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A NEW APPROACH TO THE L^2-REGULARITY THEOREMS FOR LINEAR STATIONARY NONHOMOGENEOUS STOKES SYSTEMS

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Abstract: We give a very simplified version of the L^2-regularity theorem for solutions of a nonhomogeneous system of the Stokes type (see $(1.3)_\lambda$) which coincides with the Stokes system when $\lambda = 0$ (see Theorem 1.1). We also give a corresponding approximation theorem (see Theorem 1.2).

1 – Introduction

Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$, $n \geq 2$, locally located on one side of its boundary Γ, a manifold of class $C^{k,1}$. ν denotes the unit external vector normal to Γ. In this paper we consider, in particular, the Stokes system

$$(1.1) \qquad \begin{cases} -\mu\,\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma\,, \end{cases}$$

where u and f are n-vector fields and p and g scalar fields, over Ω. φ is a vector field defined on Γ. f, g and φ are given. We assume that the compatibility condition

$$(1.2) \qquad \int_\Omega g\,dx = \int_\Gamma \varphi \cdot \nu\,d\Gamma$$

holds. We denote by $H^k \equiv H^k(\Omega)$, k integer, the Sobolev space consisting of (vector or scalar) functions that belong to $L^2 \equiv L^2(\Omega)$ together with the partial

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derivatives of order less or equal to k. $H^{-1} \equiv H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. For convenience we set $\overset{\circ}{H}{}^2 \equiv H^2 \cap H_0^1$. We define

$$L^2_\# \equiv \left\{ g \in L^2 \colon\, m(g) = 0 \right\}, \qquad m(g) \equiv \int_\Omega g\, dx \ ,$$

and also $H^k_\# \equiv H^k \cap L^2_\#$. The canonical norm in H^k is denoted by $\| \cdot \|_k$ and that in L^2 simply by $\| \cdot \|$.

In the sequel we also consider the system

$$(1.3)_\lambda \qquad \begin{cases} -\mu\, \Delta u + \nabla p = f & \text{in } \Omega, \\ \lambda p + \nabla \cdot u = g & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma \ , \end{cases}$$

where $\lambda \geq 0$ is a real parameter. When $\lambda = 0$ this system is just the usual (nonhomogeneous) Stokes system (1.1). We give here a very simple proof of the following result.

Theorem 1.1. *Let k be a nonnegative integer and λ be a nonnegative real parameter. Assume that Γ is a manifold of class $C^{k,1}$, that $f \in H^{k-1}$, $g \in H^k$, $\varphi \in H^{k+1/2}(\Gamma)$, and that (1.2) holds. Then, there is a unique solution (u_λ, p_λ) in the space $H^{k+1} \times H^k_\#$ of problem $(1.3)_\lambda$. Moreover*

$$(1.4)_\lambda \qquad \mu \|u_\lambda\|_{k+1} + (1 + \mu\lambda)\, \|p_\lambda\|_k \leq c\Big(\|f\|_{k-1} + \mu \|g\|_k + \mu \|\varphi\|_{k+1/2,\Gamma} \Big) \ .$$

Here, and in the sequel, we denote by c constants that depend at most on Ω and k. The same symbol c is used to indicate distinct constants, even in the same formula.

We remark that one easily reduces the nonhomogeneous boundary condition $u = \varphi$ on Γ to the homogeneous one $u = 0$ on Γ. In fact it is well known that there is a linear continuous map γ^{-1} from $H^{k+1/2}(\Gamma)$ into $H^{k+1}(\Omega)$ such that $(\gamma \circ \gamma^{-1})\, \varphi = \varphi$ (see [Pr], [Ne2]). Here γ denotes the usual trace operator. In view of this fact we assume in the sequel that $\varphi = 0$. The term "homogeneous" means here that $g = 0$. Note, by the way, that for the Stokes system (1.1) under the canonical assumption $g = 0$ the construction of the map γ^{-1} is quite involved (even in the case $n = 3$) due to the constraint on the divergence of u.

Let us now made some comments on our results. If $k = 0$ and $\lambda > 0$ the system $(1.3)_\lambda$ is well known in numerical analysis. In this context, the main point is the approximation of the solution (u_0, p_0) of the Stokes system with the solution (u_λ, p_λ) of $(1.3)_\lambda$ as λ goes to zero (penalty method). It is well known that $\|u_\lambda - u_0\|_1 \le c\lambda \|f\|_{-1}$ (for details see [G.R.], Ch. II, §2.4. See also [T], Ch. I, §6). Theorem 1.1 provides an easy proof of this estimate in the general case. More precisely one has the following result.

Theorem 1.2. *Under the assumptions of Theorem 1.1 one has*

$$\mu\|u_\lambda - u_0\|_{k+1} + (1+\mu\lambda)\,\|p_\lambda - p_0\|_k \le c\mu\lambda\Big(\|f\|_{k-1} + \mu\|g\|_k + \mu\|\varphi\|_{k+1/2,\Gamma}\Big)\,. \tag{1.5}$$

In particular

$$\|\nabla\cdot u_\lambda - g\|_k \le c\lambda\Big(\|f\|_{k-1} + \mu\|g\|_k + \mu\|\varphi\|_{k+1/2,\Gamma}\Big)\,. \tag{1.6}$$

Proof: By applying the estimate $(1.4)_\lambda$ to the system $-\mu\,\Delta(u_\lambda - u_0) + \nabla(p_\lambda - p_0) = 0$ in Ω, $\lambda(p_\lambda - p_0) + \nabla\cdot(u_\lambda - u_0) = \lambda\,p_0$ in Ω, $u_\lambda - u_0 = 0$ on Γ, it follows that the left hand side of (1.5) is bounded by $c\mu\lambda\|p_0\|_k$. This last quantity is bounded by the right hand side of (1.5), by Theorem 1.1 for $\lambda = 0$. This proves (1.5). ∎

When $\lambda = 0$ many distinct proofs of Theorem 1.1 are available in the literature. However our very elementary proof turns out to be simpler than the current ones (and, in any case, interesting by itself). Let us made some comments on this point.

