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OPTIMAL CONTROL FOR THE BOUNDARY FLUX TO THE CONTINUOUS CASTING PROBLEM

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Abstract: A three dimensional evolutionary continuous casting problem is considered. The problem is described by a singular parabolic equation with a singular convection term and mixed boundary conditions. The problem of optimal control for the boundary flux is discussed. Necessary conditions for the optimal control problem are derived in the absence of mushy region.

1 – Introduction

In this paper we consider the following singular parabolic equation with a singular convection term and mixed boundary conditions

$$
\begin{cases}
\partial_t \eta + b\, \partial_z \eta - \Delta u = 0, & \text{in } \mathcal{D}'(Q_T), \\
u = h_i, & \text{on } \Sigma_T^i, \quad i = 0, L, \\
-\dfrac{\partial u}{\partial n} = g, & \text{on } \Sigma_T^N, \\
\eta = \eta_0(x,y,z), & \text{on } t = 0, \\
\eta \in \beta(u) \ .
\end{cases}
\tag{1.1}
$$

Here $(x,y,z) \in Q \in \mathbb{R}^3$, $Q = \Omega \times [0,L]$, $\Omega = (0,a_1) \times (0,a_2)$, $\Gamma_i = \Omega \times \{z=i\}$, $i=0,L$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$, $\Gamma_N = \partial\Omega \times [0,L]$, $\Sigma_T^D = \Gamma_D \times [0,T]$ and $\Sigma_T^N = \Gamma_N \times [0,T]$, n is the unit outward normal to Q on Γ_N, u and η are the unknown temperature and entropy respectively, $\beta(\cdot)$ is a maximum monotone graph, b is the extracting velocity, g is the boundary flux. (1.1) is a mathematical model for the continuous casting problem in the steel industry which describes the heat conduction phenomena with phase change and singular convective effect (see [1]). The existence,

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uniqueness and regularity of the solutions as well as the asymptotic behavior were known for both steady state and evolutionary cases (see [2]). If $b = 0$, (1.1) is the Stefan problem. We refer the readers to [3], [4] for relevant results about Stefan problem and to [5] and [6] for the related optimal control problems of the two-phase Stefan problem.

The purpose of this paper is to study the optimal control of the boundary flux g according to the requirement of the industry. Some necessary conditions are derived for the optimal control and optimal state. We notice that even for the two-phase Stefan problem ($b = 0$), the necessary conditions for boundary flux control has not been obtained up to now (see [6]), p.213, Remark 7.9). Therefore setting $b = 0$ in our case one can get the necessary conditions for the two-phase Stefan problem.

Owing to the existence of convection and radiation in the boundary of secondary cooling region, it satisfies

$$-\frac{\partial T}{\partial n} = \alpha(T - T_{\mathrm{H_2O}}) + \sigma\,\varepsilon(T^4 - T_{\mathrm{ext}}^4)\ ,$$

where T is the absolute temperature, α is called the heat transfer coefficient which is interpreted as the control parameter by controlling the water quality, $T_{\mathrm{H_2O}}$ is the water temperature and T_{ext} is the temperature of the exterior environment, σ and ε are positive constants. In our formulation the flux g is the controlling term. When this is known, it is an easy task to determine α, which is important for practical purposes, since $T_{\mathrm{H_2O}}$ and T_{ext} are given, and T, the optimal state, can be obtained at same time with g.

The paper is organized as follows. In section 2 we state the assumptions, recall the weak formulation and obtain an existence result of the optimal control problem. In order to get the necessary conditions, in section 3 we construct an approximating control problem and get some necessary conditions for this approximating problem. In the last section we get the necessary conditions to the original control problem based on uniform estimates and a limit procedure.

2 – The existence of the optimal control

Let us start with the following assumptions

(H): The functions β, h_i, b and g appeared in (1.1) satisfy the following

$$\beta(u) = \begin{cases} c_1\,u + 1 & u > 0, \\ [0,1] & u = 0, \\ c_2\,u & u < 0\,. \end{cases} \tag{2.1}$$

Here we suppose the latent heat is 1, and c_i $(i = 1, 2)$ are positive constants.

$$b > 0 \tag{2.2}$$

is a constant (the extracting velocity).

$$\begin{cases} \exists\, h \in L^2(0,T;H^1(Q))\colon\ h = h_i \ \text{ on } \Sigma_T^i,\ i = 0, L; \\ |h|_{L^\infty(Q_T)} \le M \ \text{ and } \ \partial_z h \in L^1(Q_T)\,, \end{cases} \tag{2.3}$$

$$g \in L^2(\Sigma_T^N)\,, \qquad 0 \le g \le \overline{M}\,, \tag{2.4}$$

the physical background of $g \ge 0$ is the cooling in industry and $\overline{M}$ is the maximum flux.

$$\begin{cases} \eta_0 \in \beta(u_0) \quad \text{a.e. in } \ Q \ \text{ for some } u_0 \in L^\infty(Q)\,, \\ |u_0|_{L^\infty(Q)} \le M\,. \end{cases} \tag{2.5}$$

Next let us introduce the weak formulation of problem (1.1) as follows. A pair $(u,\eta) \in L^2(0,T;H^1(Q)) \times L^2(Q)$ is a weak solution of (1.1), if

$$\eta \in \beta(u) \ \text{ a.e.}\,, \qquad u = h \ \text{ on } \Sigma_T^D\,, \qquad \text{and} \tag{2.6}$$

$$-\iint\limits_{Q_T} \eta(\partial_t\zeta + b\,\partial_z\zeta) + \iint\limits_{Q_T} \nabla u\,\nabla\zeta + \int\limits_{\Sigma_T^N} g\,\zeta = \int\limits_{Q} \eta_0\,\zeta(0)\,, \qquad \forall\,\zeta \in W_0\,, \tag{2.7}$$

where the space of test function is given by

$$W_0 = \left\{\zeta \in H^1(Q_T);\ \ \zeta|_{t=T} = 0 \ \text{ and } \ \zeta = 0 \ \text{ on } \Sigma_T^D\right\}.$$

