flow of a C^1 vector-field.*

Remark 4.6. If we recall Remark 4.4, that is, assume system (4.14) with $\delta_1 = 0$ and $\delta_2 > 0$, one can argüe as in Theorem 4.5 and say that if $r > 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$, there is in $C = C^0([-r, 0], \mathbb{R}^2)$ a two-dimensional compact manifold with boundary, diffeomorphic to a disk D in $\mathbb{R}^2$, which is invariant under the flow of system (4.14) provided δ_2 is sufficiently small in the interval

$$0 < \delta_2 \leq \frac{\varepsilon_1\varepsilon_2\,\gamma_2\left[r^2 - (4\pi^2/\varepsilon_1\varepsilon_2)\right]}{4\pi^2\, r}\,. \tag{4.23}$$

Moreover, the flow of (4.14)restricted to D is the flow of a complete C^1 vector-field.

In order to understand better the flow on the two-dimensional manifold D, considered in the Remark 4.6, we take $T_k = r/k$, $k \in \mathbb{N}$, and assume that for a solution $(N_1(t), N_2(t))$ of (4.14) with $\delta_1 = 0$ one has:

$$C_k \stackrel{\text{def}}{=} \int_{-T_k}^{0} N_1(t+\theta)\, d\theta \quad \text{for all} \quad t \in \mathbb{R} \; ; \tag{4.24}$$

then,

$$C_k = \int_{-T_k}^{0} N_1(t - T_k + \theta)\, d\theta = \int_{-2T_k}^{-T_k} N_1(t+\theta)\, d\theta = \int_{-3T_k}^{-2T_k} N_1(t+\theta)\, d\theta \; , \tag{4.25}$$

and so on, one obtains

$$k\, C_k = \int_{-kT_k}^{0} N_1(t+\theta)\, d\theta = \int_{-r}^{0} N_1(t+\theta)\, d\theta \quad \text{for all} \quad t \in \mathbb{R} \; . \tag{4.26}$$

If we observe the proof of Theorem 4.2 and repeat the arguments for the system

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(\varepsilon_1 - \gamma_1 N_2) \; , \\ \frac{dN_2}{dt} &= N_2(-\varepsilon_2 + \gamma_2 N_1) + N_2\, \delta_2 \int_{-r}^{0} N_1(t+\theta)\, d\theta \; , \end{aligned} \tag{4.27}$$

we see that any solution verifying (4.24) and (4.26) satisfies the ODE system

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(\varepsilon_1 - \gamma_1 N_2) \; , \\ \frac{dN_2}{dt} &= N_2\Big[-(\varepsilon_2 - \delta_2\, k\, C_k) + \gamma_2 N_1\Big] \; . \end{aligned} \tag{4.28}$$

If the integer k satisfies the boundedness conditions $(\varepsilon_2 - \delta_2\, k\, C_k) > 0$ and

$$T_k = r/k > \frac{2\pi}{\sqrt{\varepsilon_1(\varepsilon_2 - \delta_2\, k\, C_k)}} \; , \tag{4.29}$$

system (4.28) allows to write

$$\frac{C_k}{T_k} = \frac{1}{T_k} \int_{-T_k}^{0} N_1(t+\theta)\, d\theta = \frac{\varepsilon_2 - \delta_2\, k\, C_k}{\gamma_2} \; , \tag{4.30}$$

and, necessarily, one obtains:

$$k\, C_k = \frac{\varepsilon_2\, r}{\gamma_2 + \delta_2\, r} \; . \tag{4.31}$$

If kC_k is given by (4.31), according to (4.29) and (4.30) one has $\varepsilon_2 - \delta_2 k C_k = \gamma_2 k C_k / r = \varepsilon_2 \gamma_2 / (\gamma_2 + \delta_2 r) > 0$ and so the system (4.28) does not depend on k.

