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ON THE BANACH PRINCIPLE

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Abstract: We extend the Banach principle to sequences of operators which have as range an Archimedean Riesz space (the Riesz space does not have to be a space of classes of equivalence of measurable functions). The class of Riesz spaces for which our extension works is quite large. The role played in the classical Banach principle by the almost everywhere convergence is taken by the notion of individual convergence which we have introduced in an earlier work. The absence of a measure is compensated by the use of the σ-order continuous dual of the Dedekind completion of the Riesz space involved in the extension. In order to prove our extension, we obtain a characterization of individually convergent sequences which resembles a Cauchy condition.

1 – Introduction

Our aim in this paper is to extend the Banach principle [3] to a setting which does not involve measure theoretical arguments.

The Banach principle is by far the most important tool used in the study of the almost everywhere convergence in ergodic theory (for a detailed exposition of the principle, its history, its most important applications in ergodic theory, and related results, see the books of Garsia [4] and of Krengel [5]). Extensions of the Banach principle have been obtained in the pioneering works of Yosida [11] (see also Yosida's book [12]) and von Weizsäcker [10].

Let (Ω, Σ, μ) be a finite measure space, let $\mathcal{M}(\mu)$ be the Riesz space of all classes of equivalence of real valued measurable functions defined on (Ω, Σ, μ), and let X be a Banach space.

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A mapping $T\colon X \to \mathcal{M}(\mu)$ is called continuous in measure if $Tx_n \to Tx$ in measure whenever $\|x_n - x\| \to 0$.

The Banach principle can be formulated as follows:

Theorem 1. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear operators continuous in measure, $T_n\colon X \to \mathcal{M}(\mu)$ for every $n \in \mathbb{N}$, and set $T^*x = \sup_{n\in\mathbb{N}} |T_n x|$ for every $x \in X$.*

- **a)** *If $T^*x < +\infty$ μ-a.e. for every $x \in X$, then there exists a positive monotone nonincreasing function $C : (0, +\infty) \to \mathbb{R}$ such that $\lim_{\lambda\to+\infty} C(\lambda) = 0$, and such that $\mu(\{T^*x > \lambda\|x\|\}) \le C(\lambda)$ for every $\lambda \in \mathbb{R}$, $\lambda > 0$ and for every $x \in X$.*
- **b)** *If there exists a function C as in a), then the set $\mathcal{F} = \{x \in X \mid (T_n x)_{n\in\mathbb{N}}$ converges μ-a.e.$\}$ is closed in the norm topology of X. In particular, if $T^*x < +\infty$ μ-a.e. for every $x \in X$, then $\mathcal{F}$ is closed.* ∎

The most common use of the Banach principle is in the case in which we want to show that $\mathcal{F} = X$. Then, in view of Theorem 1, it is enough to show that $T^*x < +\infty$ μ-a.e. for every $x \in X$, and that $\mathcal{F}$ contains a dense subset of X.

Let E be an order complete Riesz space, and let E'_c be the σ-order continuous dual of E; we say that E has property $\mathcal{P}$ if E'_c separates the points of E. We say that an Archimedean Riesz space F has property $\mathcal{P}$ if the Dedekind completion $\widehat{F}$ of F has property $\mathcal{P}$.

Now let X be a Banach space, and let E be an Archimedean Riesz space which has property $\mathcal{P}$. We will extend Theorem 1 to sequences of linear operators $(T_n)_{n\in\mathbb{N}}$, $T_n\colon X \to E$ for every $n \in \mathbb{N}$.

As we mentioned before, the Banach principle has been extended by Yosida [11] and von Weizsäcker [10]. However, their extensions do not have the flavor of the Banach principle since by the time the works of Yosida and von Weizsäcker were created, there was no suitable notion to be used instead of the almost everywhere convergence. We will circumvent this difficulty by using the individual convergence which we introduced in our papers [16] and [17] inspired by the works of Nakano [7] and Ornstein [8].

The terminology and results used in this paper can be found in the books of Aliprantis and Burkinshaw [1, 2], Luxemburg and Zaanen [6], Schaefer [9], Zaanen [13], and in our papers [14, 15, 16, 17].

Let E be an Archimedean Riesz space, and let $\widehat{E}$ be the Dedekind completion of E. Given $w \in \widehat{E}$, we will denote by $B(w)$ the projection band generated by the singleton $\{w\}$ in $\widehat{E}$ and by P_w the band projection associated to $B(w)$.

Let X be a Banach space, and let $T\colon X \to E$ be an operator (T need not be linear). We say that T is continuous in duality if the sequence of real numbers $(\langle P_{(|Tx_n - Tx| - \varepsilon v)} + v, z\rangle)_{n\in\mathbb{N}}$ converges to zero whenever $v \in \widehat{E}$, $v \geq 0$, $z \in E'_c$, $z \geq 0$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, $x \in X$, and whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of X such that (x_n) norm converges to x.

Given a sequence $(u_n)_{n\in\mathbb{N}}$ of positive elements of E, we will denote by $B_\infty((u_n)_n)$ the largest band in $\widehat{E}$ on which $(u_n)_n$ is unbounded (see [16] for details).

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of mappings, $T_n\colon X \to E$ for every $n \in \mathbb{N}$. We say that $(T_n)_n$ has property $\mathcal{D}$ if $B_\infty((|T_n x|)_n) = 0$ for every $x \in X$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E (throughout this paper, the notation $u < v$ means $u \leq v$ and $u \neq v$ whenever u and v are elements of a Riesz space). Set (see [16]):

$$B_{OS}((u_n)_n) = \left\{ u \in \widehat{E} \;\middle|\; \begin{array}{l} u = 0 \text{ or } u \neq 0, \text{ and for every } v \in \widehat{E},\ 0 < v \leq |u|, \\ \text{there exist } w \in \widehat{E},\ 0 < w \leq v, \text{ and } \alpha, \beta \in \mathbb{R}, \\ 0 < \beta < \alpha, \text{ such that} \\ \left(\limsup\limits_n \Big(((u_n - \beta\, w)^-) \wedge w\Big)\right) \wedge \\ \qquad\qquad \wedge \left(\limsup\limits_n \Big(((u_n - \alpha\, w)^+) \wedge w\Big)\right) \neq 0 \end{array} \right\}$$

and

$$B_{NOS}((u_n)_n) = \left\{ u \in \widehat{E} \;\middle|\; \begin{array}{l} u = 0 \text{ or } u \neq 0, \text{ and for every } v \in \widehat{E},\ 0 < v \leq |u|, \\ \text{there exists } w \in \widehat{E},\ 0 < w \leq v, \text{ such that} \\ \left(\limsup\limits_n \Big(((u_n - \beta\, s)^-) \wedge s\Big)\right) \wedge \\ \qquad\qquad \wedge \left(\limsup\limits_n \Big(((u_n - \alpha\, s)^+) \wedge s\Big)\right) = 0 \\ \text{for every } s \in \widehat{E},\ 0 \leq s \leq w, \text{ and for every} \\ \alpha, \beta \in \mathbb{R},\ 0 < \beta < \alpha. \end{array} \right\}.$$

It can be shown (see Proposition 2 of [16]) that $B_{OS}((u_n)_n)$ and $B_{NOS}((u_n)_n)$ are projection bands in $\widehat{E}$; moreover, $\widehat{E}$ is the order direct sum of $B_{OS}((u_n)_n)$ and $B_{NOS}((u_n)_n)$. As in [16], we call $B_{OS}((u_n)_n)$ the band of oscillations of the sequence $(u_n)_n$.

Let $B_d((u_n)_n)$ be the (projection) band in $\widehat{E}$ generated by $B_\infty((u_n)_n) \cup B_{OS}((u_n)_n)$ ($B_d((u_n)_n)$ is called the band of divergence of the sequence $(u_n)_n$). We say that $(u_n)_n$ converges individually (on $\widehat{E}$) if $B_d((u_n)_n) = 0$.

If $(u_n)_{n\in\mathbb{N}}$ is a sequence of (not necessarily positive) elements of E, we say that $(u_n)_n$ converges individually (on $\widehat{E}$) if both sequences $(u_n^+)_n$ and $(u_n^-)_n$ converge individually (see [17]).

The main result of the paper is the following extension of Theorem 1:

Theorem 2. *Let E be an Archimedean Riesz space which has property $\mathcal{P}$, let $\widehat{E}$ be the Dedekind completion of E, and let $\widehat{E}'_c$ be the σ-order continuous dual of $\widehat{E}$. Let X be a Banach space, let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear operators, $T_n : X \to E$ for every $n \in \mathbb{N}$ such that T_n is continuous in duality for every $n \in \mathbb{N}$. Then the following assertions are true:*

- **a**) *If the sequence $(T_n)_{n\in\mathbb{N}}$ has property $\mathcal{D}$, then for every $v \in \widehat{E}$, $v \geq 0$, and for every $z \in \widehat{E}'_c$, $z \geq 0$, there exists a monotone nonincreasing function $C_{v,z} : (0, +\infty) \to \mathbb{R}$ such that $\lim_{\lambda\to+\infty} C_{v,z}(\lambda) = 0$, and such that $\sup_n \langle P_{(\bigvee_{l=1}^n |T_l x| - \lambda\|x\|v)^+} v,\, z\rangle \leq C_{v,z}(\lambda)$ for every $x \in X$ and $\lambda \in \mathbb{R}$, $\lambda > 0$.*
- **b**) *If for every $v \in \widehat{E}$, $v \geq 0$ and for every $z \in \widehat{E}'_c$, $z \geq 0$, there exists a function $C_{v,z}$ as in a), then the set $\mathcal{F} = \{x \in X \mid$ the sequence $(T_n x)_{n\in\mathbb{N}}$ converges individually on $\widehat{E}\}$ is closed in the norm topology of X.*

Let E be an Archimedean Riesz space, and let $\widehat{E}$ be the Dedekind completion of E. It is easy to see that E has property $\mathcal{P}$ if and only if $\widehat{E}$ has property $\mathcal{P}$. If $(u_n)_n$ is a sequence of elements of E, then we may also think of $(u_n)_n$ as a sequence of elements of $\widehat{E}$; it follows that $(u_n)_n$ converges individually on $\widehat{E}$ as a sequence of elements of E if and only if it converges individually on $\widehat{E}$ whenever we think of $(u_n)_n$ as a sequence of elements of $\widehat{E}$. If X is a Banach space, if $(T_n)_{n\in\mathbb{N}}$ is a sequence of linear operators, $T_n : X \to E$ for every $n \in \mathbb{N}$, and if we think of E as a Riesz subspace of $\widehat{E}$, then we can define $\widehat{T}_n : X \to \widehat{E}$, $\widehat{T}_n(x) = T_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$; it is easy to see that T_n is continuous in duality if and only if $\widehat{T}_n$ is continuous in duality and that the sequence $(T_n)_n$ has property $\mathcal{D}$ if and only if $(\widehat{T}_n)_n$ has property $\mathcal{D}$. We conclude that in order to prove Theorem 2 it is enough to prove it under the assumption that E is an order complete Riesz space (that is, we may assume that $E = \widehat{E}$ in Theorem 2).