Proposition 2.1. *Let (2.1)–(2.5) hold, then problem (2.6)–(2.7) has at least one weak solution (u,η) with the following regularities*

$$\begin{cases} u \in L^2(0,T;H^1(Q)) \cap L^\infty(Q_T)\,, \\ \eta \in L^\infty(Q_T)\,. \end{cases} \tag{2.8}$$

Proof: Consider the following regularized problem

$$\partial_t \beta_\varepsilon(u_\varepsilon) + b\,\partial_z \beta_\varepsilon(u_\varepsilon) - \Delta u_\varepsilon = 0 \quad \text{in } Q_T\;,$$

$$u_\varepsilon = h \quad \text{on } \Sigma_T^D\;,$$

$$-\frac{\partial u_\varepsilon}{\partial n} = g \quad \text{on } \Sigma_T^N\;,$$

$$u_\varepsilon|_{t=0} = u_0(x,y,z)\;,$$

where

$$\beta_\varepsilon(z) = \begin{cases} c_1\,z+1 & \text{if } z \ge \varepsilon, \\ c_2\,z & \text{if } z \le 0\;, \end{cases}$$

and $\beta_\varepsilon(z) \in c^2$, $\min\{c_1, c_2\} < \beta'_\varepsilon \le 2\,\varepsilon^{-1}$.

The proof is composed with two parts. The first part is to get the following uniform estimates

$$|u_\varepsilon|_{L^\infty(Q_T)} \le M_1\;, \tag{2.9}$$

and

$$|u_\varepsilon|_{H^1_{\text{loc}}(Q_T)} + |u_\varepsilon|_{L^2(0,T;H^1(Q))} \le M_2\;, \tag{2.10}$$

where M_1, M_2 are constants independent of $\varepsilon > 0$.

The second part is to take the limit. The whole proof is similar to the proof of Theorem 1 in [2]. The only difference is the proof of (2.9) because the flux g is different. Therefore only the proof of (2.9) will be given.

The method of estimating the maximum of u_ε is the same as in [2] because $g \ge 0$. In the following we estimate the minimum of u_ε.

Set

$$Z(x,y,z,t) = \exp\left\{\left(x - \frac{a_1}{z}\right)^2 + \left(y - \frac{a_2}{z}\right)^2\right\} \tag{2.11}$$

and

$$u_\varepsilon = Z\,V_\varepsilon\;, \tag{2.12}$$

then V_ε satisfies

$$\begin{cases} Z^{-1}\,\partial_t\beta_\varepsilon(Z\,V_\varepsilon) + Z^{-1}\,\partial_z\beta_\varepsilon(Z\,V_\varepsilon) - \Delta V_\varepsilon - 2\dfrac{\nabla Z}{Z}\cdot\nabla V_\varepsilon - \dfrac{\Delta Z}{Z}\,V_\varepsilon = 0\;, \\ V_\varepsilon = Z^{-1}h\;, \\ -\left[\dfrac{\partial V_\varepsilon}{\partial n} + Z^{-1}\,\dfrac{\partial Z}{\partial n}\,V_\varepsilon\right] = Z^{-1}g\;, \\ V_\varepsilon = Z^{-1}\,u_0(x,y,z)\;. \end{cases} \tag{2.13}$$

Notice that

$$\frac{\Delta Z}{Z} = \left(x - \frac{a_1}{2}\right)^2 + \left(y - \frac{a_2}{2}\right)^2 + 2 > 0 \ ,$$

$$Z^{-1}\,\frac{\partial Z}{\partial n} = \begin{cases} a_1 & \text{on } x = 0, a_1, \\ a_2 & \text{on } y = 0, a_2 \ . \end{cases}$$

So the boundary condition $(2.13)_3$ can be written in the form

$$-\frac{\partial V_\varepsilon}{\partial n} = \begin{cases} a_1\left(V_\varepsilon + \dfrac{1}{a_1\, z}\, g\right), & x = 0, a_1, \\[2ex] a_2\left(V_\varepsilon + \dfrac{1}{a_2\, z}\, g\right), & y = 0, a_2 \ . \end{cases}$$

Considering $0 \le Z(x,y,z,t) = Z(x,y) \le C_0$, we multiply the equation $(2.13)_1$ by $(V_\varepsilon - N)^-$, where

$$f^- = \begin{cases} 0 & \text{if } f \ge 0, \\ f & \text{if } f < 0 \ , \end{cases}$$

$$N = \min\left\{Z^{-1}\, h,\ -\frac{1}{a_1\, Z}\, g,\ -\frac{1}{a_2\, Z}\, g\right\} \ ,$$

and integrate over Q_T. Recalling $\beta(u) = c_2\, u$ if $u < 0$, after a calculation we have, for all $\varepsilon > 0$,

$$V_\varepsilon \ge N \ .$$

By the definition (2.12) we get

$$u_\varepsilon \ge -M_1 \ .$$

Here M_1 only depends on M, $\overline{M}$ and C_0.

(2.9) and (2.10) follows that there is function $u \in L^2(0,T;H^1(Q)) \cap L^\infty(Q_T)$ such that

$$\text{(2.14)} \qquad \begin{cases} u_\varepsilon \to u & \text{strongly in } L^2(Q_T), \\ u_\varepsilon \rightharpoonup u & \text{weakly in } L^2(0,T;H^1(Q)), \\ \beta_\varepsilon(u_\varepsilon) \rightharpoonup \eta & \text{weakly in } L^2(Q_T) \ . \end{cases}$$

And (u,η) satisfies (2.6) and (2.7) (see [2]). ∎

Remark. Set $S_T^\delta = \{(x,y,z,t) \in Q_T;\ 0 \le z \le \delta\}$. If

$$\text{(2.15)} \qquad \exists\, \delta > 0;\quad u \ge \rho > 0 \quad \text{a.e. in } S_T^\delta \ ,$$

then the solution of (2.6) and (2.7) is unique (see [2]).

