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REMARKS ON THE SMOOTHNESS OF THE $L^\infty(0,T;L^3)$ SOLUTIONS OF THE 3-D NAVIER–STOKES EQUATIONS

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Abstract: In this note we consider the $L^\infty(0,T;L^3(\Omega))$ solutions of the Navier–Stokes equations, where Ω is a domain of $\mathbb{R}^3$. We give a very simple proof of a sufficient condition for regularity of solutions. This condition contains, as a quite particular case, continuity from the left on $(0,T]$ with values in $L^3(\Omega)$. See Theorem 2.2 below.

Introduction

The existence of regular global solutions and the uniqueness of weak solutions to the Navier–Stokes equations are, may be, the more famous open problems in the field of nonlinear partial differential equations. These problems are open at least from the issuing of the celebrated J. Leray's paper [L]. Fundamental papers by J. Leray, E. Hopf, O.A. Ladyzhenskaya, J. Serrin, A.A. Kiselev, G. Prodi, J.L. Lions and others, have keep alive the interest on these challenging problems. More recently, this interest has been revived by well known papers by V. Sheffer and by L. Caffarelli, R. Kohn and L. Nirenberg. It is from this days the work by J. Nečas, M. Røužička and V. Šverák where these authors show that the stationary equation (3.11) in reference [L] has no nontrivial solutions (otherwise, singularities to the evolution Navier–Stokes equations would appear in finite time; see [L]).

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The failure of all the attempts to giving satisfactory answers to the above open problems has carried to a central position the investigation of sufficient conditions for existence and uniqueness of solutions. In this direction, a main line of research starts with the pioneering paper [P] by G. Prodi followed by J. Serrin's paper [S]. It consists in looking for sufficient conditions like (2.2). Many further interesting contributions were given in this same direction. See, for instance, [F], [FJR], [K], [M], [So], [SW], [G], [GM] and many others; an extension of the sufficient condition (2.2) to values $s \leq 2$ was obtained in reference [BV1]. Note, however, the lack of whatever recent improvement in the direction of increasing the critical value (the right hand side of (2.2)) beyond the value 1. A useful basis for further developments in this direction consists in having at one's disposal simple proofs of the basic results, in particular by keeping all the hypotheses within a sensible range of generality (a not severe constraint, since the presence of the very restrictive assumption (2.2) made superfluous a wide generality on other points). Following this point of view we assume here that $n = 3$ and that the initial data and the external force fields are sufficiently general to our purposes.

Let us now introduce the problem studied below. It is well known that a weak solution $u(t)$ of the Navier–Stokes equations that belongs to $L^\infty(0, T, L^n)$ is unique; see [So] and references. Moreover such a solution is regular if it is continuous in $[0, T]$ with values in L^n; see [W] and also [G]. Below we introduce a condition (called, for convenience, condition A) such that if a weak solution belongs to $L^\infty(0, T; L^3)$ and satisfies this condition it is necessarily a strong solution (a straightforward bootstrap argument shows then that the solution is more regular provided that a, f and the boundary Γ are sufficiently smooth). Our condition contains, as a quite particular case, continuity from the left on $(0, T]$ with values in $L^3(\Omega)$.

Many of the above calculation are still valid if $n > 3$. For that reason we sometimes shall write n instead of 3.

We take the opportunity of referring the interested reader to [BV3] where we introduce another approach to the problem of establishing sufficient conditions for regularity.

Added in the press-proof: The author is grateful to Professor H. Sohr for some bibliographical information and also for the sending of the preprint to reference [KoS] where related results, obtained by completely different methods, are proved. Recently (September 97) we also received the preprint [Ko] where a condition related to the hypothesis A below is taken into consideration ([Ko], equation (1.4)).

1 – Preliminaries

In this section we recall some well-known definitions and results, useful in the sequel. Ω denotes an open bounded subset of $\mathbb{R}^3$ with boundary Γ. We assume that Γ is of class $C^{0,1}$ and that Ω is locally located on one side of Γ. The well known symbols $\mathcal{D}(\Omega)$, $H_0^m(\Omega)$, denote classical functional spaces that will be not defined here. In the sequel we drop the symbol Ω from these notations. We denote by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ the usual norms in the spaces L^p and $W^{m,p}$ respectively, and set $\|\cdot\| = \|\cdot\|_2$. We will not distinguish between spaces consisting of scalar or of vector functions. For instance, we denote $(L^2)^3$ simply by L^2. The same convention applies to norms. We denote by $(\cdot,\cdot)$ the scalar product in L^2 and by $((\cdot,\cdot))$ that in H_0^1, namely

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \, dx \ ,$$

where

$$\nabla u \cdot \nabla v = \sum_{i,j} (\partial u_i/\partial x_j)\,(\partial v_i/\partial x_j) \ .$$

We set

$$\mathcal{V} = \Big\{ v \in \mathcal{D}(\Omega) \colon \operatorname{div} v = 0 \Big\} \ ,$$

where $v = (v_1, v_2, v_3)$ and we denote by H and V the closures of $\mathcal{V}$ in L^2 and in H_0^1 respectively.