Then system (4.28) has only one T_k-periodic trajectory, provided (4.29) holds, that is, provided

$$T_k = \frac{r}{k} > \frac{2\pi\sqrt{r}}{\sqrt{\varepsilon_1\gamma_2\, k\, C_k}} = \frac{2\pi\sqrt{\gamma_2+\delta_2\, r}}{\sqrt{\varepsilon_1\varepsilon_2\,\gamma_2}}\ ; \tag{4.32}$$

moreover, this T_k-periodic trajectory is also a trajectory of (4.27). The conclusion is the following: *provided* (4.32) *holds, system* (4.27) *has a* (r/k)*-periodic solution.*

Since (4.32) is equivalent to

$$\frac{r^2}{k^2} > \frac{4\pi^2\,\gamma_2 + 4\pi^2\,\delta_2\, r}{\varepsilon_1\varepsilon_2\,\gamma_2} \tag{4.33}$$

and because (4.33) is equivalent to

$$\frac{r}{k} > g(\delta_2) \overset{\text{def}}{=} \frac{2\pi^2\, k\,\delta_2 + \sqrt{4k^2\,\pi^4\,\delta_2^2 + \varepsilon_1\varepsilon_2\,\gamma_2^2\, 4\pi^2}}{\varepsilon_1\varepsilon_2\,\gamma_2}\ , \tag{4.34}$$

one can say that: *for any integer* $k > 1$ *such that* (4.34) *holds, system* (4.27) *has a (r/k)-periodic solution.* The function $g(\delta_2)$ in (4.34) is increasing for $\delta_2 > 0$ and $\lim_{\delta_2\to 0} g(\delta_2) = 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$.

Theorem 4.7. *For any integer $k > 1$ such that $r/k > 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$, system (4.14) with $\delta_1 = 0$ has a periodic solution of minimum period r/k, for any δ_2 such that*

$$0 < \delta_2 \le \frac{\varepsilon_1\varepsilon_2\,\gamma_2\left[(r/k)^2 - 4\pi^2/\varepsilon_1\varepsilon_2\right]}{4\pi^2\, r}\ .$$

Moreover, all these periodic solutions are solutions of the same ODE (4.28) which does not depend on k. ∎

If we use Theorem 4.7, we can state the following result that generalizes the Remark 4.6:

Theorem 4.8. *Assume $r > 2\pi/\sqrt{\varepsilon_1\varepsilon_2}$ and let $k \ge 1$ be the greatest positive integer such that r/k is still greater than $2\pi/\sqrt{\varepsilon_1\varepsilon_2}$. Then, for any δ_2 sufficiently small, there exists in $C = C^0([-r,0],\mathbb{R}^2)$ a two dimensional compact manifold with boundary, diffeomorphic to a disk D in $\mathbb{R}^2$, which is invariant under the flow of system (4.27). Moreover, the flow of (4.27) restricted to D is the flow of a complete C^1 vector-field that has at least k periodic orbits with minimum periods r, $r/2$, …, r/k, the boundary ∂D being the r-periodic one.* ∎

Another example can be constructed, using Theorem 4.8, if we consider the product of two systems of the form (4.27), as a system in $\mathbb{R}^4$:

$$\begin{aligned}
&\frac{dN_1}{dt} = N_1(t)\Big[\varepsilon_1 - \gamma_1 N_1(t)\Big] ,\\
&\frac{dN_2}{dt} = N_2(t)\Big[-\varepsilon_2 + \gamma_2 N_1(t) + \delta_2 \int_{-r}^{0} N_1(t+\theta)\, d\theta\Big] ,\\
&\frac{dN_3}{dt} = N_3(t)\Big[\varepsilon_3 - \gamma_3 N_4(t)\Big] ,\\
&\frac{dN_4}{dt} = N_4(t)\Big[-\varepsilon_4 + \gamma_4 N_3(t) + \bar{\delta}_2 \int_{-r}^{0} N_3(t+\theta)\, d\theta\Big] ,
\end{aligned} \tag{4.35}$$

where, as before, ε_i, γ_i, $i = 1,2,3,4$ are positive numbers, $r > 0$ is the lag and δ_2, $\bar{\delta}_2$ are positive parameters. If we assume $r > M_0 = \max\{2\pi/\sqrt{\varepsilon_1\varepsilon_2}\,, 2\pi/\sqrt{\varepsilon_3\varepsilon_4}\}$, let $k \geq 1$ be the greatest integer such that r/k is still greater than M_0. This way, Theorem 4.8 can be applied to the two first equations, as well as, to the two last equations of the Volterra system of retarded type (4.35). We obtain two compact manifolds with boundary, both diffeomorphic to disks D and $\bar{D}$ in $\mathbb{R}^2$ invariant, respectively, under the first two equations and under the last two equations of (4.35), provided δ_2 and $\bar{\delta}_2$ are sufficiently small. It is clear that $D\times\bar{D}$ and its boundary $\partial(D\times\bar{D})$ are diffeomorphic to sets invariant under the flow of (4.35) and also we easily see that the boundary $\partial(D\times\bar{D})$ is homeomorphic to a three-dimensional sphere S^3 since we have

$$\partial(D\times\bar{D}) = (\partial D\times \operatorname{int}(\bar{D})) \cup (\operatorname{int}(D)\times\partial\bar{D}) \cup (\partial D\times\partial\bar{D}) \ ; \tag{4.36}$$