It is well known that L^2-regularity theorems for elliptic equations present special features with respect to the general L^p-case, $p \neq 2$. Usually, the proofs of the regularity of the solution for the Stokes system follow the potential theoretical approach (see [Ca], [So], [La]), as for general elliptic systems (see, for instance [ADN]). However, in the particular (but central) case $p = 2$ it is sometimes possible to apply the elementary method of the differential quotients, introduced in reference [Ni]. For the homogeneous ($g = 0$) Stokes system (1.1) this gap was filled up in [C.F.]. See theorem 3.11 in this last reference. A previous proof, independent of potential theory, is given in reference [SS], where $n = 3$. Below we give a proof of the L^2-regularity for the system $(1.3)_\lambda$ without resort to potential theory as well. However, we do not apply the differential quotients method to the Stokes system. The following are some of the advantages of this choice. If the differential quotients method is applied directly to the Stokes system one has to prove the H^2-estimate in the framework of the integral (variational) formulation of the problem. In our proof this formulation is used only for establishing the existence of the weak solution (Theorem 2.1 below). On the other hand (see [C.F.]) the differential quotients method is applied to a system of equations more involved than the original Stokes system. In our approach, the central part of the proof of the H^2-estimate is done by working directly with the original system $(1.3)_\lambda$ (moreover, part of these estimates are just obtained as an immediate consequence of the H^1-estimates).

It is worth noting that our proof do not entirely avoid the differential quotients method since we will assume the estimate (1.9) below, concerning the solution of the scalar Poisson equation, a result obtained (not necessarily, however) by the differential quotients method. But for this elementary problem the (very familiar) proof is particularly simple. Moreover this result is not used in order to prove the main estimate $(1.8)_\lambda$ (i.e., the Theorem 4.1 below) but only in order to prove the effective existence of the solution (a point, sometimes, more or less passed over). In fact, we show that the existence of a regular solution to the above Poisson's equation together with the a priori estimate for the system $(1.3)_\lambda$ allow an elegant and simple proof of the existence of the regular solution to this last problem (see section 4).

Finally we recall that the common approaches to the Stokes problem require (with respect to the theory of second order elliptic scalar equations) an additional set of non trivial results, connected to particular functional spaces, which are specific to that problem. This fact leaves the Stokes system outside the elementary theory of elliptic partial differential equations. Our proof does not require any of these particular results. In this same regard, note that the non-homogeneous Stokes system (1.1) can be reduced, as well, to the homogeneous one ($h = 0$). Also this reduction, however, requires further specific technical devices due to the constraint $\nabla \cdot u = 0$.

Lastly, we note the little regularity assumed here for the boundary Γ.

For convenience we concentrate our attention in the H^2-regularity ($k = 1$) since H^{k+1}-regularity for $k \geq 2$ follows then by more or less standard devices. Summarizing, in the sequel we give a simple proof of the following result.

Theorem 1.3. *Let Ω be an open bounded set of class $C^{1,1}$ and assume that $f \in L^2$ and $g \in H^1_\#$. Then, there is a unique solution $(u,p) \in H^2 \times H^1_\#$ of the problem*

$$(1.7)_\lambda \qquad \begin{cases} -\mu\,\Delta u + \nabla p = f & \text{in } \Omega, \\ \lambda p + \nabla \cdot u = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma\,, \end{cases}$$

for each $\lambda \geq 0$. Moreover,

$$(1.8)_\lambda \qquad \mu\|u\|_2 + (1+\mu\lambda)\,\|p\|_1 \leq c\Big(\|f\| + \mu\|g\|_1\Big)\,.$$

Note that the system $(1.7)_\lambda$ admits a unique solution for each $g \in L^2(\Omega)$. However the solution blows up as λ goes to zero, if $m(g) \neq 0$.

Below, we give a self-contained proof of Theorem 1.3. We assume just the classical theorem establishing that if Ω is as in that theorem and if $f \in L^2$ then the (unique) solution $u \in H^1_0$ of the scalar Poisson's equation $-\Delta u = f$ belongs to H^2 and satisfies the estimate

$$\|u\|_2 \le c_0 \, \|f\| \ . \tag{1.9}$$

2 – Existence of the variational solution

It is advisable to begin by proving the existence theorem in the space $H^1_0 \times L^2_\#$ for the solution (u,p) of problem $(1.7)_\lambda$. We begin by assuming that $\lambda > 0$. For convenience we set

$$Y \equiv H^{-1} \times L^2_\# \,, \qquad X_1 \equiv H^1_0 \times L^2_\# \ .$$

We denote by $(\cdot,\cdot)$ the scalar product in L^2 (for scalar and for vector fields). Set $U=(u,p)$, $V=(\varphi,\psi)$ and consider the bilinear continuous form over X_1

$$a_\lambda(U,V) \equiv \mu(\nabla u, \nabla\varphi) - (p, \nabla\cdot\varphi) + \lambda(p,\psi) + (\nabla\cdot u, \psi) \ . \tag{2.1}$$