The paper is organized as follows: in the next section (Section 2) we will discuss several technical results; in Section 3 we obtain a reformulation of the definition of the individually convergent sequences; finally, in the last section (Section 4) we will use the results obtained in Section 2 and Section 3 in order to prove Theorem 2.

2 – Some useful results

As mentioned in Introduction, we will be concerned in this section with several results we need in order to prove Theorem 2.

In the next two lemmas we will assume given a Riesz space G. For $w \in G$, we will denote by $H(w)$ the principal projection band generated by the singleton $\{w\}$ and by Q_w the band projection associated with $H(w)$ (naturally, the notations will be used provided that the principal projection band exists). The proofs of the two lemmas are straightforward: the assertion of Lemma 3 is a consequence of a well-known property of principal projection bands (see, for example, Theorem 24.7, pp. 135–136 of [6]); the proof of Lemma 4 follows from the equality $Q_{s+t} = Q_s + Q_t - Q_{s \wedge t}$ which is valid whenever s, t are projection elements in G, $s \geq 0$, $t \geq 0$ (see, for example, p. 36 of [2]).

Lemma 3. *Assume that G is order complete, and let A be a subset of positive elements of G such that* $\sup A$ *exists in G. Then $Q_{\sup A} u = \sup_{v \in A} Q_v u$ for every $u \in G$, $u \geq 0$. (Note that $\sup_{v \in A} Q_v u$ exists in G since $Q_v u$ is a component of u; hence, the set $\{Q_v u \mid v \in A\}$ is bounded above by u.)* ∎

Lemma 4. *Assume that G has the projection property, and let $u, v, w \in G$ be such that $u \geq 0$, $v \geq 0$ and $w \geq 0$.*

a) *If v is a component of u, and if $w = u - v$, then $Q_u = Q_v + Q_w$.*

b) *If $u \leq v + w$, then $Q_u \leq Q_v + Q_w$.* ∎

Let E be an order complete Riesz space, let E'_c be the vector space of all order bounded σ-order continuous linear functionals on E, and let $(E'_c)'$ be the order dual of E'_c. Consider the natural embedding $\psi:\ E \to (E'_c)'$, $\psi(u):\ E'_c \to \mathbb{R}$, $\psi(u)(x) = x(u)$ ($= \langle u, x \rangle$) for every $x \in E'_c$ and $u \in E$. It is well-known that ψ is a lattice preserving operator; also well-known is the fact that ψ is one-to-one if and only if E'_c separates the points of E, that is, ψ is one-to-one if and only if E has property $\mathcal{P}$ (for a discussion of the above-mentioned properties of the natural embedding, see pp. 58–59 of [2]).

Proposition 5. *If an order complete Riesz space E has property $\mathcal{P}$, then the natural embedding $\psi\colon\ E \to (E'_c)'$ preserves countable infima and suprema.*

Proof: Since E has property $\mathcal{P}$ we may think of E as a Riesz subspace of $(E'_c)'$. Thus, in order to prove that ψ preserves countable suprema and infima, it is enough to prove that $u_n \downarrow 0$ in E implies that $u_n \downarrow 0$ in $(E'_c)'$.

To this end, let $u_n \downarrow 0$ in E. Since $(E'_c)'$ is order complete, it follows that there exists $u' \in (E'_c)'$ such that $u_n \downarrow u'$ in $(E'_c)'$.

Let $0 \leq x \in E'_c$. By Proposition 4.2, p. 72 of [9], we obtain that $\langle x, u_n\rangle \downarrow \langle x, u''\rangle$. Since $u_n \downarrow 0$ in E, it also follows that $\langle x, u_n\rangle \downarrow 0$. We conclude that $u' = 0$. ∎

3 – A characterization of the individually convergent sequences

As pointed out in Introduction, in this section we will obtain a rather simple necessary and sufficient condition for the individual convergence of a sequence of elements of an Archimedean Riesz space.

We mentioned in Introduction that in order to prove Theorem 2, we may restrict our attention to order complete Riesz spaces, only. Thus, from now on throughout the paper (unless otherwise stated), we will assume given an order complete Riesz space E.

Lemma 6. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E, and assume that $B_\infty((u_n)_n) = 0$. Then for every $t \in E$, $t > 0$, there exists $s \in E$, $0 < s \leq t$, such that the sequence $(P_s u_n)_n$ is order bounded in E.*

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E such that $B_\infty((u_n)_n) = 0$, and let $t \in E$, $t > 0$.

Set $v_n = \bigvee_{k=1}^n u_k$ for every $n \in \mathbb{N}$.

Since $B_\infty((u_n)_n) = 0$, it follows that $(u_n)_n$ is not unbounded on $B(t)$; hence, the sequence $(v_n)_n$ is not unbounded on $B(t)$, as well.

Using Lemma 9-(a) of [14], we conclude that $(v_n)_n$ does not diverge individually to ∞ on $B(t)$.

Since $(v_n)_n$ is a monotonic nondecreasing sequence, we may apply Lemma 7 of [14]. Accordingly, there exist a nonzero component s of t and $\lambda \in \mathbb{R}$, $\lambda > 0$, such that $P_s v_n \leq \lambda s$ for every $n \in \mathbb{N}$. Clearly, the sequence $(P_s u_n)_n$ is order bounded since $0 \leq P_s u_n \leq \lambda s$ for every $n \in \mathbb{N}$. ∎

Lemma 7. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E, and suppose that $B_{OS}((u_n)_n) = 0$. Let $s \in E$, $s \geq 0$, and assume that the sequence $(P_s u_n)_n$ is order bounded in E. Then, $\liminf_n P_s u_n = \limsup_n P_s u_n$.*

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E such that $B_{OS}((u_n)_n) = 0$, let $s \in E$, $s \geq 0$, and suppose that the sequence $(P_s u_n)_n$ is order bounded in E.

Set $u_S = \limsup_n P_s u_n$ and $u_I = \liminf_n P_s u_n$, and assume that $u_I \neq u_S$.

Now set $t = u_S - u_I$. Then $t > 0$. Our goal is to prove that $t \in B_{OS}((u_n)_n)$; thus, we obtain a contradiction since we assume that $B_{OS}((u_n)_n) = 0$.

Let $q \in E$, $0 < q \leq t$; we will study the following two cases:

i) $q \wedge u_I = 0$

and

ii) $q \wedge u_I \neq 0$.

The proof of the proposition will be completed as soon as we show that in each case there exist $r \in E$, $0 < r \leq q$, $\alpha, \beta \in \mathbb{R}$, $0 < \beta < \alpha$, such that $(\limsup_n(((u_n - \beta r)^-) \wedge r)) \wedge (\limsup_n(((u_n - \alpha r)^+) \wedge r)) \neq 0$.

i) Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \beta < \alpha < 1$, and set $r = q$.

Taking into consideration that $u_S \in B(s)$, and $u_I \in B(s)$, we obtain that $q \in B(s)$; therefore, $((u_n - \beta q)^-) \wedge q \in B(s)$ for every $n \in \mathbb{N}$.

It follows that

$$
\begin{aligned}
(7.1)\qquad \limsup_n\Big(((u_n - \beta q)^-) \wedge q\Big) &= \limsup_n\Big(P_s\Big(((u_n - \beta q)^-) \wedge q\Big)\Big) = \\
&= \limsup_n\Big(((\beta q - P_s u_n)^+) \wedge q\Big) = \Big((\beta q - \liminf_n P_s u_n) \vee 0\Big) \wedge q \\
&= P_q\Big(\Big((\beta q - u_I) \vee 0\Big) \wedge q\Big) = \Big((\beta q - P_q u_I) \vee 0\Big) \wedge q = (\beta q) \wedge q = \beta q\ .
\end{aligned}
$$

It also follows that

$$
\begin{aligned}
(7.2)\qquad \limsup_n\Big(((u_n - \alpha q)^+) \wedge q\Big) &= \limsup_n\Big(P_s\Big(((u_n - \alpha q)^+) \wedge q\Big)\Big) = \\
&= \Big(\Big((\limsup_n P_s u_n) - \alpha q\Big) \vee 0\Big) \wedge q = \Big((u_S - \alpha q) \vee 0\Big) \wedge q \\
&= \Big((u_S - u_I + u_I - \alpha q) \vee 0\Big) \wedge q \geq (q - \alpha q) \wedge q = (1 - \alpha)\, q\ .
\end{aligned}
$$

Using (7.1) and (7.2), we conclude that

$$
\Big(\limsup_n\Big(((u_n - \beta q)^-) \wedge q\Big)\Big) \wedge \Big(\limsup_n\Big(((u_n - \alpha q)^+) \wedge q\Big)\Big) \geq \Big(\min\{1 - \alpha, \beta\}\Big)\, q \neq 0\ .
$$

ii) Using the Freudenthal spectral theorem (see, for example, [2, Theorem 6.8, pp. 82–83]), we obtain that there exists a nonzero component w of u_I in E and a real number a, $a > 0$, such that $aw \leq \frac{1}{2}(q \wedge u_I)$.

Set $r = aw$, $\alpha = \frac{1}{a} + 1$ and $\beta = \frac{1}{a} + \frac{1}{2}$.