$\partial D\times\operatorname{int}(\bar{D})$ and $\operatorname{int}(D)\times\partial\bar{D}$ are solid tori with the common boundary $\partial D\times\partial\bar{D}$, a two dimensional torus T^2. These three components of $\partial(D\times\bar{D})$ in (4.36) are, themselves diffeomorphic to sets, invariant under the flow of (4.35), that is, system (4.35) defines a RFDE on the compact manifold (without boundary) T^2 (see [3]). All solutions of (4.35) inside $D\times\bar{D}$ are solutions of a complete C^1 vector-field. Therefore, the set $D\times\bar{D}$ is homeomorphic to a compact invariant set contained in the compact attractor of (4.35).

5 – Behavior near equilibrium – Planar Case

The behavior of system (4.14) in a neighborhood of the equilibrium point (N_1^0, N_2^0),

$$N_1^0 = \frac{\varepsilon_2}{\gamma_2 + \delta_2\, r}\,, \qquad N_2^0 = \frac{\varepsilon_1}{\gamma_1 + \delta_1\, r}\ , \tag{5.1}$$

can be analysed after the change of coordinates

$$q_1 = N_1 - N_1^0 , \quad q_2 = N_2 - N_2^0 . \tag{5.2}$$

After this translation one sees that the origin $(0,0)$ is the only equilibrium of the system (5.3):

$$\begin{aligned} \dot q_1 &= q_1\Big[\gamma_1\, q_2 + \int_{-r}^{0} \delta_1\, q_2(t+\theta)\, d\theta\Big] + N_1^0\Big[\gamma_1\, q_2 + \int_{-r}^{0} \delta_1\, q_2(t+\theta)\, d\theta\Big] , \\ \dot q_2 &= q_2\Big[\gamma_2\, q_1 + \int_{-r}^{0} \delta_2\, q_1(t+\theta)\, d\theta\Big] + N_2^0\Big[\gamma_2\, q_1 + \int_{-r}^{0} \delta_2\, q_1(t+\theta)\, d\theta\Big] . \end{aligned} \tag{5.3}$$

The linearized system at $(0,0)$ is given by

$$\begin{aligned} \dot q_1 &= N_1^0\Big[\gamma_1\, q_2 + \int_{-r}^{0} \delta_1\, q_2(t+\theta)\, d\theta\Big] , \\ \dot q_2 &= N_2^0\Big[\gamma_2\, q_1 + \int_{-r}^{0} \delta_2\, q_1(t+\theta)\, d\theta\Big] , \end{aligned} \tag{5.4}$$

and the characteristic equation (see [2]) is

$$\lambda^2 + N_1^0 N_2^0\Big(\gamma_1 + \delta_1 \int_{-r}^{0} e^{\lambda\theta}\, d\theta\Big)\Big(\gamma_2 + \delta_2 \int_{-r}^{0} e^{\lambda\theta}\, d\theta\Big) = 0 . \tag{5.5}$$

It is clear that λ satisfies (5.5) if and only if $\bar\lambda$ satisfies (5.5). Since $\lambda = 0$ is not a root, because $N_1^0 N_2^0(\gamma_1 + \delta_1 r)(\gamma_2 + \delta_2 r) = \varepsilon_1\varepsilon_2$ is positive, the equation (5.5) is equivalent to (5.6) or (5.7):

$$\lambda^2 + N_1^0 N_2^0\Big(\gamma_1 + \frac{\delta_1}{\lambda}(1 - e^{-\lambda r})\Big)\Big(\gamma_2 + \frac{\delta_2}{\lambda}(1 - e^{-\lambda r})\Big) = 0 , \tag{5.6}$$

$$\lambda^4 + N_1^0 N_2^0\Big(\gamma_1\lambda + \delta_1(1 - e^{-\lambda r})\Big)\Big(\gamma_2\lambda + \delta_2(1 - e^{-\lambda r})\Big) = 0 . \tag{5.7}$$