Especially, the condition

$$\left\{\begin{array}{l} h\in C^0(\overline{\Sigma}_T^D)\,, \qquad u_0\in C^0(\overline{Q}) \qquad \text{and} \\ h|_{t=0}=u_0 \;\;\text{on } \Gamma_D\,, \qquad h>0 \;\;\text{on } \overline{\Sigma}_T^D\cap\{z=0\}\ , \end{array}\right. \tag{2.16}$$

implies that u is continuous in $\overline{Q}_T$ and satisfies (2.15) (see [2]).

The task of this paper is to find (u^*,g^*), such that

$$J(u^*,g^*)=\text{minimize}\, J(u,g)\,, \qquad u\in G_{\text{ad}}\ , \tag{2.17}$$

where

$$J(u,g)=\frac{1}{2}\iint\limits_{Q_T}(u-u_d)^2+\frac{1}{2}\int\limits_{\Sigma_T^N}g^2\ , \tag{2.18}$$

$u_d\in L^2(Q_T)$ is a known function.

$$G_{\text{ad}}=\left\{g\in L^2(\Sigma_T^N);\ 0\le g\le\overline{M}\right\}\ , \tag{2.19}$$

$$(u,\eta)\ \ \text{satisfies (2.6) and (2.7)}\ . \tag{2.20}$$

The existence of the problem (2.17)–(2.20) depends on the following proposition.

Proposition 2.2. *Under the assumptions (2.1)–(2.5) and (2.16), let $G_{\text{ad}}\ni g_n\rightharpoonup g$ weakly in $L^2(\Sigma_T^N)$, (u_n,η_n), (u,η) denote the solutions of (2.6) and (2.7) corresponding to g_n, g, then*

$$\begin{aligned} &u_n\rightharpoonup u \quad \text{weakly in}\ \ L^2(0,T;H^1(Q))\ ,\\ &\eta_n\rightharpoonup\eta \quad \text{weakly in}\ \ L^2(Q_T)\ . \end{aligned} \tag{2.21}$$

Proof: $g_n\in G_{\text{ad}}$ and $g_n\rightharpoonup g$ weakly in $L^2(\Sigma_T^N)$ implies $g\in G_{\text{ad}}$. Consider approximation problem

$$\left\{\begin{array}{ll} \partial_t\beta_\varepsilon(u_{\varepsilon n})+b\,\partial_z\beta_\varepsilon(u_{\varepsilon n})-\Delta u_{\varepsilon n}=0 & \text{in } Q_T,\\ u_{\varepsilon n}=h & \text{on } \Sigma_T^D,\\ -\dfrac{\partial u_{\varepsilon n}}{\partial n}=g_n & \text{on } \Sigma_T^N,\\ u_{\varepsilon n}|_{t=0}=u_0(x,y,z)\ . & \end{array}\right.$$

We can get estimate (2.9) and (2.10) which are uniformly with respect to ε and n. So we first take the limit $\varepsilon \to 0$ to get (2.6) and (2.7) corresponding to (u_n, η_n), g_n, and than let $n \to \infty$ to get (2.6) and (2.7) corresponding to (u, η) and g because of the uniqueness. ∎

Theorem 2.3. *The control problem (2.17)–(2.20) has at least one optimal pair (u^*, g^*). Foremore there exists $\eta^* \in L^2(Q_T)$, such that (u^*, η^*) is the solution of (2.6) and (2.7) corresponding to $g = g^*$.*

Proof: If $(u_n, g_n) \in L^2(0,T; H^1(Q)) \times L^2(\Sigma_T^N)$ is a minimizing sequence, we may suppose

$$g_n \rightharpoonup g^* \quad \text{weakly in} \;\; L^2(\Sigma_T^N) \ ,$$

then from Proposition 2.2, it follows that for the corresponding states $u_n \rightharpoonup u^*$ weakly in $L^2(0,T; H^1(Q))$ and u^* is the state corresponding to g^*.

The weak lower semicontinuity of J ends the proof. ∎

3 – Approximation problem

Consider the following approximating optimal control problem:

Find a pair $(u_\varepsilon^*, g_\varepsilon^*)$ such that

$$J_\varepsilon(u_\varepsilon^*, g_\varepsilon^*) = \min J_\varepsilon(u_\varepsilon, g_\varepsilon) \, , \qquad g_\varepsilon \in G_{\text{ad}} \ , \tag{3.1}$$

where

$$J_\varepsilon(u_\varepsilon, g_\varepsilon) = \frac{1}{2} \iint\limits_{\Omega_T} (u_\varepsilon - u_d)^2 + \frac{1}{2} \int\limits_{\Sigma_T^N} g_\varepsilon^2 + \frac{1}{2} \int\limits_{\Sigma_T^N} (g_\varepsilon - g^*)^2 \ , \tag{3.2}$$

subject to

$$\begin{cases} \partial_t \beta_\varepsilon(u_\varepsilon) + b\, \partial_z \beta_\varepsilon(u_\varepsilon) - \Delta u_\varepsilon = 0 & \text{in } Q_T, \\ u_\varepsilon = h & \text{on } \Sigma_T^D, \\ -\dfrac{\partial u_\varepsilon}{\partial n} = g_\varepsilon & \text{on } \Sigma_T^N, \\ u_\varepsilon = u_0(x) & \text{on } t = 0 \ . \end{cases} \tag{3.3}$$

According to the result in the previous section, the (smooth) optimal control problem (3.1)–(3.3) has an optimal pair $(u_\varepsilon^*, g_\varepsilon^*)$.