Since w is a component of u_I, and since $u_I \in B(s)$, it follows that $((u_n - \beta a w)^-) \wedge (aw) \in B(s)$ for every $n \in \mathbb{N}$. Thus, we obtain that

$$\begin{aligned}\limsup_n \Big(((u_n - \beta a w)^-) \wedge (aw) \Big) &= \limsup_n P_s \Big(((u_n - \beta a w)^-) \wedge (aw) \Big) = \\ &= \limsup_n P_s \Big(((\beta a w - u_n)^+) \wedge (aw) \Big) = \Big(\beta a w - \liminf_n P_s u_n \Big)^+ \wedge (aw) \\ &= \Big(w + \frac{a}{2} w - u_I \Big)^+ \wedge (aw) \ .\end{aligned}$$

Since $w - u_I \leq 0$, and since $(\frac{a}{2} w) \wedge (u_I - w) = 0$, it follows that $(w + \frac{a}{2} w - u_I)^+ = \frac{a}{2} w$; therefore, $(w + \frac{a}{2} w - u_I)^+ \wedge (aw) = \frac{a}{2} w$. Accordingly,

$$\limsup_n \Big(((u_n - \beta a w)^-) \wedge (aw) \Big) = \frac{a}{2} w \ . \tag{7.3}$$

Using again the fact that w is a component of u_I and $u_I \in B(s)$, we deduce that $(u_n - \alpha a w)^+ \wedge (aw) \in B(s)$ for every $n \in \mathbb{N}$. Accordingly,

$$\begin{aligned}\limsup_n \Big(((u_n - \alpha a w)^+) \wedge (aw) \Big) &= \limsup_n P_s \Big(((u_n - \alpha a w)^+) \wedge (aw) \Big) = \\ &= \bigg(\Big(\limsup_n (P_s u_n) - \alpha a w \Big)^+ \bigg) \wedge (aw) = \bigg(\Big(u_S - \Big(\frac{1}{a} + 1 \Big) aw \Big)^+ \bigg) \wedge (aw) \\ &= \Big(u_S - w - aw \Big)^+ \wedge (aw) = \Big(u_S - u_I + u_I - w - aw \Big)^+ \wedge (aw) \\ &\geq \Big((q \wedge u_I) - aw \Big)^+ \wedge (aw) \geq (2aw - aw) \wedge (aw) = aw \ .\end{aligned} \tag{7.4}$$

In view of (7.3) and (7.4), we conclude that

$$\bigg(\limsup_n \Big(((u_n - \beta a w)^-) \wedge (aw) \Big) \bigg) \wedge \bigg(\limsup_n \Big(((u_n - \alpha a w)^+) \wedge (aw) \Big) \bigg) \geq \frac{a}{2} w \ . \ \blacksquare$$

Proposition 8. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of positive elements of E, and assume that $(u_n)_n$ converges individually on E. Then for every $t \in E$, $t > 0$, there exists $s \in E$, $0 < s \leq t$, such that the sequence $(P_s u_n)_n$ is order bounded in E, and such that* $\liminf_n P_s u_n = \limsup_n P_s u_n$.

Proof: Let $t \in E$, $t > 0$. Since $B_\infty((u_n)_n) = 0$, by Lemma 6, there exists $s \in E$, $0 < s \leq t$, such that the sequence $(P_s u_n)_n$ is order bounded in E. Taking into consideration that $B_{OS}((u_n)_n) = 0$, and using Lemma 7, we obtain that $\liminf_n P_s u_n = \limsup_n P_s u_n$. $\blacksquare$

Given a doubly indexed sequence $(u_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$, set $\limsup_{(n,k)} u_{nk} = \bigwedge_{l\in\mathbb{N}} \bigvee_{\substack{n\ge l\\ k\ge l}} u_{nk}$ and $\liminf_{(n,k)} u_{nk} = \bigvee_{l\in\mathbb{N}} \bigwedge_{\substack{n\ge l\\ k\ge l}} u_{nk}$ whenever the right hand sides of the above equalities exist.

Now let $(u_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a doubly indexed sequence of elements of E such that $u_{nk}\ge 0$ for every $(n,k)\in\mathbb{N}\times\mathbb{N}$. We say that $(u_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero if $\limsup_{(n,k)}(u_{nk}\wedge v)=0$ for every $v\in E$, $v\ge 0$.

Lemma 9. *Let $(u_n)_{n\in\mathbb{N}}$ be a seequence of positive elements of E. Assume that $(u_n)_n$ is order bounded in E, and that $\liminf_n u_n = \limsup_n u_n$. Set $v_{nk} = u_n - u_k$ for every $n,k\in\mathbb{N}$. Then the doubly indexed sequence $(|v_{nk}|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero.*

Proof: Since $(u_n)_n$ is order bounded and since $0\le |v_{nk}|\le u_n+u_k$ for every $(n,k)\in\mathbb{N}\times\mathbb{N}$ it follows that $(|v_{nk}|)_{(n,k)}$ is also order bounded; therefore, $\limsup_{(n,k)}|v_{nk}|$ exists in E. Consequently, in order to prove that $(|v_{nk}|)_{(n,k)}$ converges individually to zero, it is enough to prove that $\limsup_{(n,k)}|v_{nk}|=0$.

Since

$$\begin{aligned}\limsup_{(n,k)}|v_{nk}| &= \bigwedge_{l\in\mathbb{N}}\bigvee_{\substack{n\ge l\\ k\ge l}}(v^+_{nk}+v^-_{nk})\\ &\le \Big(\bigwedge_{l\in\mathbb{N}}\bigvee_{\substack{n\ge l\\ k\ge l}} v^+_{nk}\Big)+\Big(\bigwedge_{l\in\mathbb{N}}\bigvee_{\substack{n\ge l\\ k\ge l}} v^-_{nk}\Big)\\ &= \limsup_{(n,k)} v^+_{nk}+\limsup_{(n,k)} v^-_{nk}\ ,\end{aligned}$$

it follows that the proof of the lemma is completed once we prove that $\limsup_{(n,k)} v^+_{nk}=0$ and $\limsup_{(n,k)} v^-_{nk}=0$.

Now,

$$\begin{aligned}0\le \limsup_{(n,k)} v^+_{nk} &= \bigwedge_{l\in\mathbb{N}}\bigvee_{\substack{n\ge l\\ k\ge l}}\Big((u_n-u_k)\vee 0\Big)\\ &= \bigwedge_{l\in\mathbb{N}}\bigvee_{n\ge l}\Big(\Big(u_n+\Big(\bigvee_{k\ge l}(-u_k)\Big)\Big)\vee 0\Big)\\ &= \bigwedge_{l\in\mathbb{N}}\Big(\Big(\Big(\bigvee_{n\ge l}u_n\Big)-\Big(\bigwedge_{k\ge l}u_k\Big)\Big)\vee 0\Big)\\ &= \Big(\Big(\bigwedge_{l\in\mathbb{N}}\bigvee_{n\ge l}u_n\Big)-\Big(\bigvee_{l\in\mathbb{N}}\bigwedge_{k\ge l}u_k\Big)\Big)\vee 0 = 0\ .\end{aligned}$$

Thus, $\limsup_{(n,k)} v^+_{nk}=0$.

Taking into consideration that $v_{nk}^{-} = v_{kn}^{+}$ for every $(n,k) \in \mathbb{N} \times \mathbb{N}$, we obtain that $\limsup_{(n,k)} v_{nk}^{-} = \limsup_{(n,k)} v_{kn}^{+} = \limsup_{(k,n)} v_{kn}^{+} = 0$. ∎

Proposition 10. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E, and assume that $(u_n)_n$ converges individually on E. Set $v_{nk} = u_n - u_k$ for every $n,k \in \mathbb{N}$. Then the doubly indexed sequence $(|v_{nk}|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero.*

Proof: Assume that $(|v_{nk}|)_{(n,k)}$ does not converge individually to zero. Then there exists $v \in E$, $v > 0$ such that $\limsup_{(n,k)}(|v_{nk} \wedge v|) \neq 0$.

Set $w = \limsup_{(n,k)}(|v_{nk}| \wedge v)$. Taking into consideration that $w > 0$, by Proposition 8, we obtain that there exists $s \in E$, $0 < s \leq w$, such that the sequence $(P_s u_n)_n$ is order bounded in E, and such that $\limsup_n P_s u_n = \liminf_n P_s u_n$.

It follows that $\limsup_{(n,k)} |P_s u_n - P_s u_k| \neq 0$ since

$$s = \limsup_{(n,k)}(|v_{nk}| \wedge s) \leq \limsup_{(n,k)}(P_s(|v_{nk}|)) = \limsup_{(n,k)} |P_s u_n - P_s u_k| \ .$$

We have obtained a contradiction since, by Lemma 9, the doubly indexed sequence $(|P_s u_n - P_s u_k|)_{(n,k)}$ converges individually to zero. ∎

Proposition 11. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E, and set $v_{nk} = u_n - u_k$ for every $(n,k) \in \mathbb{N} \times \mathbb{N}$. If the doubly indexed sequence $(|v_{nk}|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero, then $B_\infty((u_n)_n) = 0$.*

Proof: Assume that $B_\infty((u_n)_n) \neq 0$. Then there exists $v \in B_\infty((u_n)_n)$, $v > 0$.

Let $l \in \mathbb{N}$. Using Lemma 3 of [16] and Lemma 4 of [14], we obtain that the sequence $((u_n - u_l)^+)_{n\geq l}$ is unbounded on $B_\infty((u_n)_n)$. Thus,

$$\sup_{\substack{n\geq l\\ k\geq l}}\Big(|u_n - u_k| \wedge v\Big) \geq \sup_{n\geq l}\Big(|u_n - u_l| \wedge v\Big) \geq \sup_{n\geq l}\Big((u_n - u_l)^+ \wedge v\Big) = v \ .$$

Accordingly, $\inf_l \sup_{\substack{n\geq l\\ k\geq l}}(|u_n - u_k| \wedge v) = v \neq 0$. We have obtained a contradiction since the doubly indexed sequence $(|v_{nk}|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero. ∎

Proposition 12. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E such that the doubly indexed sequence $(|u_n - u_k|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero. If $s \in E$, $s \geq 0$ is such that the sequence $(P_s u_n)_n$ is order bounded in E, then $\limsup_n P_s u_n = \liminf_n P_s u_n$.*

Proof: Let $s \in E$, $s \geq 0$, be such that the sequence $(P_s u_n)_n$ is order bounded, and assume that $\limsup_n P_s u_n \neq \liminf_n P_s u_n$.

Set $w = \limsup_n P_s u_n - \liminf_n P_s u_n$. Clearly, $w > 0$.