Denoting $A = N_1^0 N_2^0 \gamma_1\gamma_2$, $B = N_1^0 N_2^0(\gamma_1\delta_2 + \gamma_2\delta_1)$, $C = N_1^0 N_2^0 \delta_1\delta_2$, the equation (5.7) can be written as

$$(\lambda^4 + A\lambda^2 + B\lambda + C)\, e^{2\lambda r} - (B\lambda + 2C)\, e^{\lambda r} + C = 0 . \tag{5.8}$$

Let us seek the pure imaginary roots, $\lambda = bi$, that is, look for $b \neq 0$ such that

$$(b^4 - Ab^2 + Bbi + C)\,(\cos 2br + i \sin 2br) - (Bbi + 2C)\,(\cos br + i \sin br) + C = 0 \tag{5.9}$$

and, separating real and imaginary parts:

$$
(5.10)\qquad \begin{aligned} &(b^4-Ab^2+C)\cos 2br - Bb\sin 2br - 2C\cos br + Bb\sin br = -C\ ,\\ &(b^4-Ab^2+C)\sin 2br + Bb\cos 2br - 2C\sin br - Bb\cos br = 0\ . \end{aligned}
$$

Define in $\mathbb{R}^2$ the positive orthonormal basis $(w(b), w^\perp(b))$ by

$$
(5.11)\qquad w(b) = (\cos br, \sin br)\,, \qquad w^\perp(b) = (-\sin br, \cos br)\ ;
$$

so, system (5.10) becomes

$$
(5.12)\qquad \begin{aligned} &(Bb, b^4-Ab^2+C)\cdot w(2b) - (Bb, 2C)\cdot w(b) = 0\ ,\\ &(Bb, b^4-Ab^2+C)\cdot w^\perp(2b) - (Bb, 2C)\cdot w^\perp(b) = -C\ . \end{aligned}
$$

In order to investigate better equations (5.12), we notice that if R_b is the counterclockwise rotation by the angle br, that is R_b is given by the matrix

$$
(5.13)\qquad R_b = \begin{pmatrix} \cos br & -\sin br \\ \sin br & \cos br \end{pmatrix}\ ,
$$

we have $v\cdot w(b) = (R_b v)\cdot w(2b)$ and $v\cdot w^\perp(b) = (R_b v)\cdot w^\perp(2b)$, for any vector $v\in\mathbb{R}^2$. Therefore, system (5.12) becomes equivalent to the vector equation

$$
(5.14)\qquad (Bb, b^4-Ab^2+C) - R_b(Bb, 2C) = -C w^\perp(2b)\ .
$$

We will consider only the case $b>0$ in (5.14) because the solutions of (5.5) are pairwise conjugated.

Theorem 5.1. *The equilibrium point (N_1^0, N_2^0) of system (4.14) is hyperbolic except for systems with lag $r>0$ such that either*

$$
(5.15)\qquad r = 2\left[-\frac{\delta_1}{\gamma_1} - \frac{\delta_2}{\gamma_2} + \sqrt{\left(\frac{\delta_1}{\gamma_1}+\frac{\delta_2}{\gamma_2}\right)^2 - \frac{4\delta_1\delta_2}{\gamma_1\gamma_2} + \frac{\varepsilon_1\varepsilon_2}{k^2\pi^2}}\,\right]^{-1}, \quad k=1,2,\dots\ ,
$$

or the following equality holds for some $b_0>0$:

$$
(5.16)\qquad (Bb_0, b_0^4 - Ab_0^2 + 2C) = 2C(-\sin b_0 r, \cos b_0 r)\ ,
$$

where $A = N_1^0 N_2^0\gamma_1\gamma_2$, $B = N_1^0 N_2^0(\gamma_1\delta_2+\gamma_2\delta_1)$, $C = N_1^0 N_2^0\delta_1\delta_2$ and N_1^0, N_2^0 given by (5.1).

