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ON $\varphi - |\overline{N}, p_n; \delta|_k$ SUMMABILITY FACTORS

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Abstract: In this paper a general theorem on $\varphi - |\overline{N}, p_n; \delta|_k$ summability factors, which generalizes a result of Bor [2] on $|\overline{N}, p_n|_k$ summability factors, has been proved.

1 – Introduction

Let (φ_n) be a sequence of positive real numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty\,, \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1)\ . \tag{1.1}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\, s_v \qquad (P_n \neq 0) \tag{1.2}$$

defines the sequence (T_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [3]).

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty\ . \tag{1.3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\overline{N}, p_n|$) summability.

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The series $\sum a_n$ is said to be summable $\varphi - |\overline{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |T_n - T_{n-1}|^k < \infty \ . \tag{1.4}$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |\overline{N}, p_n; \delta_k|$ summability is the same as $|\overline{N}, p_n|_k$ summability.

2 – The following theorem is known

Theorem A ([2]). *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(n\, p_n) \quad \text{as } \ n \to \infty \ . \tag{2.1}$$

Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta \lambda_n| \leq \beta_n \ , \tag{2.2}$$

$$\beta_n \to 0 \quad \text{as } \ n \to \infty \ , \tag{2.3}$$

$$\lambda_m \, X_m = O(1) \quad \text{as } \ m \to \infty \ , \tag{2.4}$$

$$\sum_{n=1}^{\infty} n \, X_n |\Delta \beta_n| < \infty \ . \tag{2.5}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as } \ m \to \infty \ , \tag{2.6}$$

then the series $\sum a_n \, \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

3. The object of this paper is to generalize above theorem in the following form.

Theorem. *Let (p_n) be a sequence of positive numbers such that condition (2.1) of Theorem A is satisfied and let (φ_n) be a sequence of positive real numbers* *such that*

$$\varphi_n \, p_n = O(P_n) \ , \tag{3.1}$$

$$\sum_{n=v+1}^{\infty} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v}\right) \ . \tag{3.2}$$

If (X_n) is a positive non-decreasing sequence and suppose that there exist sequences (λ_n) and (β_n) such that conditions (2.2)–(2.5) of Theorem A are satisfied. If

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k = O(X_m) \quad \text{as} \ \ m \to \infty \ , \tag{3.3}$$

then the series $\sum a_n \, \lambda_n$ is summable $\varphi - |\overline{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in this theorem, then we get Theorem A.
We need the following lemma for the proof of our theorem.

Lemma ([4]). *If (X_n) is a positive non-decreasing sequence and (β_n) is a positive sequence such that (2.3) and (2.5) hold, then*

$$n \, X_n \, \beta_n = o(1) \quad \text{as} \ \ n \to \infty \ , \tag{3.4}$$

$$\sum_{n=1}^{\infty} X_n \, \beta_n < \infty \ . \tag{3.5}$$

4 – Proof of the Theorem

Let (T_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum a_n \, \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \, \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) \, a_v \, \lambda_v \ .$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n \, P_{n-1}} \sum_{v=1}^{n} P_{v-1} \, a_v \, \lambda_v = \frac{p_n}{P_n \, P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \, \lambda_v}{v} \, v \, a_v \ .$$

Using Abel's transformation, we get

$$
\begin{aligned}
T_n - T_{n-1} &= \frac{(n+1)}{nP_n}\, p_n\, t_n\, \lambda_n - \frac{p_n}{P_n\, P_{n-1}} \sum_{v=1}^{n-1} p_v\, t_v\, \lambda_v\, \frac{v+1}{v} \\
&\quad + \frac{p_n}{P_n\, P_{n-1}} \sum_{v=1}^{n-1} P_v\, \Delta\lambda_v\, t_v\, \frac{v+1}{v} - \frac{p_n}{P_n\, P_{n-1}} \sum_{v=1}^{n-1} p_v\, t_v\, \lambda_{v+1}\, \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\ , \quad \text{say}\ .
\end{aligned}
$$

By Minkowski's inequality it is sufficient to show that

$$
\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |T_{n,r}|^k < \infty, \qquad \text{for} \ \ r = 1,2,3,4\ .
$$

Since $\lambda_n = O(1/X_n) = O(1)$, by (2.4), we get that

$$
\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} |\lambda_n|^{k-1}\, |\lambda_n|\, \varphi_n^{\delta k-1} |t_n|^k = O(1) \sum_{n=1}^{m} |\lambda_n|\, \varphi_n^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{\delta k-1} |t_v|^k + O(1)\, |\lambda_m| \sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \beta_n\, X_n + O(1)\, |\lambda_m|\, X_m = O(1) \qquad \text{as} \ \ m \to \infty\ ,
\end{aligned}
$$

by virtue of the hypotheses and the Lemma.

Now, when $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1}\, \frac{1}{P_{n-1}} \Big\{ \sum_{v=1}^{n-1} p_v |t_v|^k\, |\lambda_v|^k \Big\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1}\, |\lambda_v|\, p_v |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1}\, \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|\, \varphi_v^{\delta k-1} |t_v|^k = O(1) \qquad \text{as} \ \ m \to \infty\ .
\end{aligned}
$$

Since $v\,\beta_v=o(1/X_v)=O(1)$, by (3.4), using the fact that $P_v=O(v\,p_v)$, by (2.1), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\,\frac{1}{P_{n-1}^k}\Big\{\sum_{v=1}^{n-1}v\,p_v\,\beta_v\,|t_v|\Big\}^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\Big\{\sum_{v=1}^{n-1}(v\beta_v)^k\,p_v|t_v|^k\Big\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}(v\,\beta_v)^{k-1}\,v\,\beta_v\,p_v\,|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\,\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}v\,\beta_v\,\frac{p_v}{P_v}\,\varphi_v^{\delta k}\,|t_v|^k = O(1)\sum_{v=1}^{m}v\,\beta_v\,\varphi_v^{\delta k-1}\,|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\,\beta_v)\sum_{r=1}^{v}|t_r|^k\,\varphi_r^{\delta k-1}+O(1)\,m\,\beta_v\sum_{v=1}^{m}\varphi_v^{\delta k-1}\,|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\,\beta_v)|\,X_v+O(1)\,m\,\beta_m\,X_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|\,X_v+O(1)\sum_{v=1}^{m-1}\beta_{v+1}\,X_v+O(1)\,m\,\beta_m\,X_m\\
&= O(1)\quad\text{as}\quad m\to\infty\ ,
\end{aligned}$$

by virtue of the hypotheses and the Lemma.

Finally, using the fact that $P_v=O(v\,p_v)$, by (2.1), as in $T_{n,1}$ and $T_{n,2}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|T_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\,\frac{1}{P_{n-1}^k}\Big\{\sum_{v=1}^{n-1}p_v\,|t_v|\,|\lambda_{v+1}|\Big\}^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\,\frac{1}{P_{n-1}}\Big\{\sum_{v=1}^{n-1}p_v\,|t_v|^k\,|\lambda_{v+1}|^k\Big\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}p_v\,|t_v|^k\,|\lambda_{v+1}|^{k-1}\,|\lambda_{v+1}|\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\,\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k-1}\,|t_v|^k\,|\lambda_{v+1}| = O(1)\quad\text{as}\quad m\to\infty\ .
\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,r}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for } r = 1, 2, 3, 4 \ .$$

This completes the proof of the theorem. ∎

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