This form is coercive over X_1 since

$$a_\lambda(U,U) \ge \mu\|u\|_1^2 + \lambda\|p\|^2 \ .$$

On the other hand, by setting

$$\langle L, V\rangle \equiv \langle f,\varphi\rangle_{H^{-1},H^1_0} + (g,\psi) \ ,$$

the element $L \equiv (f,g) \in Y$ defines a linear continuous functional over X_1. Hence, the Lax–Milgram lemma shows the existence of a (unique) solution $U=(u,p) \in X_1$ of the problem

$$a_\lambda(U,V) = \langle L, V\rangle \,, \qquad \forall\, V \in X_1 \ . \tag{2.2}$$

By choosing $V=U$ (2.2) yields

$$\mu\|\nabla u\|^2 + \lambda\|p\|^2 \le \|f\|_{-1}\, \|u\|_1 + \|g\|\, \|p\| \ . \tag{2.3}$$

In particular,

$$\mu\|u\|_1^2 + \lambda\|p\|^2 \le \frac{c}{\mu}\|f\|_{-1}^2 + \frac{1}{\lambda}\|g\|^2 \,. \tag{2.4}$$

By setting $\psi = 0$, equation (2.2) shows that the first equation $(1.7)_\lambda$ holds in H^{-1}. On the other hand, by setting $\varphi = 0$, it follows that $(\lambda p + \nabla\cdot u, \psi) = (g, \psi)$ for each $\psi \in L^2_\#$. Since the functions p, $\nabla\cdot u$, and g belong to $L^2_\#$ it follows that the second equation $(1.7)_\lambda$ is satisfied. This shows that (2.2) is a variational formulation of problem $(1.4)_\lambda$.

The estimate (2.4) is not useful for passing to the limit as λ goes to zero. Note, in particular, that the right hand side tends to $+\infty$ if $g \not\equiv 0$. However, having in hand the existence of the solution in the space X_1, it is not difficult to get the sharp estimate. In fact, we have shown that $\nabla p = \mu\,\Delta u + f$ belongs to H^{-1}. Hence $\|\nabla p\|_{-1} \le \mu\|u\|_1 + \|f\|_{-1}$. On the other hand, since $p \in L^2_\#$ and $\nabla p \in H^{-1}$, one has $\|p\| \le c\|\nabla p\|_{-1}$ (see Lemma 2.1 below). Consequently

$$\|p\| \le c\Big(\mu\|u\|_1 + \|f\|_{-1}\Big) \,.$$

Next, by replacing in the right hand side of equation (2.3) $\|p\|$ by the right hand side of the above inequality it readily follows

$$\mu\|u\|_1^2 \le c\Big(\frac{1}{\mu}\|f\|_{-1}^2 + \mu\|g\|^2\Big) \,.$$

Hence for each $\lambda > 0$, $\mu\|u\|_1 \le c(\|f\|_{-1} + \mu\|g\|)$. By using the above estimate for $\|p\|$ one gets (see remark below)

$$\mu\|u\|_1 + \|p\| \le c\Big(\|f\|_{-1} + \mu\|g\|\Big) \,. \tag{2.5}$$

If $\lambda = 0$ the proof follows from the estimate (2.5) since this estimate shows that the solution (u_λ, p_λ) of problem $(1.7)_\lambda$, $\lambda > 0$, is weakly convergent in $H_0^1 \times L^2_\#$ to some (u, p), as $\lambda \to 0$. Note that the problem $(1.7)_\lambda$, for each $\lambda \ge 0$, has a unique solution in $H_0^1 \times L^2_\#$, in the sense of distributions, since if $f = 0$ and $g = 0$ the solution must vanish. The proof is trivial. Thus we have proved the following result.

Theorem 2.1. *Let $(f, g) \in H^{-1} \times L^2_\#$. Then, the problem $(1.7)_\lambda$, $\lambda \ge 0$, has a unique solution (u, p) in the space $H_0^1 \times L^2_\#$. Moreover, (2.5) holds.*

Remark. By using (in particular) the estimate (2.3) one easily shows that $\|p\|$ can be replaced by $(1 + \mu\lambda)\|p\|$ in the estimate (2.5).

Lemma 2.1. *There exists a positive constant c such that $\|p\| \le c\|\nabla p\|_{-1}$ for each $p \in L^2_\#$ verifying $\nabla p \in H^{-1}$.*

Proof: Assume that $\Gamma \in C^{1,1}$. By defining

$$X(\Omega) = \Big\{p \in H^{-1} \colon \ \nabla p \in H^{-1}\Big\}$$

one has $X = L^2$ as sets. A very elementary, self contained, proof of this result is given in reference [D.L.] Chap. 3, Theorem 3.2. Here $n = 3$, however the proof does not depend on this assumption as remarked in the footnote (9); see also [T], page 28, Lemma 5. Let us show further the equivalence of the norms in X and in L^2.