As usual, P denotes the orthogonal projection of L^2 onto H. We identify the dual space H' with H. Hence $V \hookrightarrow H \equiv H' \hookrightarrow V'$. All that is standard.

The symbol $C(0,T;X)$ denotes the space of functions $v(t)$, continuous in the closed interval $[0,T]$ with values in the Banach space X. $C_w(0,T;X)$ consists of continuous functions with respect to the weak topology in X. In the sequel we also use spaces $L^s(0,T;X)$, $1 \le s \le \infty$.

The unbounded operator A is defined in the following standard way. We set

$$D(A) = \Big\{ u \in V \colon \ v \to ((u,v)) \text{ is continuous on } V \text{ w.r.t. the } H\text{-topology} \Big\}$$

and define Au, for each $u \in D(A)$, as being the element of H for which

$$(Au, v) = ((u,v)) \,, \qquad \forall\, v \in V \ .$$

It is well known that $-A$ is the generator of an analytical semigroup (see, for instance, [DLi], pag. 379, example 3, or [LiM], pag. 24, theorem 3.2). Moreover, if Γ is regular then $D(A) = V \cap H^2$ and $A = -P\Delta$. This is a classical result due to L. Cattabriga and V.A. Solonnikov (for an elementary proof, where $\Gamma \in C^{1,1}$, see [BV2]). Actually the relation $D(A) = V \cap H^2$ is not essential here. In the following it is sufficient that $D(A) \hookrightarrow W^{1,6}$ together with the estimate

$$\|\nabla v\|_6 \leq c_0 \|Av\| \,, \qquad \forall\, v \in D(A) \;, \tag{1.1}$$

for some positive constant c_0.

Next we consider the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu\, \Delta u + (u \cdot \nabla) u + \nabla \pi = f, & \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0,x) = a(x) \;, & \end{cases} \tag{1.2}$$

where $Q_T = [0,T] \times \Omega$, $\Sigma_T = [0,T] \times \Gamma$. In order to avoid inessential manipulations we assume in the sequel that $a \in V$ and $f \in L^2(0,T;H)$.

For $u, v, w \in V$ we define

$$b(u,v,w) = \sum_{i,k=1}^{3} \int_\Omega u_k \, \frac{\partial v_i}{\partial x_k} \, w_i \, dx \;.$$

Note that $b(u,v,w) = -b(u,w,v)$. Hence $b(u,v,v) = 0$. Moreover, we define $B(u,v) \in V'$ (the dual space of V) by setting

$$(B(u,v), w) = b(u,v,w) \,, \qquad \forall\, w \in V \;.$$

For convenience we write $B(u) = B(u,u)$.

We say that u is a *weak solution* of the Navier–Stokes equations (1.2) if $u \in C_w(0,T;H) \cap L^2(0,T;V)$ and, moreover,

$$\begin{aligned} \int_0^T \Big[(u(t), \phi'(t)) - \mu((u(t), \phi(t))) - b(u(t), u(t), \phi(t)) + (f(t), \phi(t)) \Big] \, dt = \\ = (u(T), \phi(T)) - (a, \phi(0)) \;, \end{aligned} \tag{1.3}$$

for all $\phi \in C^1(0,T;V)$.

It is well known that there is, at least, one weak solution in $[0,T]$. References are classical.

We say that u is a *strong solution* of the Navier–Stokes equations (1.2) if

$$\begin{cases} u \in L^2(0,T;D(A)) \cap C(0,T;V), \\ u' \in L^2(0,T;H) \end{cases} \tag{1.4}$$

and

$$\begin{cases} u' + \mu\, Au + B(u) = f \quad \text{in } L^2(0,T;H), \\ u(0) = a\ . \end{cases} \tag{1.5}$$

Clearly, a strong solution satisfies (1.3). Note that the fact that u belongs to $C(0,T;V)$ follows from the other two assumptions in (1.4), since $V{=}[D(A),H]_{1/2}$.