Theorem 3.1. *Under the assumptions of (2.1)–(2.5) and (2.16),*

$$\begin{cases} g_\varepsilon^* \to g^* & \text{strongly in } L^2(\Sigma_T^N), \\ u_\varepsilon^* \to u^* & \text{strongly in } L^2(Q_T), \\ u_\varepsilon^* \rightharpoonup u^* & \text{weakly in } L^2(0,T;H^1(Q)) \ . \end{cases} \tag{3.4}$$

Proof: We may suppose $g_\varepsilon^* \rightharpoonup \overline{g}$ weakly in $L^2(\Sigma_T^N)$, and $\overline{u}$ is the solution of (2.6), (2.7) corresponding to $g = \overline{g}$. It follows that $u_\varepsilon^* \to \overline{u}$ strongly in $L^2(Q_T)$ and weakly in $L^2(0,T;H^1(Q))$ by the uniqueness of the problem (2.6) and (2.7). In the following we prove $\overline{g} = g^*$, $\overline{u} = u^*$ and $g_\varepsilon^* \to g^*$ strongly in $L^2(\Sigma_T^N)$. In fact,

$$\begin{aligned} J_\varepsilon(\overline{u},\overline{g}) &= \frac{1}{2}\iint\limits_{Q_T} (\overline{u} - u_d)^2 + \frac{1}{2}\int\limits_{\Sigma_T^N} \overline{g}^2 + \frac{1}{2}\int\limits_{\Sigma_T^N} (\overline{g} - g^*)^2 \\ &\le \varliminf_{\varepsilon\to 0}\Big[\frac{1}{2}\iint\limits_{Q_T} (u_\varepsilon^* - u_d)^2 + \frac{1}{2}\int\limits_{\Sigma_T^N} g_\varepsilon^{*2} + \frac{1}{2}\int\limits_{\Sigma_T^N} (g_\varepsilon^* - g^*)^2\Big] \\ &\le \lim_{\varepsilon\to 0}\Big[\frac{1}{2}\iint\limits_{Q_T} (\widehat{u}_\varepsilon - u_d)^2 + \frac{1}{2}\int\limits_{\Sigma_T^N} g^{*2}\Big]. \end{aligned}$$

Here $\widehat{u}_\varepsilon$ is the solution of (3.3) corresponding to $g = g^*$ and we used the property $J_\varepsilon(u_\varepsilon^*, g_\varepsilon^*) \le J_\varepsilon(\widehat{u}_\varepsilon, g^*)$ for every $\varepsilon > 0$ in the last step. Considering $\widehat{u}_\varepsilon \to u^*$ strongly in $L^2(Q_T)$, so we have

$$\begin{aligned} J_\varepsilon(\overline{u},\overline{g}) &\le \frac{1}{2}\iint\limits_{Q_T} (u^* - u_d)^2 + \frac{1}{2}\int\limits_{\Sigma_T^N} g^{*2} \\ &= J(u^*, g^*) \\ &\le J(\overline{u},\overline{g}) \ . \end{aligned}$$

It follows that $\overline{g} = g^*$ and then $\overline{u} = u^*$.

At last, we prove $g_\varepsilon^* \to g^*$ strongly in $L^2(\Sigma_T^N)$. From $J_\varepsilon(u_\varepsilon^*, g_\varepsilon^*) \le J_\varepsilon(\widehat{u}_\varepsilon, g^*)$ we have

$$\varlimsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon^*, g_\varepsilon^*) \le \lim_{\varepsilon\to 0} J_\varepsilon(\widehat{u}_\varepsilon, g^*) \ .$$

Considering $u_\varepsilon^* \to u^*$, $\widehat{u}_\varepsilon \to u^*$ strongly in $L^2(Q_T)$, we obtain

$$\varlimsup_{\varepsilon\to 0}\Big[\int\limits_{\Sigma_T^N} g_\varepsilon^{*2} + \int\limits_{\Sigma_T^N} (g_\varepsilon^* - g^*)^2\Big] \le \int\limits_{\Sigma_T^N} g^{*2} \ .$$

On other hand

$$\overline{\lim_{\varepsilon\to 0}}\Big[\int_{\Sigma_T^N} g_\varepsilon^{*2} + \int_{\Sigma_T^N} (g_\varepsilon^* - g^*)^2\Big] \geq \underline{\lim}_{\varepsilon\to 0} \int_{\Sigma_T^N} g_\varepsilon^{*2} + \overline{\lim_{\varepsilon\to 0}} \int_{\Sigma_T^N} (g_\varepsilon^* - g^*)^2$$

and

$$\int_{\Sigma_T^N} g^{*2} \leq \underline{\lim}_{\varepsilon\to 0} \int_{\Sigma_T^N} g_\varepsilon^{*2} \;.$$

So we get

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Sigma_T^N} (g_\varepsilon^* - g^*)^2 \leq 0 \;.$$

This completes the proof. ∎

Now we study the necessary conditions for the optimal pair $(u_\varepsilon^*, g_\varepsilon^*)$ of the problem (3.1)–(3.3).

Define

$$\widetilde{G}_\varepsilon = \Big\{g \in L^2(\Sigma_T^N);\; g_\varepsilon^* + \lambda g \in G_{\mathrm{ad}},\ \text{for all sufficiently small } \lambda > 0\Big\} \;.$$

Suppose u_ε^λ satisfy

$$\text{(3.5)}\qquad \begin{cases} \partial_t\beta_\varepsilon(u_\varepsilon^\lambda) + b\,\partial_z\beta_\varepsilon(u_\varepsilon^\lambda) - \Delta u_\varepsilon^\lambda = 0 & \text{in } Q_T,\\ u_\varepsilon^\lambda = h & \text{on } \Sigma_T^D,\\ -\dfrac{\partial u_\varepsilon^\lambda}{\partial n} = g_\varepsilon^* + \lambda\, g & \text{on } \Sigma_T^N,\\ u_\varepsilon^\lambda = u_0(x,y,z) & \text{on } t=0\;. \end{cases}$$

Proposition 3.2. *Define* $W_\varepsilon^\lambda = \frac{u_\varepsilon^\lambda - u_\varepsilon^*}{\lambda}$. *Then*

$$|W_\varepsilon^\lambda|_{L^2(Q_T)} \leq C|g|_{L^2(\Sigma_T^N)}\;,$$

where C is independent of ε and λ.