It is obvious that $|||p||| \equiv \|p\|_{-1} + \|\nabla p\|_{-1} \le c\|p\|$ for each $p \in L^2$. Moreover $|||\cdot|||$ is a norm in L^2. Hence the norms $|||\cdot|||$ and $\|\cdot\|$ are equivalent in L^2 if $\{L^2, |||\cdot|||\}$ is complete. Let $\{u_n\}$ be a Cauchy sequence in this space. Then $u_n \to u$ in H^{-1} and $\nabla u_n \to \nabla u$ in H^{-1}, for some $u \in X = L^2$. Hence $\{L^2, |||\cdot|||\}$ is complete. In particular

$$\|p\| \le c\Big(\|p\|_{-1} + \|\nabla p\|_{-1}\Big), \qquad \forall\, p \in L^2_\# \,. \tag{2.6}$$

In order to prove the estimate claimed in the lemma it is sufficient to show that to each $\varepsilon > 0$ there corresponds a $c_\varepsilon > 0$ such that $\|p\|_{-1} \le \varepsilon\|p\| + c_\varepsilon\|\nabla p\|_{-1}$, for each $p \in L^2_\#$. If this were false, it would follow that there exist some $\varepsilon_0 > 0$ and a sequence p_n in $L^2_\#$ for which

$$\|p_n\|_{-1} \ge \varepsilon_0\|p_n\| + n\|\nabla p\|_{-1}\,, \qquad \forall\, n \in \mathbb{N}\,,$$

where $\|p_n\|_{-1} = 1$. It readily follows that (for a subsequence) p_n converges to some p in H^{-1} (since $H^{-1} \hookrightarrow L^2$ is compact) and weakly in L^2 (hence $p \in L^2_\#$). Moreover ∇p_n converges to 0 in H^{-1}, hence $\nabla p = 0$. Consequently $p = 0$, which contradicts $\|p\|_{-1} = 1$.

Finally, we note that the above lemma holds even if Γ belongs only to the class $C^{1,1}$. For a proof see [Nel].

3 – The local a priori estimate

In this section we prove $(1.8)_\lambda$ as an a priori estimate in the "half space" (see (3.2)) when $\lambda > 0$. The simplicity of the proof seems remarkable. For convenience we set $x = (x', x_n)$, where $x' = (x_1, ..., x_{n-1})$. Moreover

$$Q \equiv \Big\{x\colon\ |x'| < 1,\ |x_n| < 1\Big\},$$
$$Q^+ \equiv \Big\{x \in Q\colon\ 0 < x_n\Big\},$$
$$\Lambda \equiv \Big\{x\colon\ |x'| < 1,\ x_n = 0\Big\}\ .$$

Theorem 3.1. *Let be $\Omega = Q^+$, and assume that $u \in \overset{\circ}{H}{}^2$, $p \in H^1$, $f \in L^2$, and $g \in H^1$ are functions with compact support in $Q^+ \cup \Lambda$. Moreover, assume that these functions solve the system of equations $(1.7)_\lambda$, i.e.,*

$$\begin{cases} -\mu\,\Delta u + \nabla p = f & \text{in } Q^+, \\ \lambda p + \nabla\cdot u = g & \text{in } Q^+, \\ u = 0 & \text{on } \Lambda\ . \end{cases} \tag{3.1}$$

Then, the following estimate holds

$$\mu\|u\|_2 + (1+\mu\lambda)\|\nabla p\| \le c\Big(\|f\| + \mu\|\nabla g\|\Big)\ . \tag{3.2}$$

Proof: By differentiation with respect to x_j, $j = 1, ..., n-1$, one gets

$$\begin{cases} -\mu\,\Delta\dfrac{\partial u}{\partial x_j} + \nabla\dfrac{\partial p}{\partial x_j} = \frac{\partial f}{\partial x_j} & \text{in } Q^+, \\ \lambda\dfrac{\partial p}{\partial x_j} + \nabla\cdot\dfrac{\partial u}{\partial x_j} = \dfrac{\partial g}{\partial x_j} & \text{in } Q^+, \\ \dfrac{\partial u}{\partial x_j} = 0 & \text{on } \Gamma\ . \end{cases} \tag{3.3}$$

Moreover $\partial u/\partial x_j \in H^1_0$ and $\partial p/\partial x_j \in L^2_{\#}$, since p vanishes near the lateral boundary of the cylinder Q^+. Similarly, $\partial g/\partial x_j \in L^2_{\#}$. Clearly, $\partial f/\partial x_j \in H^{-1}$. Hence, by the uniqueness of the solution of problem $(1.7)_\lambda$ in the above class and by (2.5), the solution $(\partial u/\partial x_j,\ \partial p/\partial x_j)$ of (3.3) satisfies the estimate

$$\mu\|D^2_* u\| + \|\nabla_* p\| \le c\Big(\|f\| + \mu\|\nabla g\|\Big)\ . \tag{3.4}$$

Here, D^2_* denotes second order partial derivatives except for $\partial^2/\partial x_n^2$, and ∇_* denotes first order derivatives except for $\partial/\partial x_n$. Next, consider the linear system consisting of the n^{th} equation (3.1) and of the equation obtained by differentiation of the $(n+1)^{\text{th}}$ equation (3.1) with respect to x_n namely

$$(3.5)\qquad \begin{cases} -\mu\,\dfrac{\partial^2 u_n}{\partial x_n^2}+\dfrac{\partial p}{\partial x_n}=f_n+\mu\,\Delta_* u_n\;, \\[2ex] \dfrac{\partial^2 u}{\partial x_n^2}+\lambda\,\dfrac{\partial p}{\partial x_n}=\dfrac{\partial g}{\partial x_n}-\dfrac{\partial}{\partial x_n}(\nabla_*\cdot u_*)\;, \end{cases}$$

where $\nabla_*\cdot u_*=(\partial u_1/\partial x_1)+...+(\partial u_{n-1}/\partial x_{n-1})$ and Δ_* is the Laplace operator with respect to x'. By solving this algebraic system for $\partial^2 u_n/\partial x_n^2$ and $\partial p/\partial x_n$, and by using (3.4), it readily follows that

$$(3.6)\qquad \mu\left\|\frac{\partial^2 u}{\partial x_n^2}\right\|+(1+\mu\lambda)\left\|\frac{\partial p}{\partial x_n}\right\|\le c\Big(\|f\|+\mu\|\nabla g\|\Big)\;.$$