We now note that since $w \in B(s)$, it follows that $(u_n - u_k)^+ \wedge w \in B(s)$, so $P_s((u_n - u_k)^+ \wedge w) = (u_n - u_k)^+ \wedge w$ for every $(n,k) \in \mathbb{N} \times \mathbb{N}$. Thus, we obtain that

$$\begin{aligned}
0 &= \limsup_{(n,k)} \Big(|u_n - u_k| \wedge w \Big) \geq \limsup_{(n,k)} \Big((u_n - u_k)^+ \wedge w \Big) \\
&= \limsup_{(n,k)} P_s \Big((u_n - u_k)^+ \wedge w \Big) = \bigwedge_l \bigvee_{\substack{n \geq l \\ k \geq l}} \Big(\Big((P_s u_n - P_s u_k) \vee 0 \Big) \wedge w \Big) \\
&= \Big(\Big(\bigwedge_l \bigvee_{\substack{n \geq l \\ k \geq l}} (P_s u_n - P_s u_k) \Big) \vee 0 \Big) \wedge w \\
&= \Big(\Big(\Big(\bigwedge_l \bigvee_{n \geq l} P_s u_n \Big) - \Big(\bigvee_l \bigwedge_{k \geq l} P_s u_k \Big) \Big) \vee 0 \Big) \wedge w \\
&= \Big(\limsup_n P_s u_n - \liminf_n P_s u_n \Big)^+ \wedge w = w \ .
\end{aligned}$$

We have obtained a contradiction since we have assumed that $w \geq 0$, $w \neq 0$. ∎

The next proposition offers a converse to Proposition 10.

Proposition 13. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of positive elements of E. If the doubly indexed sequence $(|u_n - u_k|)_{(n,k) \in \mathbb{N} \times \mathbb{N}}$ converges individually to zero, then the sequence $(u_n)_n$ converges individually.*

Proof: By Proposition 11, $B_\infty((u_n)_n) = 0$. Thus, in order to prove that the sequence $(u_n)_n$ converges individually we have to prove that $B_{OS}((u_n)_n) = 0$.

To this end, assume that $B_{OS}((u_n)_n) \neq 0$. Then, there exists $t \in B_{OS}((u_n)_n)$, $t > 0$. By Lemma 6, there exists $s \in B_{OS}((u_n)_n)$, $0 < s \leq t$, such that the sequence $(P_s u_n)_n$ is order bounded. By Proposition 12, $\limsup_n P_s u_n = \liminf_n P_s u_n$.

Our goal is to prove that $s \in B_{NOS}((u_n)_n)$; hence, we will obtain a contradiction. Thus, we have to prove that

$$\Big(\limsup_n \Big(((u_n - \beta w)^-) \wedge w \Big) \Big) \wedge \Big(\limsup_n \Big(((u_n - \alpha w)^+) \wedge w \Big) \Big) = 0$$

for every $w \in E$, $0 \leq w \leq s$, and for every $\alpha, \beta \in \mathbb{R}$, $0 < \beta < \alpha$.

To this end, let $w \in E$, $0 \leq w \leq s$, and let $\alpha, \beta \in \mathbb{R}$, $0 < \beta < \alpha$.

Taking into consideration that $w \in B(s)$, we obtain that $(u_n - \beta w)^- \wedge w \in B(s)$, and $(u_n - \alpha w)^+ \wedge w \in B(s)$; therefore, $P_s((u_n - \beta w)^- \wedge w) = (u_n - \beta w)^- \wedge w$ and $P_s((u_n - \alpha w)^+ \wedge w) = (u_n - \alpha w)^+ \wedge w$ for every $n \in \mathbb{N}$. Accordingly,

$$\begin{aligned}
&\Big(\limsup_n \Big(((u_n - \beta w)^-) \wedge w\Big)\Big) \wedge \Big(\limsup_n \Big(((u_n - \alpha w)^+) \wedge w\Big)\Big) = \\
&\quad = \Big(\limsup_n P_s\Big(((u_n - \beta w)^-) \wedge w\Big)\Big) \wedge \Big(\limsup_n P_s\Big(((u_n - \alpha w)^+) \wedge w\Big)\Big) \\
&\quad = \Big(\limsup_n \Big(((P_s u_n - \beta w)^-) \wedge w\Big)\Big) \wedge \Big(\limsup_n \Big(((P_s u_n - \alpha w)^+) \wedge w\Big)\Big) \\
&\quad = \Big(\limsup_n \Big(((\beta w - P_s u_n) \vee 0) \wedge w\Big)\Big) \wedge \Big(\limsup_n \Big(((P_s u_n - \alpha w) \vee 0) \wedge w\Big)\Big) \\
&\quad = \Big(\Big((\beta w - \liminf_n P_s u_n) \vee 0\Big) \wedge w\Big) \wedge \Big(\Big(\Big((\limsup_n P_s u_n) - \alpha w\Big) \vee 0\Big) \wedge w\Big) \\
&\quad = \Big((\beta w - \limsup_n P_s u_n)^+ \wedge w\Big) \wedge \Big(\Big((\limsup_n P_s u_n) - \alpha w\Big)^+ \wedge w\Big) \\
&\quad \leq (\alpha w - \limsup_n P_s u_n)^+ \wedge \Big((\limsup_n P_s u_n) - \alpha w\Big)^+ = 0\ .
\end{aligned}$$

We conclude that

$$\Big(\limsup_n \Big(((u_n - \beta w)^-) \wedge w\Big)\Big) \wedge \Big(\limsup_n \Big(((u_n - \alpha w)^+) \wedge w\Big)\Big) = 0\ . \ \blacksquare$$

Proposition 10 and Proposition 13 can be summarized as follows:

Theorem 14. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E. Then $(u_n)_n$ converges individually if and only if the doubly indexed sequence $(|u_n - u_k|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero.* $\blacksquare$

The characterization of individually convergent sequences described in the last theorem is valid even in the more general situation in which we do not assume the positivity of the terms of the sequence $(u_n)_{n\in\mathbb{N}}$. We conclude the section with a theorem which discusses this more general situation.

Theorem 15. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of (not necessarily positive) elements of E. Then $(u_n)_n$ converges individually if and only if the doubly indexed sequence $(|u_n - u_k|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero.*

Proof: Assume first that the sequence $(u_n)_n$ converges individually. Accordingly, the sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge individually, as well. By Theorem 14, the doubly indexed sequences $(|u_n^+ - u_k^+|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ and $(|u_n^- - u_k^-|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converge individually to zero. Thus, using Corollary, p. 53 of [9], we obtain that

$$\begin{aligned}
0 &\le \bigwedge_{l\in\mathbb{N}} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n - u_k| \wedge v\Big) \le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(\Big(|u_n^+ - u_k^+| + |u_n^- - u_k^-|\Big) \wedge v\Big)\\
&\le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(\Big(|u_n^+ - u_k^+| \wedge v\Big) + \Big(|u_n^- - u_k^-| \wedge v\Big)\Big)\\
&\le \bigwedge_{l} \Big(\Big(\bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^+ - u_k^+| \wedge v\Big)\Big) + \Big(\bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^- - u_k^-| \wedge v\Big)\Big)\Big)\\
&= \Big(\bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^+ - u_k^+| \wedge v\Big)\Big) + \Big(\bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^- - u_k^-| \wedge v\Big)\Big) = 0
\end{aligned}$$

for every $v \in E$, $v \ge 0$.

It follows that $\limsup_{(n,k)}(|u_n - u_k| \wedge v) = 0$ for every $v \in E$, $v \ge 0$; that is, the doubly indexed sequence $(|u_n - u_k|)_{(n,k)}$ converges individually to zero.

Now, assume that the doubly indexed sequence $(|u_n - u_k|)_{(n,k)}$ converges individually to zero.

By Theorem 1.6, pp. 6–7 of [2], it follows that $|u_n^+ - u_k^+| \le |u_n - u_k|$, and $|u_n^- - u_k^-| \le |u_n - u_k|$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$; therefore,

$$0 \le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^+ - u_k^+| \wedge v\Big) \le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n - u_k| \wedge v\Big) = 0$$

and

$$0 \le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n^- - u_k^-| \wedge v\Big) \le \bigwedge_{l} \bigvee_{\substack{n\ge l\\ k\ge l}} \Big(|u_n - u_k| \wedge v\Big) = 0$$

for every $v \in E$, $v \ge 0$. Accordingly, $\limsup_{(n,k)}(|u_n^+ - u_k^+| \wedge v) = 0$, and $\limsup_{(n,k)}(|u_n^- - u_k^-| \wedge v) = 0$ for every $v \in E$, $v \ge 0$; therefore, the doubly indexed sequences $(|u_n^+ - u_k^+|)_{(n,k)}$ and $(|u_n^- - u_k^-|)_{(n,k)}$ converge individually to zero. By Theorem 14, the sequences $(u_n^+)_n$ and $(u_n^-)_n$ converge individually; hence, $(u_n)_n$ converges individually, as well. ∎

Observation. Let G be an Archimedean Riesz space and let $\widehat{G}$ be the Dedekind completion of G. All the results of this section can be stated for sequences of elements of G. To this end, we think of G as a Riesz subspace of $\widehat{G}$

and we take all suprema and infima in $\widehat{G}$. Thus, if $(u_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ is a doubly indexed sequence of elements of G (or $\widehat{G}$) we set $\limsup_{(n,k)} u_{nk} = \bigwedge_{l\in\mathbb{N}} \bigvee_{\substack{n\geq l\\ k\geq l}} u_{nk}$ and $\liminf_{(n,k)} u_{nk} = \bigvee_{l\in\mathbb{N}} \bigwedge_{\substack{n\geq l\\ k\geq l}} u_{nk}$ whenever the suprema and infima exist in $\widehat{G}$; if $u_{nk} \geq 0$ for every $(n,k) \in \mathbb{N}\times\mathbb{N}$, then we say that $(u_{nk})_{(n,k)}$ converges individually to zero if $\limsup_{(n,k)}(u_{nk}\wedge v) = 0$ for every $v \in \widehat{G}$, $v \geq 0$. In this setting, Theorem 15 states that a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of G converges individually if and only if the doubly indexed sequence $(|u_n - u_k|)_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero.

4 – The individual convergence in Riesz spaces with property $\mathcal{P}$ and the Banach principle

As stated in Introduction, our goal in this section is to prove Theorem 2. To this end, we need several lemmas.