Proof: Assume $\varphi \in H^1(Q_T)$, $\Delta\varphi \in L^2(Q_T)$ and $\varphi = 0$ on Σ_T^D.

Multiplying the equation $(3.5)_1$ by φ and integrating over Q_T we have

$$\text{(3.6)}\qquad -\iint_{Q_T} \beta_\varepsilon(u_\varepsilon^\lambda)\,(\partial_t\varphi + b\,\partial_z\varphi) - \iint_{Q_T} u_\varepsilon^\lambda\,\Delta\varphi + {} + \int_0^T\!\!\int_{\partial\Omega} \frac{\partial\varphi}{\partial n}\,u_\varepsilon^\lambda + \int_{\Sigma_T^N} (g_\varepsilon^* + \lambda g)\varphi = 0\;.$$

For u_ε^*, the similar equality holds:

$$(3.7)\qquad -\iint\limits_{Q_T} \beta_\varepsilon(u_\varepsilon^*)\,(\partial_t\varphi + b\,\partial_z\varphi) - \iint\limits_{Q_T} u_\varepsilon^*\,\Delta\varphi + \int_0^T \iint\limits_{\partial\Omega} \frac{\partial\varphi}{\partial n}\,u_\varepsilon^* + \int\limits_{\Sigma_T^N} g_\varepsilon^*\,\varphi\ .$$

From (3.6) and (3.7) we get, if $\frac{\partial\varphi}{\partial n} = 0$ on Σ_T^N,

$$(3.8)\qquad \iint\limits_{Q_T} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big]\,(\partial_t\varphi + b\,\partial_z\varphi + \alpha\,\Delta\varphi) = \int\limits_{\Sigma_T^N} \lambda\,g\,\varphi\ ,$$

where

$$\alpha = \frac{u_\varepsilon^\lambda - u_\varepsilon^*}{\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)}\ .$$

Consider the following problem:

$$(3.9)\qquad \begin{cases} \partial_t\varphi + b\,\partial_z\varphi + \alpha\,\Delta\varphi = u_\varepsilon^\lambda - u_\varepsilon^* & \text{in } Q_T,\\ \varphi = 0 & \text{on } \Sigma_T^D,\\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \Sigma_T^N,\\ \varphi(x,y,z,T) = 0\ . \end{cases}$$

For fixed $\varepsilon > 0$, in spite of the boundary ∂Q is only Lipschitz continuous we still have $\varphi \in W_2^{2,1}(Q_T)$, due to u_ε^λ and u_ε^* are continuous in $\overline{Q}_T$. In fact the regularity on the corner can be obtained by odd extension with respect to Σ_T^D and even extension with respect to Σ_T^N.

From (3.8) and (3.9) we have

$$(3.10)\qquad \begin{aligned} \iint\limits_{Q_T} (u_\varepsilon^\lambda - u_\varepsilon^*)^2 &\le \iint\limits_{Q_T} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big](u_\varepsilon^\lambda - u_\varepsilon^*)\\ &= \int\limits_{\Sigma_T^N} \lambda\,g\,\varphi\\ &\le \lambda|g|_{L^2(\Sigma_T^N)}\,|\varphi|_{L^2(\Sigma_T^N)}\ . \end{aligned}$$

In the following we prove

$$(3.11)\qquad |\varphi|_{L^2(\Sigma_T^N)} \le C\Big\{\iint\limits_{Q_T} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big](u_\varepsilon^\lambda - u_\varepsilon^*)\Big\}^{1/2}\ ,$$

Here C is independent of ε and λ. (3.10) and (3.11) end the proof. ∎

Multiplying the equation $(3.9)_1$ by $\Delta\varphi$ and integrating over $Q_t = Q \times [t,T]$, we obtain

$$\iint_{Q_t} \partial_t\varphi\,\Delta\varphi + b\iint_{Q_t} \partial_z\varphi\,\Delta\varphi + \iint_{Q_t} \alpha(\Delta\varphi)^2 =$$

$$= \iint_{Q_t} (u_\varepsilon^\lambda - u_\varepsilon^*)\,\Delta\varphi$$

$$\le \frac{1}{2}\iint_{Q_t} \alpha(\Delta\varphi)^2 + \frac{1}{2}\iint_{Q_t} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big](u_\varepsilon^\lambda - u_\varepsilon^*)\ .$$

Notice that

$$\iint_{Q_t} \partial_t\varphi\,\Delta\varphi = \frac{1}{2}\int_Q |\nabla\varphi|^2(\cdot,t)\ ,$$

$$\iint_{Q_t} \partial_z\varphi\,\Delta\varphi = -\frac{1}{2}\int_t^T\int_{z=0} |\nabla\varphi|^2 + \frac{1}{2}\int_t^T\int_{z=L} |\nabla\varphi|^2\ .$$

Therefore

$$\int_Q |\nabla\varphi|^2(\cdot,t) + \iint_{Q_t} \alpha|\Delta\varphi|^2 \le \tag{3.12}$$

$$\le \iint_{Q_t} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big](u_\varepsilon^\lambda - u_\varepsilon^*) + b\int_t^T\int_{z=0} |\partial_z\varphi|^2\ .$$

On other hand, according to (2.15),

$$\alpha = C_1^{-1} \quad \text{in } S_T^\delta$$

if ε is small enough.