Finally, from the first $n-1$ equations (3.1) one gets

$$(3.7)\qquad \mu\,\frac{\partial^2 u_j}{\partial x_n^2}=f_j-\mu\,\Delta_* u_j-\frac{\partial p}{\partial x_j}\;,$$

for each $j\ne n$. The estimates (3.4) and (3.6) show that the L^2-norm of each term in the right hand side of (3.7) is bounded by the right hand side of (3.6). It readily follows (3.2).

Note that $\|\nabla^2 u\|$ can be replaced by $\|u\|_2$, and similarly for $\|\nabla p\|$ and $\|\nabla g\|$, since these functions vanish near the lateral boundary (and the top) of the cylinder Q^+.

Obviously, the above estimate holds, in particular, if the functions u, p, f and g have compact support contained in Q^+. In this case (3.3) also holds for $j=n$, and this ends the proof. This last case corresponds to the "interior regularity".

4 – Proof of Theorem 1.3

In the sequel $\Omega\subset\mathbb{R}^n$ is an open bounded set of class $C^{1,1}$. This means here that the open bounded set Ω is connected and locally situated on one side of its boundary Γ, a manifold of class $C^{1,1}$. More precisely, given a point $P\in\Gamma$ there are positive numbers a and b, an orthonormalized system of cartesian coordinates $(x_1,...,x_n)=(x',x_n)$ with origin at P, and a function $\psi(x')$ defined and Lipschitz continuous on the sphere $\{x'\colon\,|x'|\le b\}$ together with its first order derivatives, such that: the points x for which $x_n=\psi(x')$ belong to Γ; the points x for which $\psi(x')<x_n<a+\psi(x')$ belong to Ω; the points x for which $-a+\psi(x')<x_n<$

$\psi(x')$ belong to $\mathbb{R}^n/\overline{\Omega}$. Moreover, the tangent plane to Γ at P coincides with the x'-plane. Without loss of generality, we assume that $a \le 1$ and $b \le 1$.

We define here the sets

$$\sigma \equiv \Big\{x\colon\, |x'| < b,\ -a + \psi(x') < x_n < a + \psi(x')\Big\}\ ,$$

$$\sigma^+ \equiv \Big\{x \in \sigma\colon\, x_n > \psi(x')\Big\}\,, \qquad \dot{\sigma} = \Big\{x \in \sigma\colon\, x_n = \psi(x')\Big\}\ .$$

These sets will be used only in the appendix.

One has the following a priori estimate.

Theorem 4.1. *Let Ω be an open, bounded set of class $C^{1,1}$ and let $(f,g) \in L^2 \times H^1_{\#}$. Assume that the variational solution $(u,p) \in H^1_0 \times L^2_{\#}$ of problem (1.7)$_\lambda$, $\lambda > 0$, belongs to $H^2 \times H^1_{\#}$. Then*

$$\mu\|u\|_2 + (1+\mu\lambda)\|p\|_1 \le c\Big(\|f\| + \mu\|g\|_1\Big)\ , \tag{4.1}$$

where c depends only on Ω.

Note that this result does not include the existence of the regular solution. The proof of Theorem 4.1 is done by the standard method of localization followed by flattening the boundary. These devices reduce the global problem to a finite number of problems like that treated in section 3, for which estimates similar to (3.2) apply. By mapping back, we get the desired a priori estimate (4.1). For the reader's convenience we present a detailed proof of Theorem 4.1 in the appendix.

Next we prove the Theorem 1.3, for $\lambda > 0$, by using the a priori estimate (4.1). Let $t \in [0,1]$ and consider the problem

$$\begin{cases} -\mu\,\Delta u + t\,\nabla p = f & \text{in } \Omega, \\ \lambda p + t\,\nabla\cdot u = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma\ , \end{cases} \tag{4.2$_t$}$$

where $(f,g) \in L^2 \times H^1_{\#}$. Assume that $(u,p) \in \overset{\circ}{H}{}^2 \times H^1_{\#}$. For $t > 0$, by dividing the equations (4.2)$_t$ by t and by applying the estimate (4.1) one obtains

$$\mu\|u\|_2 + \Big(t + \frac{\mu\lambda}{t}\Big)\|p\|_1 \le c\Big(\|f\| + \frac{\mu}{t}\|g\|_1\Big)\ . \tag{4.3}$$

It is worth noting that this estimate is not sufficient to our aim, since it is not uniform near $t = 0$. We overcome this obstacle as follows. The limit problem

$(4.2)_t$ for $t=0$ splits into two independent problems

$$(4.4)\qquad \begin{cases} -\mu\,\Delta u = f & \text{in } \Omega,\\ \lambda p = g & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma\ . \end{cases}$$