Lemma 16. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and assume that $(u_n)_n$ converges individually on E. Set $v_{n,k} = |u_n - u_k|$ for every $n \in \mathbb{N}$, $k \in \mathbb{N}$. Let $v \in E$, $v \geq 0$, and let z be a positive σ-order continuous linear functional on E. Then for every $\delta \in \mathbb{R}$, $\delta > 0$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $s \in \mathbb{N}$ such that $\langle P_{(\bigvee_{\substack{n=s\\ k=s}}^{t} v_{nk}-\delta v)^+}v,\, z\rangle < \varepsilon$ for every $t \in \mathbb{N}$, $t \geq s$.*

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E such that $(u_n)_n$ converges individually, let $v \in E$, $v \geq 0$, and let z be a positive σ-order continuous linear functional on E. Set $v_{nk} = |u_n - u_k|$ for every $n \in \mathbb{N}$, $k \in \mathbb{N}$; also set $w_{st} = \bigvee_{\substack{n=s\\ k=s}}^{t} v_{nk}$ for every $s \in \mathbb{N}$, $t \in \mathbb{N}$, $s \leq t$.

Assume that the assertion of the lemma fails to be true for $(u_n)_n$, v and z. Then there exist $\delta \in \mathbb{R}$, $\delta > 0$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, such that for every $s \in \mathbb{N}$, there exists $t \in \mathbb{N}$, $t \geq s$, with the property that $\langle P_{(w_{st}-\delta v)^+}v,\, z\rangle \geq \varepsilon$.

We now note that we may assume that $0 < \delta < 1$ since $\langle P_{(w_{st}-\delta v)^+}v,\, z\rangle \leq \langle P_{(w_{st}-\delta' v)^+}v,\, z\rangle$ for every $s \in \mathbb{N}$, $t \in \mathbb{N}$, $s \leq t$, and for every $\delta \in \mathbb{R}$, $\delta' \in \mathbb{R}$, $0 < \delta' < \delta$.

Clearly, $\langle \sup_{t\geq s} P_{(w_{st}-\delta v)^+}v,\, z\rangle \geq \varepsilon$ for every $s \in \mathbb{N}$. Taking into consideration that the sequence $(\sup_{t\geq s} P_{(w_{st}-\delta v)^+}v)_{s\in\mathbb{N}}$ is an order bounded monotone nonincreasing sequence of elements of E, and since z is a σ-order continuous linear functional on E, we obtain that

$$\varepsilon \leq \inf_{s\in\mathbb{N}} \Big\langle \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v,\, z \Big\rangle = \Big\langle \inf_{s\in\mathbb{N}} \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v,\, z \Big\rangle\ .$$

Consequently, $\inf_{s\in\mathbb{N}} \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v \neq 0$.

Set $\rho = \inf_{s\in\mathbb{N}} \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v$.

Taking into consideration that $0 \leq \rho \leq \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v \leq v$ for every $s\in\mathbb{N}$, and using Exercise 10-(b), p. 41 of [2], we obtain that

$$\delta_\rho \leq \delta\, P_\rho\Big(\sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_{(w_{st}-\delta v)^+} v\Big) = \delta\Big(\sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_\rho P_{(w_{st}-\delta v)^+} v\Big) = \tag{16.1}$$

$$= \delta\Bigg(\sup_{\substack{t\in\mathbb{N}\\ t\geq s}} P_\rho\Big(\sup_{l\in\mathbb{N}}\Big(p\wedge l((w_{st}-\delta v)^+)\Big)\Big)\Bigg) = \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} \sup_{l\in\mathbb{N}} \delta\, P_\rho\Big(v\wedge\Big(l((w_{st}-\delta v)^+)\Big)\Big)$$

$$= \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} \sup_{l\in\mathbb{N}}\Big((\delta\rho)\wedge\Big(\delta l((P_\rho w_{st}-\delta\rho)^+)\Big)\Big) = \sup_{\substack{t\in\mathbb{N}\\ t\geq s}} \sup_{l\in\mathbb{N}}\Big((\delta\rho)\wedge\Big(l((P_\rho w_{st}-\delta\rho)^+)\Big)\Big)$$

$$= \Bigg(\sup_{\substack{t\in\mathbb{N}\\ t\geq s}} \sup_{l\in\mathbb{N}}\Big((\delta\rho)\wedge\Big(l(P_\rho w_{st}-\delta\rho)\Big)\Big)\Bigg)^+$$

for every $s\in\mathbb{N}$.

By Theorem 15, the doubly indexed sequence $(v_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ converges individually to zero. Accordingly, $\inf_{s\in\mathbb{N}} \sup_{\substack{l\geq s\\ t\geq l}} (w_{lt}\wedge(\delta\rho)) = 0$.

Since $\rho \neq 0$, it follows that $\delta\rho \neq 0$, so there exists $s_0 \in \mathbb{N}$ such that $\sup_{\substack{l\geq s_0\\ t\geq l}} (w_{lt}\wedge(\delta\rho)) \neq \delta\rho$.

Clearly, $0 \leq \delta\rho - \sup_{\substack{l\geq s_0\\ t\geq l}} (w_{lt}\wedge(\delta\rho)) \leq \delta\rho$; therefore, using again Exercise 10-(b), p. 41 of [2], we obtain that

$$\delta\rho - \sup_{\substack{l\geq s_0\\ t\geq l}} (w_{lt}\wedge(\delta\rho)) = P_\rho\Big(\delta\rho - \sup_{\substack{l\geq s_0\\ t\geq l}} (w_{lt}\wedge(\delta\rho))\Big) = \tag{16.2}$$

$$= \delta\rho - \sup_{\substack{l\geq s_0\\ t\geq l}} P_\rho(w_{lt}\wedge(\delta\rho)) = \delta\rho - \sup_{\substack{l\geq s_0\\ t\geq l}} \Big((P_\rho w_{lt})\wedge(\delta\rho)\Big) =$$

$$
\begin{aligned}
&= \inf_{\substack{l \geq s_0 \\ t \geq l}} \Big(\delta\rho - \Big((P_\rho w_{lt}) \wedge (\delta\rho)\Big)\Big) = \inf_{\substack{l \geq s_0 \\ t \geq l}} \Big((\delta\rho - P_\rho w_{lt})^+\Big) \\
&\leq \inf_{\substack{l \geq s_0 \\ t \geq l}} \inf_{k \in \mathbb{N}} \Big(\Big(k((\delta\rho - P_\rho w_{lt})^+)\Big) \vee (-\delta\rho)\Big) \\
&= \inf_{\substack{l \geq s_0 \\ t \geq l}} \inf_{k \in \mathbb{N}} \Bigg(\Big(\Big(k(\delta\rho - P_\rho w_{lt})\Big) \vee (-\delta\rho)\Big)^+\Bigg) \\
&\leq \inf_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \inf_{k \in \mathbb{N}} \Bigg(\Big(\Big(k(\delta\rho - P_\rho w_{s_0 t})\Big) \vee (-\delta\rho)\Big)^+\Bigg) \\
&= \Bigg(\inf_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \inf_{k \in \mathbb{N}} \Big(\Big(k(\delta\rho - P_\rho w_{s_0 t})\Big) \vee (-\delta\rho)\Big)\Bigg)^+ \\
&= \Bigg(\sup_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \sup_{k \in \mathbb{N}} \Big(\Big(k(P_\rho w_{s_0 t} - \delta\rho)\Big) \wedge (\delta\rho)\Big)\Bigg)^- .
\end{aligned}
$$

Using (16.1) and (16.2), we conclude that

$$
\begin{aligned}
&\Bigg(\Bigg(\sup_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \sup_{k \in \mathbb{N}} \Big(\Big(k(P_\rho w_{s_0 t} - \delta\rho)\Big) \wedge (\delta\rho)\Big)\Bigg)^+\Bigg) \wedge \\
&\qquad \wedge \Bigg(\Bigg(\sup_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \sup_{k \in \mathbb{N}} \Big(\Big(k(P_\rho w_{s_0 t} - \delta\rho)\Big) \wedge (\delta\rho)\Big)\Bigg)^-\Bigg) \geq \\
&\qquad \geq (\delta\rho) \wedge \Big(\delta\rho - \sup_{\substack{l \geq s_0 \\ t \geq l}} \Big(w_{lt} \wedge (\delta\rho)\Big)\Big) .
\end{aligned}
$$

Since $0 \leq \delta\rho - \sup_{\substack{l \geq s_0 \\ t \geq l}} (w_{lt} \wedge (\delta\rho)) \leq \delta\rho$, $\delta\rho - \sup_{\substack{l \geq s_0 \\ t \geq l}} (w_{lt} \wedge (\delta\rho)) \neq 0$, it follows that

$$
\begin{aligned}
&\Bigg(\Bigg(\sup_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \sup_{k \in \mathbb{N}} \Big(\Big(k(P_\rho w_{s_0 t} - \delta\rho)\Big) \wedge (\delta\rho)\Big)\Bigg)^+\Bigg) \wedge \\
&\qquad \wedge \Bigg(\Bigg(\sup_{\substack{t \in \mathbb{N} \\ t \geq s_0}} \sup_{k \in \mathbb{N}} \Big(\Big(k(P_\rho w_{s_0 t} - \delta)\Big) \wedge (\delta\rho)\Big)\Bigg)^-\Bigg) \neq 0 ;
\end{aligned}
$$

therefore, we have obtained a contradiction. ∎

Lemma 17. *Let $(u_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a doubly indexed sequence of positive elements of E, let $v\in E$, $v\geq 0$, and let z be a σ-order continuous positive linear functional on E. Assume that $\sup_{\substack{n\geq l\\ k\geq l}}(u_{nk}\wedge v)=v$ for some $l\in\mathbb{N}$. Set $w_{st}=\bigvee_{\substack{n=s\\ k=s}}^{t}u_{nk}$ for every $s\in\mathbb{N}$, $t\in\mathbb{N}$, $t\geq s$. Then,*

$$\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}\Big\langle P_{(w_{ls}-\varepsilon v)^{+}}v,\ z\Big\rangle=\Big\langle\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v,\ z\Big\rangle=\langle v,z\rangle$$

for every $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$.

Proof: We first note that the sequence $(w_{ls})_{\substack{s\in\mathbb{N}\\ s\geq l}}$ is a monotonic nondecreasing sequence; therefore, $(P_{(w_{ls}-\varepsilon v)^{+}}v)_{\substack{s\in\mathbb{N}\\ s\geq l}}$ is also monotonic nondecreasing for every $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$. Since z is a σ-order continuous linear functional, it follows that

$$\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}\Big\langle P_{(w_{ls}-\varepsilon v)^{+}}v,\ z\Big\rangle=\Big\langle\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v,\ z\Big\rangle$$

for every $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$.