Let $\zeta = \zeta(z)$ be a cut off function such that

$$\zeta = \begin{cases} 1, & 0 \le z \le \delta/2, \\ 0, & \delta \le z \le L\ . \end{cases}$$

Multiply (3.9) by ζ; we have

$$\begin{cases} \partial_t(\zeta\,\varphi) + b\,\partial_z(\zeta\,\varphi) + C_1^{-1}\,\Delta(\zeta\,\varphi) = \\ \qquad = \zeta(u_\varepsilon^\lambda - u_\varepsilon^*) + b(\partial_z\zeta)\,\varphi + C_1^{-1}(2\,\nabla\zeta\,\nabla\varphi + \Delta\zeta\varphi), \\ \zeta\,\varphi = 0 \qquad \text{on } \Sigma_T^D\ , \\ \dfrac{\partial(\zeta\varphi)}{\partial n} = 0 \quad \text{on } \Sigma_T^N\ , \\ \zeta\,\varphi|_{t=T} = 0\ . \end{cases} \tag{3.13}$$

The trace theorem and the $W_2^{2,1}$-estimate of (3.13) produce

$$\begin{aligned}\int_t^T \int\limits_{z=0} |\partial_z\varphi|^2 &= \int_t^T \int\limits_{z=0} |\partial_z(\zeta\,\varphi)|^2 \\ &\le C\int_t^T |\varphi|^2_{H^2(Q)} \\ &\le C\Big(|u_\varepsilon^\lambda - u_\varepsilon^*|^2_{L^2(Q_t)} + |\nabla\varphi|^2_{L^2(Q_t)}\Big)\ .\end{aligned}$$

Substitute it into (3.12), by the Gronwall inequality, we have

$$|\nabla\varphi|^2_{L^2(Q_T)} \le C \iiint\limits_{Q_T} \Big[\beta_\varepsilon(u_\varepsilon^\lambda) - \beta_\varepsilon(u_\varepsilon^*)\Big](u_\varepsilon^\lambda - u_\varepsilon^*)\ ,$$

which is just (3.11) by the trace theorem.

From the result of Proposition 3.2, there exists a $W_\varepsilon \in L^2(Q_T)$ such that

$$W_\varepsilon^\lambda \rightharpoonup W_\varepsilon \quad \text{weakly in } \ L^2(Q_T) \ \ (\lambda\to 0)\ , \tag{3.14}$$

and W_ε is the $L^2(Q_T)$ weak solution of the following problem

$$\begin{cases} \partial_t[\beta'_\varepsilon(u_\varepsilon^*)\,W_\varepsilon] + b\,\partial_z[\beta'_\varepsilon(u_\varepsilon^*)\,W_\varepsilon] - \Delta W_\varepsilon = 0, \\ W_\varepsilon = 0 \qquad \text{on } \Sigma_T^D, \\ -\dfrac{\partial W_\varepsilon}{\partial n} = g \quad \text{on } \Sigma_T^N, \\ W_\varepsilon = 0 \qquad \text{on } t=0\ . \end{cases} \tag{3.15}$$

In fact, from (3.6) and (3.7), we have

$$\iiint\limits_{Q_T} \bigg\{ \Big[\int_0^1 \beta'_\varepsilon\Big(\tau\,u_\varepsilon^\lambda + (1-\tau)\,u_\varepsilon^*\Big)\,d\tau\Big](\partial_t\varphi + b\,\partial_z\varphi) + \Delta\varphi \bigg\} W_\varepsilon^\lambda = \int\limits_{\Sigma_T^N} g\,\varphi\ ,$$

for any $\varphi \in C^{2,1}(Q_T)$, $\varphi = 0$ on $t=T$ and Σ_T^D, $\frac{\partial\varphi}{\partial n} = 0$ on Σ_T^N.

Since $\beta_\varepsilon \in C^2$, $\min\{c_1,c_2\} < \beta'_\varepsilon \le \frac{2}{\varepsilon}$, $u_\varepsilon^\lambda \to u_\varepsilon^*$ strongly in $L^2(Q_T)$, so for a.e. $(x,y,z,t)\in Q_T$,

$$\tau\,u_\varepsilon^\lambda + (1-\tau)\,u_\varepsilon^* \to u_\varepsilon^* \quad \text{a.e. in } \ [0,1] \ \ (\lambda\to 0)\ ,$$

the Lebesgue theorem shows that

$$\int_0^1 \beta'_\varepsilon\Big(\tau\,u_\varepsilon^\lambda + (1-\tau)\,u_\varepsilon^*\Big)\,d\tau \to \beta'_\varepsilon(u_\varepsilon^*) \quad \text{a.e. in } \ Q_T \tag{3.16}$$

and then strongly in $L^2(Q_T)$ if $\lambda \to 0$.

Sending $\lambda \to 0$, considering (3.14), we get

$$\iint\limits_{Q_T} \Big[\beta_\varepsilon'(u_\varepsilon^*)\,(\partial_t\varphi + b\,\partial_z\varphi) + \Delta\varphi\Big]\, W_\varepsilon = \int\limits_{\Sigma_T^N} g\,\varphi\ . \tag{3.17}$$

This is just the weak formulation of (3.15).

Now we introduce the following adjoint problem

$$\begin{cases} \beta_\varepsilon'(u_\varepsilon^*)\,(\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^*) + \Delta p_\varepsilon^* = u_\varepsilon^* - u_d & \text{in } Q_T,\\ p_\varepsilon^* = 0 & \text{on } \Sigma_T^D,\\ -\dfrac{\partial p_\varepsilon^*}{\partial n} = 0 & \text{on } \Sigma_T^N,\\ p_\varepsilon^*(x,y,z,T) = 0\ , \end{cases} \tag{3.18}$$

where p_ε^* is the adjoint state.

Proposition 3.3. *For the problem (3.17), we have the following uniform estimates*

$$\begin{cases} |\nabla p_\varepsilon^*|_{L^\infty(0,T;L^2(Q))} \le C\ ,\\ |\partial_t p_\varepsilon^*|_{L^2(Q_T)} \le C,\\ \displaystyle\iint\limits_{Q_T} \frac{1}{\beta_\varepsilon'(u_\varepsilon^*)}\,(\Delta p_\varepsilon^*)^2 \le C\ , \end{cases} \tag{3.19}$$

where C is independent of $\varepsilon > 0$.