As (1.1) holds, it is easily shown that

$$\|B(u,v)\| \le c\|u\|_V\, \|v\|_V^{1/2}\, \|v\|_{D(A)}^{1/2}\ . \tag{1.6}$$

It is not difficult (and well known) to use this last estimate in order to show (for instance, by a fixed point argument) that there is a positive constant c such that if

$$T \le c\Big(\|a\|_V^4 + \|f\|_{L^2(0,T;H)}^4\Big)^{-1}$$

then there is a (unique) strong solution u of problem (1.2) in $[0,T]$.

2 – Existence of the strong solutions

In this section we essentially consider weak solutions u that satisfy the assumption

$$u \in L^\infty(0,T;L^n)\ , \tag{2.1}$$

for $n = 3$. Since weak solutions belong to $C_w(0,T;L^2)$ it readily follows that weak solutions in the class (2.1) belong to $C_w(0,T;L^n)$. In particular $u(t)$ is well defined in L^n for each $t \in [0,T]$. It is known [So] that an uniqueness theorem hold (even for $n > 3$) if there is a weak solution satisfying the assumption (2.1). However, the following (very weak) uniqueness result is largely sufficient to our purposes here. If there is a weak solution u_1 satisfying (2.1) and a strong solution u_2 then necessarily $u_1 = u_2$. A very simple proof of this result can be done by adapting the proof of the theorem 2.9, chap. I, in reference [Li].

The above version of the uniqueness result together with the existence of the local strong solution yield the following (trivial) continuation property which, for convenience, we state as a lemma. A proof is given just for the reader's convenience.

Lemma 2.1. *Let be $n = 3$ and let u be a weak solution of the Navier–Stokes equations (1.2) satisfying (2.1). Assume, moreover, that for each $\bar{t} \in (0,T]$ the function u satisfies the following additional hypothesis: "If u is a strong solution in $[0,\tau]$, for each $\tau \in [0,\bar{t})$, u belongs to $C(0,\bar{t};V)$". Then u is a strong solution in $[0,T]$.*

Proof: Let $\overline{t}$ denote the supremum of the set of values τ for which u is a strong solution in $[0,\tau]$. The local existence theorem of a strong solution together with the above uniqueness result show that $\overline{t}>0$. The additional hypothesis in the lemma guarantees that u is a strong solution in $[0,\overline{t}]$. If it were $\overline{t}<T$, the same argument used above to show that $\overline{t}>0$ proves here that u is a strong solution in $[0,\overline{t}+\epsilon)$, for some $\epsilon>0$. Hence $\overline{t}=T$. ∎

It is easy to show (see below) that the additional property required in the above lemma is necessarily satisfied if the assumption (2.1) is replaced by the following one:

$$u \in L^s(0,T;L^r), \qquad \text{where } \frac{2}{s}+\frac{n}{r}=1 \ \text{ and } \ r>n \tag{2.2}$$

(here, n may be arbitrary). Hence, if $n=3$, *any weak solution satisfying (2.2) is necessarily strong*.

We point out that this last result is well known (even for arbitrarily large n) at least if Ω is smooth (see [So] and references). But, for $n=3$, a very elementary proof can be done by exploiting the local existence of a strong solution together with the classical manipulations developed in [P], lemma 5. Since the very short proof helps to clarify the borderline case (2.1), we present it here (without any claim of originality). As $u\in L^2(0,\tau;D(A))$ and $u'\equiv du/dt\in L^2(0,\tau;H)$ one has $(u',Au)=(1/2)\,d\|u\|_V^2/dt$. Consequently, it readily follows from equation (1.5) that

$$\frac{1}{2}\frac{d}{dt}\|u\|_V^2+\mu\|Au\|^2\le\|B(u)\|\,\|Au\|+\|f\|\,\|Au\| \tag{2.3}$$

in $[0,\overline{t})$. On the other hand, since $1/2=1/r+(r-2)/2r$ Hölder's inequality shows that

$$\|B(u)\|\le\|(u\cdot\nabla)u\|\le\|u\|_r\,\|\nabla u\|_{\frac{2r}{r-2}}\ ,$$

where $2r/(r-2)=2$ if $r=\infty$. Moreover

$$\|\nabla u\|_{\frac{2r}{r-2}}\le\|\nabla u\|^{1-\frac{n}{r}}\,\|\nabla u\|_{2^*}^{\frac{n}{r}}\le c\|u\|_V^{1-\frac{n}{r}}\,\|Au\|^{\frac{n}{r}}\ ,$$

since $(r-2)/2r=(1-n/r)/2+(n/r)/2^*$. Here $2^*=2n/(n-2)$ is a Sobolev embedding exponent. Consequently

$$\|B(u)\|\,\|Au\|\le c\|u\|_r\,\|u\|_V^{1-\frac{n}{r}}\,\|Au\|^{1+\frac{n}{r}}\ . \tag{2.4}$$