Thus, in order to complete the proof of the lemma, it is enough to show that $\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v=v$ for every $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$.

To this end, let $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$, and assume that $\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v\neq v$.

Set $w=\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v$.

Taking into consideration that $(w_{ls}-\varepsilon v)^{+}\wedge v\leq\sup_{k\in\mathbb{N}}((k(w_{ls}-\varepsilon v)^{+})\wedge v)=P_{(w_{ls}-\varepsilon v)^{+}}v\leq w$ for every $s\in\mathbb{N}$, $s\geq l$, we conclude that $\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}((w_{ls}-\varepsilon v)^{+}\wedge v)\leq w$.

Now,

$$\begin{aligned}\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}\Big((w_{ls}-\varepsilon v)^{+}\wedge v\Big)&=\Big(\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}\Big((w_{ls}-\varepsilon v)\wedge v\Big)\Big)^{+}=\Big(\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}\Big(\Big(w_{ls}\wedge(v+\varepsilon v)\Big)-\varepsilon v\Big)\Big)^{+}\geq\\ &\geq\Big(\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}(w_{ls}\wedge v)\Big)-\varepsilon v=\Big(\sup_{\substack{n\geq l\\ k\geq l}}(u_{nk}\wedge v)\Big)-\varepsilon v=v-\varepsilon v\ .\end{aligned}$$

Thus, $(1-\varepsilon)\,v\leq\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}((w_{ls}-\varepsilon v)^{+}\wedge v)\leq w$.

Since $(P_{(w_{ls}-\varepsilon v)^{+}}v)_{\substack{s\in\mathbb{N}\\ s\geq l}}$ is a sequence of components of v, Theorem 3.15, pp. 37–38 of [2] implies that w is a component of v, so $0\leq((1-\varepsilon)v)\wedge(v-w)\leq w\wedge(v-w)=0$; therefore, $((1-\varepsilon)v)\wedge(v-w)=0$.

We have obtained a contradiction since $((1-\varepsilon)v)\wedge(v-w)\geq(1-\varepsilon)\,(v-w)$, and since $(1-\varepsilon)\,(v-w)\neq 0$, $(1-\varepsilon)\,(v-w)\geq 0$.

Accordingly, $\sup_{\substack{s\in\mathbb{N}\\ s\geq l}}P_{(w_{ls}-\varepsilon v)^{+}}v=v$. ∎

We will assume from now on throughout this section given a Banach space X and a sequence $(T_n)_{n\in\mathbb{N}}$ of linear operators, $T_n\colon X\to E$ for every $n\in\mathbb{N}$.

For every $n\in\mathbb{N}$, we define a mapping $T_n^*: X\to E$ as follows: $T_n^*x = \bigvee_{k=1}^n |T_kx|$ for every $x\in X$.

Lemma 18. *If T_n is continuous in duality for every $n\in\mathbb{N}$, then T_n^* is also continuous in duality for every $n\in\mathbb{N}$.*

Proof: Let $l\in\mathbb{N}$, let $v\in E$, $v\geq 0$, let $z\in E_c'$, $z\geq 0$, let $x\in X$, let $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements of X which norm converges to x in X, and let $\varepsilon\in\mathbb{R}$, $\varepsilon>0$.

Using Proposition 1.4-(6), p. 51 of [9], we obtain that

$$\Big(|T_l^*x_k - T_l^*x| - \varepsilon v\Big)^+ \leq \left(\sum_{n=1}^{l} |T_nx_k - T_nx| - l\,\frac{\varepsilon v}{l}\right)^+ \leq \sum_{n=1}^{l}\left(|T_nx_k - T_nx| - \frac{\varepsilon v}{l}\right)^+$$

for every $n\in\mathbb{N}$.

Now let $\delta>0$. Since T_n is continuous in duality for every $n=1,2,...,l$, it follows that there exists $k_\delta\in\mathbb{N}$ large enough such that $\langle P_{(|T_nx_k-T_nx|-\frac{\varepsilon}{l}v)^+}v,\, z\rangle < \frac{\delta}{l}$ for every $n=1,2,...,l$ and $k\in\mathbb{N}$, $k\geq k_\delta$.

By Lemma 4-b),

$$\begin{aligned} 0 &\leq \Big\langle P_{(|T_l^*x_k-T_l^*x|-\varepsilon v)^+}v,\, z\Big\rangle \leq \Big\langle P_{\sum_{n=1}^l(|T_nx_k-T_nx|-\frac{\varepsilon}{l}v)^+}v,\, z\Big\rangle \\ &\leq \sum_{n=1}^{l}\Big\langle P_{(|T_nx_k-T_nx|-\frac{\varepsilon}{l}v)^+}v,\, z\Big\rangle < l\,\frac{\delta}{l} = \delta \end{aligned}$$

for every $k\geq k_\delta$.

Thus, for every $\delta>0$, there exists $k_\delta\in\mathbb{N}$ such that $0\leq \langle P_{(|T_l^*x_k-T_l^*x|-\varepsilon v)^+}v,\, z\rangle < \delta$ for every $k\geq k_\delta$; accordingly, the sequence $(\langle P_{(|T_l^*x_k-T_l^*x|-\varepsilon v)^+}v,\, z\rangle)_{k\in\mathbb{N}}$ converges to zero.

We have therefore proved that for every $l\in\mathbb{N}$ the mapping T_l^* is continuous in duality. ∎

Lemma 19. *Assume that the sequence $(T_n)_{n\in\mathbb{N}}$ has property $\mathcal{D}$, let $v\in E$, $v\geq 0$, and let $z\in E_c'$, $z\geq 0$. Then, for every $\varepsilon\in\mathbb{R}$, $\varepsilon>0$, and for every $x\in X$, there exists $n_0\in\mathbb{N}$ such that $\sup_{l\in\mathbb{N}}\langle P_{(T_l^*x-n_0v)^+}v,\, z\rangle<\varepsilon$.*

Proof: Let $\varepsilon\in\mathbb{R}$, $\varepsilon>0$, let $x\in X$, and set $V_n=\{w\in E\mid w\wedge(v-w)=0$ and $P_w|T_ix|\leq nw$ for every $i\in\mathbb{N}\}$, and $w_n=\sup V_n$ for every $n\in\mathbb{N}$ (the suprema exist in E since E is order complete, and since V_n is a set of components of v in E for every $n\in\mathbb{N}$).

Let $n \in \mathbb{N}$. Clearly, $P_w|T_i x| \leq nw \leq nw_n$ for every $w \in V_n$ and $i \in \mathbb{N}$; therefore, by Lemma 3, $P_{w_n}|T_i x| = \sup_{w \in V_n} P_w |T_i x| \leq nw_n$ for every $i \in \mathbb{N}$. Accordingly, $w_n \in V_n$ (that is, $w_n = \max V_n$).

We will now prove that $\sup_{n \in \mathbb{N}} w_n = v$.

Clearly, $\sup_{n \in \mathbb{N}} w_n$ is a component of v. Assume that $\sup_{n \in \mathbb{N}} w_n \neq v$, and set $s = v - \sup_{n \in \mathbb{N}} w_n$. Then, s is a nonzero component of v. Since the sequence $(T_i)_i$ has property $\mathcal{D}$, it follows that the sequence $(|T_i x|)_i$ is not unbounded on $B(s)$; hence, the sequence $(T_i^* x)_i$ is not unbounded on $B(s)$, as well.

By Lemma 9-(a) of [14], the sequence $(T_i^* x)_i$ does not diverge individually to ∞ on $B(s)$. Since $(T_i^* x)_i$ is obviously a monotone nondecreasing sequence, we may apply Lemma 7 of [14]; accordingly, there exist a nonzero component w' of s in E and $\lambda \in \mathbb{R}$, $\lambda > 0$, such that $0 \leq P_{w'}(T_i^* x) \leq \lambda w'$ for every $i \in \mathbb{N}$.

Let $n \in \mathbb{N}$, $n \geq \lambda$. Then, $w' \in V_n$ since $0 \leq P_{w'}(T_i^* x) \leq nw'$ for every $i \in \mathbb{N}$. Consequently, $w' \leq \sup_{m \in \mathbb{N}} w_m$.

On the other hand, $w' \wedge (\sup_{m \in \mathbb{N}} w_m) = 0$ since $0 \leq w' \wedge (\sup_{m \in \mathbb{N}} w_m) \leq s \wedge (\sup_{m \in \mathbb{N}} w_m) = 0$.

Clearly, we have obtained a contradiction since $w' \geq 0$, $w' \neq 0$. Accordingly, $\sup_{m \in \mathbb{N}} w_m = v$.

Set $w'_n = v - w_n$ for every $n \in \mathbb{N}$.

Let $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Since $w_n \in V_n$, it follows that $P_{w_n}|T_i x| \leq nw_n$ for every $i = 1, 2, ..., l$, so $P_{w_n}(T_l^* x) \leq nw_n$; hence, $P_{w_n}((T_l^* x - nv)^+) = (P_{w_n}(T_l^* x) - nw_n)^+ = 0$.

Using Lemma 4-a), we obtain that

$$\begin{aligned}
P_{(T_l^* x - nv)^+} v &= \sup_{k \in \mathbb{N}} \Big(v \wedge \Big(k((T_l^* x - nv)^+) \Big) \Big) \\
&= \sup_{k \in \mathbb{N}} \bigg(P_v \Big(v \wedge \Big(k((T_l^* x - nv)^+) \Big) \Big) \bigg) \\
&= \sup_{k \in \mathbb{N}} \bigg((P_{w_n} + P_{w'_n}) \Big(v \wedge \Big(k((T_l^* x - nv)^+) \Big) \Big) \bigg) \\
&\leq \bigg(\sup_{k \in \mathbb{N}} P_{w_n} \Big(v \wedge \Big(k((T_l^* x - nv)^+) \Big) \Big) \bigg) \\
&\quad + \bigg(\sup_{k \in \mathbb{N}} P_{w'_n} \Big(v \wedge \Big(k((T_l^* x - nv)^+) \Big) \Big) \bigg) =
\end{aligned}$$

$$= \left(\sup_{k\in\mathbb{N}}\left(w_n \wedge \left(k\Big(P_{w_n}((T_l^* x - nv)^+)\Big)\right)\right)\right)$$
$$+ \left(\sup_{k\in\mathbb{N}}\left(w_n' \wedge \left(k\Big(P_{w_n'}((T_l^* x - nv)^+)\Big)\right)\right)\right)$$
$$= \sup_{k\in\mathbb{N}}\left(w_n' \wedge \left(k\Big(P_{w_n'}((T_l^* x - nv)^+)\Big)\right)\right) \le w_n' \ .$$

Thus, $P_{(T_l^* x - nv)^+} v \le w_n'$ for every $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Since $(w_n)_n$ is a monotonic nondecreasing sequence of components of v such that $\sup_{n\in\mathbb{N}} w_n = v$, it follows that $(w_n')_n$ is monotonic nonincreasing and $\inf_{n\in\mathbb{N}} w_n' = 0$.