Proof: The equation $(3.18)_1$ is equivalent to

$$\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^* + \frac{1}{\beta_\varepsilon'(u_\varepsilon^*)}\,\Delta P_\varepsilon^* = \frac{1}{\beta_\varepsilon'(u_\varepsilon^*)}\,(u_\varepsilon^* - u_d)\ . \tag{3.20}$$

Multiplying the equation (3.20) by Δp_ε^*, integrating over Q_t, the similar procedure to the problem (3.9) produces the estimate $(3.19)_1$ and $(3.19)_3$. Multiplying the equation (3.20) by $\partial_t p_\varepsilon^*$, integrating over Q_t yields $(3.19)_2$.

The proof of Proposition 3.3 is completed. ∎

On other hand, by the optimality of $(u_\varepsilon^*, g_\varepsilon^*)$, we have

$$0 \le \frac{1}{\lambda}\Bigg\{\frac{1}{2}\iint\limits_{Q_T}\Big(|u_\varepsilon^\lambda - u_d|^2 - |u_\varepsilon^* - u_d|^2\Big) + \frac{1}{2}\int\limits_{\Sigma_T^N}\Big[(g_\varepsilon^* + \lambda\,g)^2 - g_\varepsilon^{*2}\Big]$$
$$+ \frac{1}{2}\int\limits_{\Sigma_T^N}\Big[(g_\varepsilon^* + \lambda\,g - g^*)^2 - (g_\varepsilon^* - g^*)^2\Big]\Bigg\}\ .$$

Letting $\lambda \to 0$, we obtain

$$0 \le \iint\limits_{Q_T} (u_\varepsilon^* - u_d)\, W_\varepsilon + \int\limits_{\Sigma_T^N} g_\varepsilon^*\, g + \int\limits_{\Sigma_T^N} (g_\varepsilon^* - g^*)\, g \quad \text{for any} \quad g \in \widetilde{G}_\varepsilon \ . \tag{3.21}$$

Taking into account of $(3.18)_1$ and (3.17), from (3.21), we get

$$0 \le \iint\limits_{Q_T} (p_\varepsilon^* + 2\, g_\varepsilon^* - g^*)\, g \quad \text{for any} \quad g \in \widetilde{G}_\varepsilon \ . \tag{3.22}$$

Up to now we get that the necessary (optimal) conditions for the smooth optimal control problem (3.1)–(3.3) are (3.18) and (3.22).

4 – Necessary conditions

In this section we take the limits in (3.18) and (3.22) to get the necessary conditions for the optimal control problem (2.17)–(2.20).

At first, from the uniform estimates (3.19), we know that there exists a $p^* \in L^\infty(0,T;H^1(Q))$, such that $\partial_t p^* \in L^2(Q_T)$ and

$$\begin{cases} p_\varepsilon^* \to p^* & \text{strongly in } L^2(Q_T), \\ \nabla p_\varepsilon^* \rightharpoonup \nabla p^* & \text{weakly in } L^\infty(0,T;L^2(Q)), \\ \partial_t p_\varepsilon^* \rightharpoonup \partial_t p^* & \text{weakly in } L^2(Q_T) \ . \end{cases} \tag{4.1}$$

Taking $\varepsilon \to 0$ in (3.22) and recalling $(3.4)_1$, we get

$$0 \le \iint\limits_{Q_T} (p^* + g^*)\, g \quad \text{for any} \quad g \in \widetilde{G} \ .$$

Here

$$\widetilde{G} = \Big\{ g \in L^2(\Sigma_T^N),\ g^* + \lambda g \in G_{\rm ad},\ \text{for all sufficiently small } \lambda > 0 \Big\} \ ,$$

because $g_\varepsilon^* + \lambda g \in \widetilde{G}_\varepsilon$ follows $g^* + \lambda g \in \widetilde{G}$ by $(3.4)_1$.

In the following we want to take the limit in (3.18). It is a very difficult problem even if $b = 0$ and

$$\text{meas}\Big\{ (x,y,z,t) \in Q_T\colon\ u^*(x,y,z,t) = 0 \Big\} = 0 \tag{4.2}$$

because of the absence of the uniform estimate for $|\partial_t u_\varepsilon^*|_{L^2(Q_T)}$ (see [6], p. 213, Remark 7.9).

In order to take the limit in (3.18), we let

$$\beta_\varepsilon(\tau) = \begin{cases} c_1\,\tau + 1 & \text{if } \tau \ge \dfrac{\varepsilon}{1 - c_1\,\varepsilon}, \\ \dfrac{1}{\varepsilon}\,\tau & 0 < \tau < \dfrac{\varepsilon}{1 - c_1\,\varepsilon}, \\ c_2\,\tau & \text{if } \tau \le 0\;. \end{cases} \tag{4.3}$$

Here β_ε is not a C^2-function and it is only Lipschitz continuous. The unique place in previous proofs where we need $\beta_\varepsilon \in C^2$ is in (3.16). Notice that under the condition (4.2), the limit procedure in (3.16) is also true if β_ε is defined as (4.3). In this case

$$\tau\,\beta_\varepsilon'(\tau) = \begin{cases} c_1\,\tau & \text{if } \tau \ge \dfrac{\varepsilon}{1 - c_1\,\varepsilon}, \\ \dfrac{1}{\varepsilon}\,\tau & \text{if } 0 < \tau < \dfrac{\varepsilon}{1 - c_1\,\varepsilon}, \\ c_2\,\tau & \text{if } \tau \le 0\;. \end{cases}$$

So

$$\tau\,\beta_\varepsilon'(\tau) = \beta_\varepsilon(\tau) - \frac{1}{1 - c_1\,\varepsilon}\,H\Big(\tau - \frac{\varepsilon}{1 - c_1\,\varepsilon}\Big)\;, \tag{4.4}$$

where H is the Heaviside function.