Thus, by Young's inequality,

$$\|B(u)\|\,\|Au\|\le c\|u\|_r^s\,\|u\|_V^2+(\mu/4)\,\|Au\|^2\ . \tag{2.5}$$

From (2.3) and (2.5) we get

$$\frac{d}{dt}\|u\|_V^2 + \mu\|Au\|^2 \le c_0\|u\|_r^s\,\|u\|_V^2 + (2/\mu)\,\|f\|^2$$

in $[0,\bar{t})$. In particular

$$\|u(t)\|_V^2 \le \left(\|a\|_V^2 + \frac{2}{\mu}\int_0^t \|f(\tau)\|^2\,d\tau\right)\cdot \exp\Bigl\{c_0\|u\|_{L^s(0,\bar{t};L^r)}\Bigr\}\ , \tag{2.6}$$

for each $t\in[0,\bar{t})$. Finally, from (2.3) we obtain an estimate for Au in $L^2(0,\bar{t};H)$ and from (1.5) an estimate for u' in $L^2(0,\bar{t};H)$. Hence $u\in C(0,\bar{t};V)$. ∎

Remark 2.1. If in equation (2.4) one has $r=n$ (i.e. if (2.2) is replaced by (2.1)) then a smallness assumption on the norm of u in $L^\infty(0,T;L^n)$ is required in order to get a sufficiently small coefficient for $\|Au\|^2$ in that same equation. In this case the additional property in Lemma 2.1 is superfluous. Consequently, weak solutions with a sufficiently small norm in $L^\infty(0,T;L^3)$ are strong (a well known result).

At the light of Lemma 2.1, our aim is now establishing conditions that imply the additional property described in that lemma. For each $k\ge 0$ and each $t\in[0,T]$ we set

$$A(t,k)=\Bigl\{x\in\Omega\colon\ |u(t,x)|>k\Bigr\}\ .$$

Hypothesis A. *We say that u satisfies the hypothesis A at $\bar{t}$ (with respect to the constant C) if (2.1) holds and, moreover, if there are $\delta>0$ and a real nonnegative function $k(t)$ defined and square integrable on $(\bar{t}-\delta,\bar{t})$ such that*

$$\int_{A(t,k(t))} |u(t,x)|^n\,dx \le C^n, \qquad \text{a.e. in}\ \ (\bar{t}-\delta,\bar{t})\ . \tag{2.7}$$

We say that u satisfies the hypothesis A in $[0,T]$ if it satisfies the hypothesis A at each $\bar{t}\in(0,T]$; here δ and $k(t)$ may depend on the particular point $\bar{t}$.

Note that u necessarily satisfies the hypothesis A in $[0,T]$ with respect to its norm in the class (2.1); in this case $k\equiv 0$. Below, we show that (for $n=3$) weak solutions satisfying the hypothesis A with respect to the constant $C_0=\mu/2c_0$ (see (1.1)) are necessarily strong (hence regular). It is worth noting that continuity from the left implies the condition A (with respect to *any arbitrarily small* positive constant C and for a *constant function* k). In the sequel we consider a slightly more general case.

Proposition 2.1. *Assume that a function u, that belongs to the class (2.1), is left continuous in $(0,\bar{t}]$ with respect to the weak topology in L^n and, moreover, that*

$$\limsup_{t\to\bar{t}-0} \|u(t)\|_n^n < \|u(\bar{t})\|_n^n + 4(C/4)^n \ . \tag{2.8}$$

Then the hypothesis A holds at $\bar{t}$ (with a constant function k). Hence it holds, in particular, if u is left continuous with respect to the strong topology in L^n.

The proof of the Proposition 2.1 is postponed to the end of this chapter. Next we state our main result.

Theorem 2.1. *Let u be a weak solution of problem (1.2). Assume that for some $\bar{t} \in (0,T]$ u is a strong solution in $[0,\tau]$ for each $\tau < \bar{t}$ and, moreover, u satisfies the hypothesis A at $\bar{t}$ with respect to the constant C_0. Then $u \in C(0,\bar{t};V)$.*

The above theorem shows that the additional hypothesis in the Lemma 2.1 holds if u satisfies the hypothesis A in $[0,T]$. Hence, for $n=3$, one has the following result.