Since z is a σ-order continuous linear functional on E, we obtain that $\langle w_{n_0}', z\rangle < \varepsilon$ for some $n_0 \in \mathbb{N}$.

Accordingly, $\sup_{l\in\mathbb{N}} \langle P_{(T_l^* x - n_0 v)^+} v, z\rangle \le \langle w_{n_0}', z\rangle < \varepsilon$. ∎

Lemma 20. *Assume that T_m is continuous in duality for every $m \in \mathbb{N}$. Let $v \in E$, $v \ge 0$, let $z \in E_c'$, $z \ge 0$, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, let l, n be two natural numbers, and set $\mathcal{B}_{l,n}(\varepsilon) = \{x \in X \mid \langle P_{(T_l^* x - nv)^+} v,\ z\rangle \le \varepsilon\}$. Then, $\mathcal{B}_{l,n}(\varepsilon)$ is a closed set (in the norm topology of X).*

Proof: We have to prove that for every norm convergent sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{B}_{l,n}(\varepsilon)$, we obtain that $\lim_{k\to+\infty} x_k$ is also an element of $\mathcal{B}_{l,n}(\varepsilon)$.

To this end, let $(x_k)_{k\in\mathbb{N}}$ be a norm convergent sequence of elements of $\mathcal{B}_{l,n}(\varepsilon)$, and set $x = \lim_{k\to+\infty} x_k$.

Set $w_h = (T_l^* x - (n + \frac{1}{h})v)^+$ for every $h \in \mathbb{N}$.

The sequence $(w_h)_{h\in\mathbb{N}}$ is order bounded in E since $0 \le w_h \le (T_l^* x - nv)^+$ for every $h \in \mathbb{N}$; thus, $\bigvee_{h\in\mathbb{N}} w_h$ exists since E is order complete. Taking into consideration that E is Archimedean, we obtain that

$$\bigvee_{h\in\mathbb{N}} w_h = \left(\bigvee_{h\in\mathbb{N}}\left(T_l^* x - \left(n + \frac{1}{h}\right)v\right)\right) \vee 0$$
$$= \left(T_l^* x - nv - \bigwedge_{h\in\mathbb{N}}\left(\frac{1}{h}\, v\right)\right) \vee 0 = (T_l^* x - nv)^+ \ .$$

By Lemma 3, $P_{(T_l^* x - nv)^+} v = \bigvee_{s\in\mathbb{N}} P_{(T_l^* x - (n+\frac{1}{s})v)^+} v$. In order to prove that $x \in \mathcal{B}_{l,n}(\varepsilon)$, we have to show that $\langle P_{(T_l^* x - nv)^+} v,\ z\rangle \le \varepsilon$; therefore, it follows that the lemma will be completely proved if we prove that $\langle P_{(T_l^* x - (n+\frac{1}{s})v)^+} v,\ z\rangle \le \varepsilon$ for every $s \in \mathbb{N}$.

Accordingly, let $s \in \mathbb{N}$. Let also $\eta \in \mathbb{R}$ be such that $\eta > 0$.

By Lemma 18, T_l^* is continuous in duality; therefore, there exists $n_\eta \in \mathbb{N}$ such that $\langle P_{(|T_l^* x - T_l^* x_k| - \frac{1}{s} v)^+} v,\, z\rangle < \eta$ for every $k \in \mathbb{N}$, $k \geq n_\eta$.

Using Lemma 4-b), we obtain that

$$\begin{aligned} 0 &\leq \Big\langle P_{(T_l^* x - (n+\frac{1}{s})v)^+} v,\, z\Big\rangle = \Big\langle P_{(T_l^* x - T_l^* x_{n_\eta} - \frac{1}{s} v + T_l^* x_{n_\eta} - nv)^+} v,\, z\Big\rangle \\ &\leq \Big\langle P_{(T_l^* x - T_l^* x_{n_\eta} - \frac{1}{s} v)^+ + (T_l^* x_{n_\eta} - nv)^+} v,\, z\Big\rangle \\ &\leq \Big\langle P_{(T_l^* x - T_l^* x_{n_\eta} - \frac{1}{s} v)^+} v,\, z\Big\rangle + \Big\langle P_{(T_l^* x_{n_\eta} - nv)^+} v,\, z\Big\rangle \\ &\leq \Big\langle P_{(|T_l^* x - T_l^* x_{n_\eta}| - \frac{1}{s} v)^+} v,\, z\Big\rangle + \Big\langle P_{(T_l^* x_{n_\eta} - nv)^+} v,\, z\Big\rangle < \eta + \varepsilon\ . \end{aligned}$$

Since $\langle P_{(T_l^* x - (n+\frac{1}{s})v)^+} v,\, z\rangle < \eta + \varepsilon$ for every $\eta \in \mathbb{R}$, $\eta > 0$, it follows that $\langle P_{(T_l^* x - (n+\frac{1}{s})v)^+} v,\, z\rangle \leq \varepsilon$. ∎

The results obtained so far in this paper now allow us to prove Theorem 2.

Proof of Theorem 2: **a**) We will first note that given $v \in E$, $v \geq 0$ and $z \in E_c'$, $z \geq 0$, it is enough to find a positive monotone nonincreasing function $C_{v,z} : (0, +\infty) \to \mathbb{R}$ such that $\lim_{\lambda \to +\infty} C_{v,z}(\lambda) = 0$, and such that $\sup_n \langle P_{(T_n^* x - \lambda \|x\|\, v)^+} v,\, z\rangle \leq C_{v,z}(\lambda)$ for every $x \in X$, $\|x\| = 1$ and $\lambda \in \mathbb{R}$, $\lambda > 0$.

Indeed, let $C_{v,z}$ be such a function, and let $y \in X$. If $y = 0$, then $\sup_n \langle P_{(T_n^* y - \lambda \|y\|\, v)^+} v,\, z\rangle = 0 \leq C_{v,z}(\lambda)$ for every $\lambda \in \mathbb{R}$, $\lambda > 0$. If $y \neq 0$, set $x = \frac{y}{\|y\|}$. Taking into consideration that

$$P_{(T_n^* x - \lambda v)^+} = P_{\frac{n}{\|y\|}(T_n^* y - \lambda \|y\|\, v)^+} = P_{(T_n^* y - \lambda \|y\|\, v)^+}\ ,$$

we conclude that

$$\Big\langle P_{(T_n^* y - \lambda \|y\|\, v)^+} v,\, z\Big\rangle = \Big\langle P_{(T_n^* x - \lambda v)^+} v,\, z\Big\rangle \leq C_{v,z}(\lambda)$$

for every $\lambda \in \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$.

Now, let $v \in E$, $v \geq 0$, and $z \in E_c'$, $z \geq 0$. Define $C_{v,z}\colon (0, +\infty) \to \mathbb{R}$ by

$$C_{v,z}(\lambda) = \sup_{\substack{y \in X \\ \|y\| \leq 1}} \sup_{l \in \mathbb{N}} \Big\langle P_{(T_l^* y - \lambda v)^+} v,\, z\Big\rangle$$

for every $\lambda \in \mathbb{R}$, $\lambda > 0$.

Clearly, the function $C_{v,z}$ is monotone nonincreasing.

Since

$$\sup_{l \in \mathbb{N}} \Big\langle P_{(T_l^* x - \lambda v)^+} v, z\Big\rangle \leq \sup_{\substack{y \in X \\ \|y\| \leq 1}} \sup_{l \in \mathbb{N}} \Big\langle P_{(T_l^* y - \lambda v)^+} v,\, z\Big\rangle = C_{v,z}(\lambda)$$

for every $x \in X$, $\|x\| = 1$ and for every $\lambda \in \mathbb{R}$, $\lambda > 0$, it follows that the function $C_{v,z}$ satisfies all the required properties provided that we prove that $\lim_{\lambda\to+\infty} C_{v,z}(\lambda) = 0$.

Since $C_{v,z}$ is a monotone nonincreasing function, it is obvious that in order to prove that $\lim_{\lambda\to+\infty} C_{v,z}(\lambda) = 0$, it is enough to show that for every $\varepsilon > 0$, there exists $\rho > 0$ such that $C_{v,z}(\rho) \le \varepsilon$.

To this end, let $\varepsilon > 0$, and set $\varepsilon' = \frac{\varepsilon}{2}$.

As in Lemma 20, set $\mathcal{B}_{l,n}(\varepsilon') = \{x \in X \mid \langle P_{(T_l^* x - nv)^+} v,\, z\rangle \le \varepsilon'\}$ for every $l \in \mathbb{N}$ and $n \in \mathbb{N}$; also set $\mathcal{A}_n(\varepsilon') = \{x \in X \mid \langle P_{(T_l^* x - nv)^+} v,\, z\rangle \le \varepsilon'$ for every $l \in \mathbb{N}\}$ for every $n \in \mathbb{N}$.

Clearly, $\mathcal{A}_n(\varepsilon') = \bigcap_{l\in\mathbb{N}} \mathcal{B}_{l,n}(\varepsilon')$ for every $n \in \mathbb{N}$; by Lemma 20, the sets $\mathcal{B}_{l,n}(\varepsilon')$, $l \in \mathbb{N}$, $n \in \mathbb{N}$, are closed, so $\mathcal{A}_n(\varepsilon')$ is a closed set for every $n \in \mathbb{N}$.

By Lemma 19, $X = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n(\varepsilon')$; thus, by the Baire category theorem, at least one of the sets $\mathcal{A}_n(\varepsilon')$, $n \in \mathbb{N}$, has nonempty interior. Accordingly, there exist $n \in \mathbb{N}$, $x_0 \in X$, and $\delta > 0$ such that $x \in \mathcal{A}_n(\varepsilon')$ whenever $x \in X$, $\|x - x_0\| < \delta$.