Proof: Replacing (5.13) in (5.14), a straightforward computation leads to

$$\begin{aligned}(5.17)\qquad &\Big(Bb(1-\cos br),\, b^4 - Ab^2 + C - Bb\sin br\Big) = \\ &\qquad = C\Big(2\sin br\,(\cos br - 1),\, \sin^2 br - \cos^2 br + 2\cos br\Big)\ .\end{aligned}$$

A solution of (5.17) is supplied by $\cos br = 1$ and $b^4 - Ab^2 = 0$, that is

$$(5.18)\qquad \begin{aligned} &br = 2k\pi\,, \qquad k = 1,2,\ldots\,, \\ &b^2 = A = \frac{\varepsilon_1\varepsilon_2\,\gamma_1\gamma_2}{(\gamma_2+\delta_2\,r)\,(\gamma_1+\delta_1\,r)}\ . \end{aligned}$$

Conditions (5.18) mean that $b>0$ satisfies (5.14) if, and only if,

$$(5.19)\qquad b = 2k\pi/r\,, \qquad k = 1,2,\ldots\,,$$

provided that

$$(5.20)\qquad \frac{4k^2\pi^2}{r^2} = \frac{\varepsilon_1\varepsilon_2\,\gamma_1\gamma_2}{(\gamma_2+\delta_2\,r)\,(\gamma_1+\delta_1\,r)}\,, \qquad k = 1,2,\ldots\ .$$

Equation (5.20) is equivalent to

$$(5.21)\qquad \left(\frac{1}{r}\right)^2 + \frac{\gamma_1\delta_2+\gamma_2\delta_1}{\gamma_1\gamma_2}\left(\frac{1}{r}\right) + \frac{\delta_1\delta_2}{\gamma_1\gamma_2} - \frac{\varepsilon_1\varepsilon_2}{4k^2\pi^2} = 0\ .$$

Solving (5.21) we get (5.15) and this concludes the first part of the proof.

Let us suppose now $\cos br \neq 1$. In this case equation (5.17) gives the equations

$$(5.22)\qquad \sin br = -\frac{B}{2C}\,b\ ,$$

$$(5.23)\qquad b^4 - Ab^2 - Bb\sin br = 2C\sin^2 br + 2C\cos br - 2C\ ,$$

that are equivalent to the system

$$(5.24)\qquad \begin{aligned} b^4 - Ab^2 + 2C &= 2C\cos br\ , \\ -Bb &= 2C\sin br\ . \end{aligned}$$

System (5.24) can be rewritten as

$$(5.25)\qquad (Bb, b^4 - Ab^2 + 2C) = 2C(-\sin br, \cos br)$$

and this completes the second part of the proof. ∎

Remark 5.2. The equation (5.15) is subjected to the condition

$$\frac{4\delta_1\delta_2}{\gamma_1\gamma_2} < \frac{\varepsilon_1\varepsilon_2}{\pi^2}$$

in order to define a positive r and in this case there exists only a finite number of lags given by the values of k making r positive in (5.15).

If there exists $b_0 > 0$ such that (5.16) defines a lag r, then there is a sequence of lags $r_k = (2k\pi/b_0) + r$, $k = 1, 2, \ldots$ satisfying (5.16).

We will analyse, in the sequel, the cases $\delta_1 = 0$, $\delta_2 > 0$ and $\delta_1 = \delta_2 = \delta > 0$.

The special case $\delta_1 = 0$, $\delta_2 > 0$ can be studied easily because $C = 0$, that is, (5.14) reduces to

$$(Bb, b^4 - Ab^2) = R_b(Bb, 0) \tag{5.26}$$

that means $R_b =$ Identity and $b^4 - Ab^2 = 0$, so, from (5.26) we obtain:

$$\begin{aligned} br &= 2k\pi\,, \qquad k = 1, 2, \ldots\,, \\ b^2 &= A = N_1^0 N_2^0\, \gamma_1\gamma_2\,. \end{aligned} \tag{5.27}$$

Conditions (5.27) are precisely equations (5.18) with $\delta_1 = 0$. So, one can apply Theorem 5.1 with the lag given by (5.15) where we make $\delta_1 = 0$, that is equivalent to

$$r = \frac{2k\pi\left[k\pi\,\delta_2 + \sqrt{(k\pi\,\delta_2)^2 + \gamma_2^2\,\varepsilon_1\varepsilon_2}\right]}{\gamma_2\,\varepsilon_1\varepsilon_2}\,. \tag{5.28}$$

So, (5.27) and (5.28) imply

$$b = \frac{\gamma_2\,\varepsilon_1\varepsilon_2}{k\pi\,\delta_2 + \sqrt{(k\pi\,\delta_2)^2 + \gamma_2^2\,\varepsilon_1\varepsilon_2}}\,, \qquad k = 1, 2, \ldots\,. \tag{5.29}$$