Note the crucial rôle of the positive parameter λ. Clearly, the problem (4.4) has a unique solution $u \in \overset{\circ}{H}{}^2$, $p \in H^1_{\#}$. Moreover, $\mu\|u\|_2 \le c_0\|f\|$ and $\lambda\|p\|_1 \le \|g\|_1$. Next we show that there is a $\delta_0 > 0$ such that the problem $(4.2)_t$ admits a unique solution $(u,p) \in \overset{\circ}{H}{}^2 \times H^1_{\#}$, for each $t \in [0,\delta_0]$. Moreover,

$$(4.5)\qquad \|u\|_2 + \|p\|_1 \le \frac{c_0}{\mu}\|f\| + \frac{1}{\lambda}\|g\|_1\ .$$

This result follows by considering the problem

$$(4.6)\qquad \begin{cases} -\mu\,\Delta u = f - t\,\nabla q & \text{in } \Omega,\\ \lambda p = g - t\,\nabla\cdot v & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma\ , \end{cases}$$

where $(v,q) \in \overset{\circ}{H}{}^2 \times H^1_{\#}$. Note that $\nabla\cdot v = H^1_{\#}$. This problem admits (by the above result for the system (4.4)) a unique solution $(u,p) \in \overset{\circ}{H}{}^2 \times H^1_{\#}$. Let $(\overline{u},\overline{p})$ be the solution corresponding to a datum $(\overline{q},\overline{v})$. By taking the difference, side by side, between the equations (4.6) and the corresponding equations for the labelled variables, one shows at once that

$$\|u-\overline{u}\|_2 + \|p-\overline{p}\|_1 \le \delta_0\Big(\frac{1}{\lambda}\|v-\overline{v}\|_2 + \frac{c_0}{\mu}\|q-\overline{q}\|_1\Big).$$

In particular, by choosing $\delta_0 = \frac{1}{2}\min\{\lambda,\mu/c_0\}$, the map $(v,q) \to (u,p)$ is a strict contraction. Hence, for each $t \in [0,\delta_0]$, there is a unique solution of problem $(4.2)_t$ in the space $\overset{\circ}{H}{}^2 \times H^1_{\#}$. Moreover, from $(4.2)_t$ and (1.9) it readily follows that

$$\frac{1}{2}\Big(\|u\|_2 + \|p\|_1\Big) \le \frac{c_0}{\mu}\|f\| + \frac{1}{\lambda}\|g\|_1$$

(note that this estimate blows up if $\lambda \to 0$). In particular, the problem $(4.2)_t$ has a unique solution $(u,p) \in \overset{\circ}{H}{}^2 \times H^1_{\#}$, for $t=\delta_0$. Next, we show that this problem has a unique solution in the desired functional space, for each $t \in [\delta_0,1]$. Since this result holds for $t=\delta_0$, it is sufficient to show that there is a fixed $\delta_1 > 0$ such that if our thesis holds for some $t_0 \in [\delta_0, 1]$ then it also holds for each $t = t_0 + \Delta t$, where $0 \le \Delta t \le \delta_1$. Let us prove this last assertion. Consider the system

$$\begin{cases} -\mu\,\Delta u + t_0\,\nabla p = f - (\Delta t)\,\nabla q & \text{in } \Omega, \\ \lambda p + t_0\,\nabla\cdot u = g - (\Delta t)\,\nabla\cdot v & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma\ , \end{cases}$$

where $(v,q) \in \overset{\circ}{H}{}^2 \times H^1_\#$. By the assumption on t_0 there is a unique solution $(u,p) \in \overset{\circ}{H}{}^2 \times H^1_\#$. Let $(\overline{u},\overline{p})$ be a second solution, corresponding to a datum $(\overline{v},\overline{q})$, and take the difference between the above equations and the corresponding equations for the labelled variables (in the system obtained in that way it is convenient to consider $t_0(p-q)$ as the "pressure" and to divide the second equation by t_0). By applying (4.1) to this system it readily follows that $\mu\|u-\overline{u}\|_2 + (t_0+\mu\lambda)\|p-\overline{p}\|_1 \le c\,\Delta t(\|q-\overline{q}\|_1 + (\mu/t_0)\|v-\overline{v}\|_2)$. By choosing $\Delta t \le \delta_1 \equiv \delta_0/2c$ one gets

$$\mu\|u-\overline{u}\|_2 + \delta_0\|p-\overline{p}\|_1 \le \frac{1}{2}\Big(\mu\|v-\overline{v}\|_2 + \delta_0\|q-\overline{q}\|_1\Big)\ .$$

Hence the map $(q,v) \to (p,u)$ has a (unique) fixed point, the solution of $(4.2)_t$. Step by step we arrive to the value $t=1$. Consequently $(1.7)_\lambda$ has a unique solution in the space $\overset{\circ}{H}{}^2 \times H^1_\#$. Since this solution satisfies (4.1), it also satisfies $(1.8)_\lambda$. As this last estimate is uniform with respect to λ, the solution (u_λ, p_λ) of $(1.7)_\lambda$ is weakly convergent in $\overset{\circ}{H}{}^2 \times H^1_\#$ to the solution (u,p) of $(1.7)_\lambda$ for $\lambda = 0$. Clearly, the limit (u,p) satisfies $(1.8)_\lambda$ for $\lambda = 0$. ∎