Thus, $\langle P_{(T_l^*(x_0+\delta y) - nv)^+} v,\, z\rangle \le \varepsilon'$ for every $l \in \mathbb{N}$ and $y \in X$, $\|y\| \le 1$.

Set $\rho = \frac{2n}{\delta}$.

Taking into consideration that

$$\begin{aligned}T_l^* y = \sup_{1\le k\le l} |T_k y| = \sup_{1\le k\le l} \left|\frac{1}{\delta}\, T_k(x_0+\delta y) - \frac{1}{\delta}\, T_k x_0\right| \le \\ \le \sup_{1\le k\le l} \frac{1}{\delta}\left|T_k(x_0+\delta y)\right| + \sup_{1\le k\le l}\frac{1}{\delta}\,|T_k x_0| = \frac{1}{\delta}\, T_l^*(x_0+\delta y) + \frac{1}{\delta}\, T_l^* x_0\end{aligned}$$

for every $l \in \mathbb{N}$, $y \in X$, and using Lemma 4-b), we obtain that

$$\begin{aligned}\left\langle P_{(T_l^* y - \frac{2n}{\delta} v)^+} v,\, z\right\rangle = \left\langle P_{(\delta T_l^* y - 2nv)^+} v,\, z\right\rangle \le \left\langle P_{(T_l^*(x_0+\delta y) + T_l^* x_0 - 2nv)^+} v,\, z\right\rangle \le \\ \le \left\langle P_{(T_l^*(x_0+\delta y) - nv)^+} v,\, z\right\rangle + \left\langle P_{(T_l^* x_0 - nv)^+} v,\, z\right\rangle \le \varepsilon' + \varepsilon' = \varepsilon\end{aligned}$$

for every $l \in \mathbb{N}$ and $y \in X$, $\|y\| \le 1$.

Accordingly,

$$C_{v,z}(\rho) = \sup_{\substack{y\in X\\ \|y\|\le 1}} \sup_{l\in\mathbb{N}} \left\langle P_{(T_l^* y - \frac{2n}{\delta} v)^+} v,\, z\right\rangle \le \varepsilon\ .$$

b) Assume that $\mathcal{F}$ is not a norm closed set in X. Then there exists x in the norm closure of $\mathcal{F}$ in X such that the sequence $(T_n x)_{n\in\mathbb{N}}$ does not converge individually on E. Then, by Theorem 15, the doubly indexed sequence $(|T_n x - T_m x|)_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ does not converge individually to zero on E; that is, $\limsup_{(n,k)}(|T_n x - T_k x| \wedge w) \ne 0$ for some $w \in E$, $w \ge 0$, $w \ne 0$.

Set $v = \limsup_{(n,k)}(|T_n x - T_k x| \wedge w)$.

Since $0 \le v \le w$, it follows that

$$\begin{aligned}\limsup_{(n,k)}\Big(|T_n x - T_k x| \wedge v\Big) &= \limsup_{(n,k)}\Big(|T_n x - T_k x| \wedge w \wedge v\Big)\\ &= \Big(\limsup_{(n,k)}\Big(|T_n x - T_k x| \wedge w\Big)\Big) \wedge v = v\ .\end{aligned}$$

Accordingly, $\sup_{\substack{n\ge l\\ k\ge l}}(|T_n x - T_k x| \wedge v) = v$ for every $l \in \mathbb{N}$.

Taking into consideration that E has property $\mathcal{P}$, we obtain that there exists $z \in E_c'$, $z \ge 0$, such that $\langle v, z\rangle \ne 0$. Set $\alpha = \langle v, z\rangle$.

Let $C_{v,z}$ be the function described at a), and let $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < \frac{1}{2}$, be small enough such that $C_{v,z}(\frac{1}{2\varepsilon}) < \alpha/3$. Then, there exists $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon^2$.

By Lemma 16, there exists $l_0 \in \mathbb{N}$ such that $\langle P_{(\bigvee_{\substack{n=l_0\\ k=l_0}}^t |T_n y - T_k y| - \varepsilon v)^+} v,\, z\rangle <$ $\alpha/3$ for every $t \in \mathbb{N}$, $t \ge l_0$.

Since

$$\begin{aligned}&\Big\langle P_{\big(\big(\bigvee_{\substack{n=l_0\\ k=l_0}}^t |T_n(x-y) - T_k(x-y)|\big) - \frac{1}{\varepsilon}\|x-y\|\, v\big)^+} v,\ z\Big\rangle \le\\ &\qquad\le \Big\langle P_{\big(2\big(\bigvee_{k=l_0}^t |T_k(x-y)|\big) - \frac{1}{\varepsilon}\|x-y\|\, v\big)^+} v,\ z\Big\rangle\\ &\qquad= \Big\langle P_{\big(\big(\bigvee_{k=l_0}^t |T_k(x-y)|\big) - \frac{1}{2\varepsilon}\|x-y\|\, v\big)^+} v,\ z\Big\rangle \le C_{v,z}\left(\frac{1}{2\varepsilon}\right)\ ,\end{aligned}$$

and since

$$\begin{aligned}\bigvee_{\substack{n=l_0\\ k=l_0}}^t |T_n x - T_k x| &\le \bigvee_{\substack{n=l_0\\ k=l_0}}^t \Big(\Big|T_n(x-y) - T_k(x-y)\Big| + |T_n y - T_k y|\Big)\\ &\le \Big(\bigvee_{\substack{n=l_0\\ k=l_0}}^t \Big|T_n(x-y) - T_k(x-y)\Big|\Big) + \Big(\bigvee_{\substack{n=l_0\\ k=l_0}}^t |T_n y - T_k y|\Big)\end{aligned}$$

for every $t \in \mathbb{N}$, $t \ge l_0$, it follows that

$$\Big\langle P_{\big(\big(\bigvee_{\substack{n=l_0\\ k=l_0}}^t |T_n x - T_k x|\big) - 2\varepsilon v\big)^+} v,\ z\Big\rangle \le$$

$$\leq \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_n(x-y)-T_k(x-y)|\big)+(\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_ny-T_ky|)-2\varepsilon v\big)^+} v,\, z\Big\rangle$$

$$\leq \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_n(x-y)-T_k(x-y)|\big)-\varepsilon v\big)^+ + \big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_ny-T_ky|\big)-\varepsilon v\big)^+} v,\, z\Big\rangle$$

$$\leq \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_n(x-y)-T_k(x-y)|\big)-\varepsilon v\big)^+} v,\, z\Big\rangle + \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_ny-T_ky|\big)-\varepsilon v\big)^+} v,\, z\Big\rangle$$

$$\leq \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_n(x-y)-T_k(x-y)|\big)-\frac{1}{\varepsilon}\|x-y\|\, v\big)^+} v,\, z\Big\rangle + \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_ny-T_ky|\big)-\varepsilon v\big)^+} v,\, z\Big\rangle$$

$$< C_{v,z}\left(\frac{1}{2\varepsilon}\right) + \frac{\alpha}{3} < \frac{2\alpha}{3}$$

for every $t \in \mathbb{N}$, $t \geq l_0$.

Thus,

$$\sup_{\substack{t\in\mathbb{N}\\t\geq l_0}} \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_nx-T_kx|\big)-2\varepsilon v\big)^+} v,\, z\Big\rangle \leq \frac{2\alpha}{3} < \alpha\ .$$

On the other hand, by Lemma 17, we obtain that

$$\sup_{\substack{t\in\mathbb{N}\\t\geq l_0}} \Big\langle P_{\big((\bigvee_{\substack{n=l_0\\k=l_0}}^{t} |T_nx-T_kx|\big)-2\varepsilon v\big)^+} v,\, z\Big\rangle = \langle v, z\rangle = \alpha\ .$$

We have obtained a contradiction; accordingly, we conclude that $\mathcal{F}$ is a norm closed set in X. ∎

REFERENCES

[1] Aliprantis, C.D. and Burkinshaw, O. – *Locally Solid Riesz Spaces*, Academic Press, New York/San Francisco/London, 1978.

[2] Aliprantis, C.D. and Burkinshaw, O. – *Positive Operators*, Academic Press, New York/London, 1985.

[3] Banach, S. – Sur la convergence presque partout des fonctionelles linéaires, *Bull. Sci. Math.,* 50(2) (1926), 27–32, 36–43.

[4] Garsia, A.M. – *Topics in Almost Everywhere Convergence*, Markham, Chicago, 1970.

[5] Krengel, U. – *Ergodic Theorems*, de Gruyter, Berlin/New York, 1985.

[6] Luxemburg, W.A.J. and Zaanen, A.C. – *Riesz Spaces, I*, North-Holland, Amsterdam/London, 1971.

[7] Nakano, H. – Ergodic theorems in semi-ordered linear spaces, *Ann. Math.,* 49 (1948), 538–556.

[8] Ornstein, D. – *The sum of iterates of a positive operator*, in "Advances in Probability and Related Topics" (P. Ney, Ed.), Vol. 2, pp. 85–115, Dekker, New York, 1970.

[9] Schaefer, H.H. – *Banach Lattices and Positive Operators*, Springer, Berlin/Heidelberg/New York, 1974.

[10] von Weizsäcker, H. – Sublineare Abbildungen und ein Konvergenzsatz von Banach, *Math. Ann.,* 212 (1974), 165–171.

[11] Yosida, K. – An abstract treatment of the individual ergodic theorems, *Proc. Imp. Acad. Tokyo,* 16 (1940), 280–284.

[12] Yosida, K. – *Functional Analysis*, 2nd printing, Springer, Berlin/Heidelberg/New York, 1966.

[13] Zaanen, A.C. – *Riesz Spaces II*, North-Holland, Amsterdam/New York/Oxford, 1983.

[14] Zaharopol, R. – On several types of convergence and divergence in Archimedean Riesz spaces, *J. Math. Anal. Appl.,* 169 (1992), 453–475.

[15] Zaharopol, R. – The Hopf decomposition and the individual divergence to ∞ on a band, *J. Math. Anal. Appl.,* 171 (1992), 471–476.

[16] Zaharopol, R. – On the ergodic theorem of E. Hopf, *J. Math. Anal. Appl.,* 178 (1993), 70–86.

[17] Zaharopol, R. – On the Nakano individual convergence, *Z. Anal. Anwendungen,* 13 (1994), 181–189.

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