Multiplying the equation $(3.18)_1$ by u_ε^* and using (4.4), we have

$$\Big[\beta_\varepsilon(u_\varepsilon^*) - \frac{1}{1 - c_1\,\varepsilon}\,H\Big(u_\varepsilon^* - \frac{\varepsilon}{1 - c_1\,\varepsilon}\Big)\Big](\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^*) + u_\varepsilon^*\,\Delta p_\varepsilon^* = (u_\varepsilon^* - u_d)\,u_\varepsilon^*\;. \tag{4.5}$$

Notice that

$$\begin{aligned} u_\varepsilon^*\,\Delta p_\varepsilon^* &= \nabla(u_\varepsilon^*\,\nabla p_\varepsilon^*) - \nabla u_\varepsilon^*\,\nabla p_\varepsilon^* \\ &= \nabla(u_\varepsilon^*\,\nabla p_\varepsilon^*) - \nabla(p_\varepsilon^*\,\nabla u_\varepsilon^*) + p_\varepsilon^*\,\Delta u_\varepsilon^*\;. \end{aligned} \tag{4.6}$$

Taking into account of the equation $(3.3)_1$, we obtain

$$\begin{aligned} p_\varepsilon^*\,\Delta u_\varepsilon^* &= p_\varepsilon^*\Big[\partial_t\beta_\varepsilon(u_\varepsilon^*) + b\,\partial_z\beta_\varepsilon(u_\varepsilon^*)\Big] \\ &= \partial_t[\beta_\varepsilon(u_\varepsilon^*)\,p_\varepsilon^*] + b\,\partial_z[\beta_\varepsilon(u_\varepsilon^*)\,p_\varepsilon^*] \\ &\quad - \beta_\varepsilon(u_\varepsilon^*)\,(\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^*)\;. \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) we get

$$\begin{aligned} \partial_t[\beta_\varepsilon(u_\varepsilon^*)\,p_\varepsilon^*] + b\,\partial_z[\beta_\varepsilon(u_\varepsilon^*)\,p_\varepsilon^*] - \frac{1}{1 - c_1\,\varepsilon}\,H\Big(u_\varepsilon^* - \frac{\varepsilon}{1 - c_1\,\varepsilon}\Big)(\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^*) + \\ + \nabla(u_\varepsilon^*\,\nabla p_\varepsilon^*) - \nabla(p_\varepsilon^*\,\nabla u_\varepsilon^*) = u_\varepsilon^*(u_\varepsilon^* - u_d)\;. \end{aligned} \tag{4.8}$$

Since

$$(4.9)\qquad \begin{cases} \beta_\varepsilon(u_\varepsilon^*) \to \beta(u^*) & \text{strongly in } L^2(Q_T),\\ H\Big(u_\varepsilon^* - \dfrac{\varepsilon}{1-c_1\,\varepsilon}\Big) \to H(u^*) & \text{strongly in } L^2(Q_T)\ , \end{cases}$$

by (4.2) and Lebesgue theorem. Notice that $\beta(u^*)$ has a meaning by (4.2).

So if we multiply the equation (4.8) by a test function $\zeta \in C^1(\overline{Q}_T)$ satisfying $\zeta = 0$ on $t = 0$ and Σ_T^D, we get

$$-\iint_{Q_T} \beta_\varepsilon(u_\varepsilon^*)\, p_\varepsilon^*(\partial_t\zeta + b\,\partial_z\zeta) - \frac{1}{1-c_1\,\varepsilon}\iint_{Q_T} H\Big(u_\varepsilon^* - \frac{\varepsilon}{1-c_1\,\varepsilon}\Big)(\partial_t p_\varepsilon^* + b\,\partial_z p_\varepsilon^*)\,\zeta -$$
$$-\iint_{Q_T} u_\varepsilon^*\,\nabla p_\varepsilon^*\,\nabla\zeta + \iint_{Q_T} p_\varepsilon^*\,\nabla u_\varepsilon^*\,\nabla\zeta + \int_{\Sigma_T^N} p_\varepsilon^*\,g_\varepsilon^*\,\zeta = \iint_{Q_T} u_\varepsilon^*(u_\varepsilon^* - u_d)\,\zeta\ .$$

Considering (3.4), (4.1) and (4.9), let $\varepsilon \to 0$, we obtain at last

$$(4.10)\qquad \iint_{Q_T} \beta(u^*)\,p^*(\partial_t\zeta + b\,\partial_z\zeta) + \iint_{Q_T} H(u^*)\,(\partial_t p^* + b\,\partial_z p^*)\,\zeta +$$
$$+\iint_{Q_T} u^*\,\nabla p^*\,\nabla\zeta - \iint_{Q_T} p^*\,\nabla u^*\,\nabla\zeta - \int_{\Sigma_T^N} p^*\,g^*\,\zeta + \iint_{Q_T} u^*(u^* - u_d)\,\zeta = 0\ .$$

Up to now we get

Theorem 4.1. *If (u^*, g^*) is an optimal pair for the problem (2.17)–(2.20) from Theorem 2.3, under the assumption (4.2), there is a function $p^* \in L^\infty(0,T;H^1(Q))$ such that $\partial_t p^* \in L^2(Q_T)$, moreover*

$$(4.11)\qquad \begin{cases} \partial_t[\beta(u^*)\,p^*] + b\,\partial_z[\beta(u^*)\,p^*] - H(u^*)\,(\partial_t p^* + b\,\partial_z p^*) + \\ \qquad + \nabla(u^*\,\nabla p^*) - \nabla(p^*\,\nabla u^*) = u^*(u^* - u_d),\\ p^* = 0 \qquad \text{on } \ \Sigma_T^D,\\ \dfrac{\partial p^*}{\partial n} = 0 \qquad \text{on } \ \Sigma_T^N,\\ p^*(x,y,z,T) = 0\ , \end{cases}$$

in the weak sense of (4.10) and

$$(4.12)\qquad 0 \le \int_{\Sigma_T^N} (p^* + g^*)\,g \quad \text{for any } \ g \in \widetilde{G}\ ,$$

(4.11) and (4.12) are necessary conditions of the optimal control problem (2.17)–(2.20).

Remark. Condition (4.2) means no mushy region. Under some reasonable conditions, one can get (4.2) (see [7]).

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