5 – Appendix

Here, we prove the Theorem 4.1. The proof follows the well known technique of localization and flattening the boundary. Let $P \in \Gamma$ and consider the neighbourhood σ of P and the set σ^+ introduced at the beginning of section 4. Eventually by choosing a smaller σ, we assume that the size parameters a and b are less or equal to 1. Let ϑ be a fixed $C_0^\infty(\mathbb{R}^n)$ function, $0 \le \vartheta \le 1$, with compact support contained in the set σ. In the sequel we will use the cartesian coordinates with origin at P, introduced in section 4 (note that the equations $(1.7)_\lambda$ are invariant to orthonormal changes of coordinates, since so are the Laplacian, the gradient and the divergence). Localization is done by multiplication by ϑ. We define

$$v = \vartheta\,u\,, \qquad q = \vartheta p\ . \tag{5.1}$$

For convenience, in the sequel vectors $u = (u_1, ..., u_n)$ are regarded as column vectors. $\Delta u = (\Delta u_1, ..., \Delta u_n)$. Moreover, $\nabla u \equiv [\nabla u_1, ..., \nabla u_n]$ is the square matrix whose i^{th} row is ∇u_i. A dot denotes the usual matrix multiplication. One has $\Delta(\vartheta u) = \vartheta(\Delta u) + 2\nabla u \cdot \nabla\vartheta + (\Delta\vartheta)u$. It readily follows from equations $(1.4)_\lambda$ that

$$\begin{cases} -\mu\,\Delta v + \nabla q = \vartheta f - 2\mu\,\nabla u \cdot \nabla\vartheta - \mu(\Delta\vartheta)\,u + p\,\nabla\vartheta, & \text{in } \sigma^+, \\ \lambda q + \nabla\cdot v = \vartheta g + \nabla\vartheta\cdot u, & \text{in } \sigma^+, \\ v = 0, & \text{on } \dot\sigma\ . \end{cases} \tag{5.2}$$

Note that the functions v and q, as well as each single term in the above equations, have compact support in $\sigma^+ \cup \dot\sigma$. The above equations hold almost everywhere (and in the L^2 sense) due to the regularity assumptions on u and p.

Next, we make a change of independent variables $y = \Psi(x)$, in order to get a flat boundary. Consider the function $\psi(x')$, section 4, concerning the Cartesian equation of the boundary Γ near the new origin P. We define a map $y = \Psi(x)$ given by $y_j = \psi_j(x) = x_j$ if $j = 1, ..., n-1$, $y_n = \psi_n(x) = x_n - \psi(x')$. Note that the inverse map $x = \Psi^{-1}(y)$ is simply given by $x_j = y_j$ if $j \neq n$, $x_n = y_n + \psi(y')$. Also note that $\partial y_j/\partial x_i = \delta_{ij} - \delta_{nj}(\partial\psi(x')/\partial x_i)$. One has

$$\Psi(\sigma^+) = \widehat{Q}^+ \equiv \Big\{x\colon\ |x'| < a,\ 0 < x_n < b\Big\}\ ,$$

$$\Psi(\dot\sigma) = \widehat{\Lambda} \equiv \Big\{x\colon\ |x'| < a,\ x_n = 0\Big\}\ .$$

In the sequel, if h is a function defined on σ^+ we denote by $\widetilde{h}$, or by $(h)^\sim$, the function $h(x)$ written with respect to the y-coordinates, i.e. $\widetilde{h}(y) \equiv h(\Psi^{-1}(y))$.

Now, we write the equations (5.2) with respect to the y_i variables, $y \in \widehat{Q}^+$. Since each term in equations (5.2) has compact support in $\widehat{Q}^+ \cup \widehat{\Lambda}$ we assume that they are defined on Q^+, by setting them equal to zero outside $\widehat{Q}^+$.

From now on the symbols Δ and ∇ concern the y variables. Otherwise, we use symbols Δ_x and ∇_x. One has (summation with respect to repeated indices is assumed)

$$\widetilde{\Delta_x v} = a_{k\ell}\,\frac{\partial^2\widetilde{v}}{\partial_{y_k}\,\partial_{y_\ell}} - (\widetilde{\Delta_{x'}\psi})\,\frac{\partial\widetilde{v}}{\partial y_n}\ , \tag{5.3}$$

where $\Delta_{x'}$ denotes the Laplacian with respect to the variables $x_1, ..., x_{n-1}$ and

$$a_{k\ell} \equiv \left(\frac{\partial\psi_k}{\partial x_j}\,\frac{\partial\psi_\ell}{\partial x_j}\right)^{\sim} = \delta_{k\ell} + b_{k\ell}\ ,$$

where $b_{k\ell} = b_{\ell k}$ is given by $b_{k\ell} = 0$ if $k, \ell \leq n-1$; $b_{kn} = b_{nk} = -\partial\psi(x')/\partial x_k$ if $k \leq n-1$; $b_{nn} = |\nabla_{x'}\psi(x')|^2$.

By using this notation, the equation (5.3) becomes

$$\widetilde{\Delta_x v} = \Delta\widetilde{v} + b_{k\ell}\,\frac{\partial^2\widetilde{v}}{\partial y_k\,\partial_{y_\ell}} - (\widetilde{\Delta_{x'}\psi})\,\frac{\partial\widetilde{v}}{\partial y_n}\ .$$

Similarly, $(\widetilde{\nabla_x q}) = \nabla\widetilde{q} - (\widetilde{\nabla_{x'}\psi})\,(\partial\widetilde{q}/\partial y_n)$ and $(\widetilde{\nabla_x \cdot v}) = \nabla\cdot\widetilde{v} - (\widetilde{\nabla_{x'}\psi})\cdot(\partial\widetilde{v}/\partial y_n)$. By convention, the n^{th} component of the vector field $\nabla_{x'}\psi(x')$ vanishes identically. Hence, the equations (5.2) became