Theorem 2.2. *Assume that $n=3$, $\Gamma \in C^{0,1}$, $a \in V$ and $f \in L^2(0,T;H)$. Let u be a weak solution of problem (1.2) which satisfies the hypothesis A in $[0,T]$ with $C = C_0$. Then u is a strong solution in $[0,T]$. In particular u is a strong solution if (2.8) holds with $C = C_0$, hence if u is strongly continuous from the left in $(0,T]$.*

Proof of Theorem 2.1: By the hypothesis A there is a $t_0 = \bar{t} - \delta$ and a function $k(t)$ in $L^2(t_0,\bar{t})$ such that (2.7) holds. From (2.3) it readily follows that

$$\frac{d}{dt}\|u\|_V^2 + \mu\|Au\|^2 \le \frac{1}{\mu}\|B(u)\|^2 + 2\|f\|\,\|Au\| \ . \tag{2.9}$$

Moreover,

$$\|B(u)\|^2 \le \int_{\Omega/A(t)} |u|^2\,|\nabla u|^2\,dx + \int_{A(t)} |u|^2\,|\nabla u|^2\,dx$$

where, for convenience, we set $A(t) = A(t,k(t))$. By using Hölder's inequality it follows that

$$\|B(u)\|^2 \le k^2(t)\int_\Omega |\nabla u|^2\,dx + \Big(\int_{A(t)} |u|^n\,dx\Big)^{2/n}\Big(\int_\Omega |\nabla u|^{2^*}\,dx\Big)^{2/2^*},$$

where $2^* = 2n/(n-2)$. Hence, by the hypothesis A,

$$\|B(u)\|^2 \le k^2(t)\,\|u\|_V^2 + C^2\,c_0^2\|Au\|^2\ .$$

This estimate together with (2.9) shows that

$$\frac{d}{dt}\|u\|_V^2 + \frac{\mu}{4}\|Au\|^2 \le \frac{k^2(t)}{\mu}\|u\|_V^2 + \frac{2}{\mu}\|f\|^2 \qquad \text{a.e. in } \ (t_0,\bar t)\ .$$

Therefore $u \in L^2(t_0,\bar t; D(A)) \cap L^\infty(t_0,\bar t; V)$, moreover $u' \in L^2(t_0,\bar t; H)$. This shows that $u \in C(t_0,\bar t; V)$. ∎

Remark 2.2. It is worth noting that the hypotheses of Theorem 2.1 by themselves do not allow us to use a compactness argument. In fact, let X be any *infinite* dimensional Hilbert space. Assume that $u \in L^\infty(0,T;X)$ is weakly continuous in $[0,T]$ and strongly continuous from the left in $[0,T)$ with values in X. It does not follow from these assumptions that there is a $\delta > 0$ such that the set $\{v(t)\colon\ t \in (T-\delta,T)\}$ is relatively compact in X.

Proof of Proposition 2.1: Assume that the hypotheses in this proposition hold but that (2.7) is false. Then

$$\int_{A(t,k)} |u(t,x)|^n\,dx \le C^n \qquad \text{a.e. in } \ (\bar t - k^{-1}, \bar t)$$

is false for each positive integer k. Hence there is a sequence t_k, $\bar t - k^{-1} < t_k < \bar t$, such that

$$\int_{A(k)} |u(t_k,x)|^n\,dx \ge C^n\,, \qquad \forall\, k \in \mathbb{N}\ ,$$

where $A_k = A(t_k,k)$. Set $u_k(x) = u(t_k,x)$ and $v = u(\bar t)$. Clearly

$$C^n \le \int_{A_k} |u_k|^n\,dx \le 2^{n-1}\int_{A_k} |u_k - v|^n\,dx + 2^{n-1}\int_{A_k} |v|^n\,dx\ . \tag{2.10}$$

In particular, by using Clarkson inequality, one gets

$$C^n \le 2^{n-1}\Big[2^{n-1}\Big(\|u_k\|_n^n + \|v\|_n^n\Big) - \|u_k + v\|_n^n\Big] + 2^{n-1}\int_{A_k} |v|^n\,dx\ .$$

Next, by passing to the limit as k goes to infinity, by using (2.8), and by taking into account that u_k is weakly convergent in L^n to v, it readily follow that

$$C^n < C^n + 2^{n-1}\lim_{k\to\infty}\int_{A_k} |v|^n\,dx\ . \tag{2.11}$$

On the other hand

$$k^n |A_k| \le \int_{A_k} |u_k|^n \, dx \le \|u\|^n_{L^\infty(0,T;L^n)} \ .$$

Thus, $|A_k| \le c/k^{-n}$. Hence, by the absolute continuity of the integral with respect to the measure it follows that

$$\lim_{k\to\infty} \int_{A_k} |v|^n \, dx = 0 \ ,$$

which, together with (2.11), shows a contradiction. ∎

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