These computations prove the following

Corollary 5.3. *The equilibrium point (N_1^0, N_2^0) of system (4.14) with $\delta_1 = 0$ and $\delta_2 > 0$ is hyperbolic except for systems with lag*

$$r = \frac{2k\pi\left[k\pi\,\delta_2 + \sqrt{(k\pi\,\delta_2)^2 + \gamma_2^2\,\varepsilon_1\varepsilon_2}\right]}{\gamma_2\,\varepsilon_1\varepsilon_2}\,, \qquad k = 1, 2, \ldots\,. \blacksquare$$

Let us consider now system (4.14) with parameters $\delta_1 = \delta_2 = \delta > 0$. Theorem 5.1 applies to this case with the conditions (5.15) and (5.16) properly rewritten, respectively, as

$$r = 2\left[-\left(\frac{\delta}{\gamma_1}+\frac{\delta}{\gamma_2}\right) + \sqrt{\left(\frac{\delta}{\gamma_1}+\frac{\delta}{\gamma_2}\right)^2 - \frac{4\delta^2}{\gamma_1\gamma_2} + \frac{\varepsilon_1\varepsilon_2}{k^2\pi^2}}\right]^{-1}, \quad k=1,2,\dots, \tag{5.30}$$

and, for some $b_0 > 0$,

$$(Bb_0, b_0^4 - Ab_0^2 + 2C) = 2C(-\sin b_0 r, \cos b_0 r), \tag{5.31}$$

where $A = N_1^0 N_2^0 \gamma_1\gamma_2$, $B = N_1^0 N_2^0 \delta(\gamma_1+\gamma_2)$, $C = N_1^0 N_2^0 \delta^2$, $N_1^0 N_2^0 = \frac{\varepsilon_1\varepsilon_2}{(\gamma_1+\delta_r)(\gamma_2+\delta_r)}$.

However we can say more in this special case. The characteristic equation (5.5) now is given by

$$\lambda^2 + N_1^0 N_2^0 \left(\gamma_1 + \delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\right)\left(\gamma_2 + \delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\right) = 0. \tag{5.32}$$

From now on, our discussion is based in considering δ as a parameter.

Theorem 5.4. *Suppose $r\sqrt{\varepsilon_1\varepsilon_2} \neq 2k\pi$, $k = 1,2,\dots$. Then for $\delta > 0$ sufficiently small, the critical point (N_1^0, N_2^0) of system (5.14) is a hyperbolic critical point and the unstable manifold has dimension two.*

Proof: Although we are interested in $\delta > 0$, it is convenient to consider equation (5.32) with δ varying in a neighborhood of $\delta = 0$. Call $\lambda = a + ib$, then the characteristic equation gives

$$\begin{aligned} H_1(\delta,a,b) &\overset{\text{def}}{=} a^2 - b^2 + \alpha\Big[\big(\gamma_2 + \delta f(a,b)\big)\big(\gamma_1 + \delta f(a,b)\big) - \delta^2 g^2(a,b)\Big] = 0, \\ H_2(\delta,a,b) &\overset{\text{def}}{=} 2ab + \alpha\delta g(a,b)\Big[\gamma_1 + \gamma_2 + 2\delta f(a,b)\Big] = 0, \end{aligned} \tag{5.33}$$

where

$$\alpha = \alpha(\delta) \overset{\text{def}}{=} \frac{\varepsilon_1\varepsilon_2}{(\gamma_1+\delta r)(\gamma_2+\delta r)},$$

$$f(a,b) \overset{\text{def}}{=} \int_{-r}^0 e^{a\theta}\cos b\theta\, d\theta,$$

$$g(a,b) \overset{\text{def}}{=} \int_{-r}^0 e^{a\theta}\sin b\theta\, d\theta.$$

Let us point out the following remarks:

1) $\dot\alpha(0) = -(\gamma_1+\gamma_2)\, r\, \varepsilon_1\varepsilon_2/(\gamma_1\gamma_2)^2$ and $\alpha(0) = \varepsilon_1\varepsilon_2/\gamma_1\gamma_2$.

2) For $\delta = 0$ we get $a = 0$ and $b^2 = \varepsilon_1\varepsilon_2$.

Consider the map

$$M\colon\ (\delta,a,b) \in \mathbb{R}\times\mathbb{R}^2 \longmapsto \Big(H_1(\delta,a,b),\ H_2(\delta,a,b)\Big) \in \mathbb{R}^2$$

where $M(0,0,b_0) = M(0,0,-b_0) = 0$ and $b_0 = \sqrt{\varepsilon_1\varepsilon_2} > 0$.