(5.4)
$$\begin{cases} -\mu\,\Delta\widetilde{v} + \nabla\widetilde{q} = \mu\, b_{k\ell}\,\dfrac{\partial^2\widetilde{v}}{\partial y_k\,\partial_{y_\ell}} - \mu(\widetilde{\Delta_{x'}\psi})\,\dfrac{\partial\widetilde{v}}{\partial y_n} + (\widetilde{\nabla_{x'}\psi})\,\dfrac{\partial\widetilde{q}}{\partial y_n} + \nabla\widetilde{F}, & \text{in } Q^+,\\ \lambda\widetilde{q} + \nabla\cdot\widetilde{v} = (\widetilde{\nabla_{x'}\psi})\cdot\dfrac{\partial\widetilde{v}}{\partial y_n} + (\vartheta g + \nabla\vartheta\cdot u)^{\sim}, & \text{in } Q^+,\\ \widetilde{v} = 0, & \text{on } \Lambda\ , \end{cases}$$

where $F \equiv \vartheta f - 2\mu\,\nabla u\cdot\nabla\vartheta - \mu(\Delta\vartheta)u + p\,\nabla\vartheta$.

Next, note that in the above argument we can choose the size parameters a and b as small as we want. Since $\nabla_{x'}\psi(0) = 0$ it follows that, given $L > 0$ there is a $b > 0$ (the radius of the cylinder σ) such that $|\nabla_{x'}\psi(x')| \le L$ for $|x'| \le b$. By the regularity $C^{1,1}$ of ψ, there is also a real positive M such that $|(D^2_{x'}\psi)^{\sim}| \le M$ for $|x'| \le b$. L will be fixed below. However, for convenience, we assume from now on that b is such that $L \le 1$. Also note that an upper bound for M depends only on Γ, hence on Ω. In other words, $L \le 1$ and M can be replaced by constants c that depend only on Ω.

Now, note that the solutions $\widetilde{v}$ and $\widetilde{q}$ of the problem (5.4) are just in the situation described in section 3. They belong to the spaces $\overset{\circ}{H}{}^2$ and H^1 (since $y = \Psi(x)$ and $x = \Psi^{-1}(y)$ preserve H^2-regularity) and they have compact support on $Q^+ \cup \Lambda$ (note that in section 3 this last assumption replaces the requirement $m(\widetilde{q}) = 0$). Hence, the estimate (3.2) together with the above remarks about L and M show that

(5.5)
$$\begin{aligned} \mu\|\widetilde{v}\|_2 + (1+\mu\lambda)\|\nabla\widetilde{q}\| &\le c\Big[\mu(L+L^2)\,\|\widetilde{v}\|_2 + \mu\|\widetilde{v}\|_1 + L\|\nabla\widetilde{q}\|\Big]\\ &\quad + \widehat{c}\|\widetilde{f}\| + \widehat{c}\mu\|\widetilde{u}\|_1 + \widehat{c}\|p\| + c\mu L\|\widetilde{v}\|_2 + \widehat{c}\mu\Big[\|\widetilde{g}\|_1 + \|\widetilde{u}\|_1\Big]\ , \end{aligned}$$

where norms concern the domain Q^+. The constants $\widehat{c}$ depend on ϑ. We are careful about this point just to avoid misunderstandings. Next, we choose b "so small" that $2c\mu(L+L^2) \le \mu/2$ and $cL \le 1/2$. Hence, we drop from the right hand side of (5.5) the terms with L and take one half of the left hand side. We denote the equation obtained by this way by (5.5bis).

The above construction is done in correspondence to each point $P \in \Gamma$. Next, we fix a finite covering $\{\sigma_\alpha\}$ of Γ by sets of the above type. We add to this covering of Γ a finite family $\{\sigma_\beta\}$ of cylinders, $\overline{\sigma}_\beta \subset \Omega$, such that $(\bigcup \sigma_\alpha) \cup (\bigcup \sigma_\beta)$ covers $\overline{\Omega}$. Next, we fix a partition of unity $\{\vartheta_\alpha\} \cup \{\vartheta_\beta\}$ subordinate to the above covering of $\overline{\Omega}$. Now, each ϑ_α (or ϑ_β) is fixed, hence the corresponding $\widehat{c}$ that appear in equation (5.5bis) are fixed (and, in fact, depend only on Ω). Clearly, we are using here the fact that estimates like (5.5bis) also hold in correspondence to the cylinders σ_β (interior regularity). By taking into account that (with obvious notations)

$$u = \Sigma\, \vartheta_\alpha u + \Sigma\, \vartheta_\beta u \equiv \Sigma\, v_\alpha + \Sigma\, v_\beta \ ,$$

and that

$$p = \Sigma\, \vartheta_\alpha p + \Sigma\, \vartheta_\beta p = \Sigma\, q_\alpha + \Sigma\, q_\beta \ ,$$

by using the local estimates (5.5bis) in order to bound $\|\widetilde{v}_\alpha\|_2$, $\|\nabla \widetilde{q}_\alpha\|$, $\|\widetilde{v}_\beta\|_2$, $\|\nabla \widetilde{q}_\beta\|$, by mapping back from Q^+ to each σ_α^+ (or σ_β) and by collecting all these estimates, one gets

$$\mu \|u\|_2 + (1+\lambda\mu)\|\nabla p\| \le c\Big(\|f\| + \mu\|g\|_1\Big) + c\Big(\mu\|u\|_1 + \|p\|\Big) \ .$$

This estimate together with (2.5) yields (4.1). ∎

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