The jacobian determinant

$$\det\begin{pmatrix} \partial H_1/\partial a & \partial H_1/\partial b \\ \partial H_2/\partial a & \partial H_2/\partial b \end{pmatrix}$$

is equal to $4\varepsilon_1\varepsilon_2$ at the points $(0,0,b_0)$ and $(0,0,-b_0)$.

According to the Implicit Function Theorem applied in the neighborhood of $(0,0,b_0)$ and $(0,0,-b_0)$, there are $a(\delta)$ and $b(\delta)$ for $|\delta|$ small satisfying

$$\begin{aligned} &H_1\Big(\delta,a(\delta),b(\delta)\Big) = 0\ , \\ &H_2\Big(\delta,a(\delta),b(\delta)\Big) = 0\ , \end{aligned} \tag{5.34}$$

which give the solutions of (5.33) for $\delta > 0$ small. To obtain $\dot a(0)$ we compute the derivative of $H_2(\delta,a(\delta),b(\delta))$ with respect to δ:

$$\begin{aligned} 2\dot a(\delta)\, b(\delta) + 2a(\delta)\, \dot b(\delta) + \alpha(\delta)\, g(a(\delta),b(\delta))\, \Big[\gamma_1+\gamma_2+2\delta f(a(\delta),b(\delta))\Big] + \\ + \delta\,\frac{d}{d\delta}\Big[\alpha(\delta)\, g(a(\delta),b(\delta))\, \Big[\gamma_1+\gamma_2+2\delta f(a(\delta),b(\delta))\Big]\Big] = 0 \end{aligned} \tag{5.35}$$

and then

$$2\dot a(0)\, b(0) + \alpha(0)\, g(a(0),b(0))\, [\gamma_1+\gamma_2] = 0\ . \tag{5.36}$$

But $g(a(0),b(0)) = g(0,b_0) = \int_{-r}^0 \sin b_0\theta\, d\theta = (\cos rb_0 - 1)/b_0$. Finally

$$\dot a(0) = -\frac{(\varepsilon_1\varepsilon_2/\gamma_1\gamma_2)\,(\gamma_1+\gamma_2)\,(\cos rb_0 - 1)}{2b_0^2}\ . \tag{5.37}$$

Since $b_0 = \sqrt{\varepsilon_1\varepsilon_2}$, our hypotheses give $\cos rb_0 \neq 1$ and we get $\dot a(0) > 0$. The theorem is, then, proved since $\lambda = \pm ib_0$ are simple imaginary roots. In fact, if

$$h(\lambda,\delta) \stackrel{\text{def}}{=} \lambda^2 + N_1^0 N_2^0 \Big(\gamma_1 + \delta\int_{-r}^0 e^{\lambda\theta}\, d\theta\Big)\, \Big(\gamma_2 + \delta\int_{-r}^0 e^{\lambda\theta}\, d\theta\Big)\ , \tag{5.38}$$

then, denoting with $'$ the derivative with respect to λ,

$$
\begin{aligned}
h'(\lambda,\delta) &= 2\lambda + N_1^0 N_2^0 \Big(\lambda\delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\Big) \Big(\gamma_2 + \delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\Big) \\
&\quad + N_1^0 N_2^0 \Big(\gamma_1 + \delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\Big) \Big(\lambda\delta \int_{-r}^0 e^{\lambda\theta}\, d\theta\Big)
\end{aligned}
\tag{5.39}
$$

and $h'(ib_0, 0) = 2ib_0 \neq 0$. ∎

Since there are no pure imaginary roots of (5.32) for $\delta = 0$, except for $\lambda = \pm ib_0$, the proof of Theorem 5.4 shows that all the hypotheses of the Hopf Bifurcation Theorem for RFDE (see [2]) are verified. Therefore, we have the

Corollary 5.5. *Under the hypotheses of Theorem 5.4, $\delta = 0$ gives us a Hopf bifurcation for system (4.14) with $\delta_1 = \delta_2 = \delta$ varying in a small neighborhood of $\delta = 0$.* ∎

Remark 5.6. If one increases $\delta > 0$ properly, we could try, with a more involved analysis, to use equations (5.30) and (5.31) in order to discover how the unstable manifold increases its dimension, with other possible Hopf bifurcations.

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