SOME REMARKS ON TARDIFF’S FIXED POINT THEOREM ON MENGER SPACES

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1 – Introduction

Let $D_+ = \{F : \mathbb{R} \to [0, 1] \mid F(0) = 0\}$ be the family of all distribution functions $F : \mathbb{R} \to [0, 1]$ such that $F(0) = 0$, and $H_0$ be the element of $D_+$ which is defined by

$$H_0 = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A $t$-norm $T$ is a binary operation on $[0, 1]$ which is associative, commutative, has 1 as identity, and is non-decreasing in each place. We say that $T'$ is stronger than $T''$ and we write $T' \succeq T''$ if $T'(a, b) \geq T''(a, b)$, $\forall a, b \in [0, 1]$.

**Definition 1.1.** Let $X$ be a set, $\mathcal{F} : X^2 \to D_+$ a mapping ($\mathcal{F}(x, y)$ will be denoted $F_{xy}$) and $T : [0, 1] \times [0, 1] \to [0, 1]$ a $t$-norm. The triple $(X, \mathcal{F}, T)$ is called a Menger space if it satisfies the following properties:

1. **(PM0)** If $x \neq y$ then $F_{xy} \neq H_0$;
2. **(PM1)** If $x = y$ then $F_{xy} = H_0$;
3. **(PM2)** $F_{xy} = F_{yx}$, $\forall x, y \in X$;
4. **(M)** $F_{xy}(u + v) \geq T(F_{xz}(u), F_{zy}(v))$, $\forall x, y, z \in X$, $\forall u, v \in \mathbb{R}$.

Let $f : [0, 1] \to [0, \infty]$ be a continuous function which is strictly decreasing and vanishes at 1.

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Definition 1.2 ([6]). The pair \((X, \mathcal{F})\) which has the properties (PM0)–(PM2) is called a probabilistic \(f\)-metric structure iff
\[
\forall t > 0 \quad \exists s > 0 \quad \text{such that} \quad \left[ f \circ F_{xy} (s) < s, \ f \circ F_{xy} (s) < s \right] \Rightarrow f \circ F_{xy} (t) < t .
\]

Remark 1.3. If \((X, \mathcal{F})\) is a probabilistic \(f\)-metric structure then the family
\[
W^f = \{ \mathcal{W}_r \}_{r \in (0, f(0))}, \quad \text{where} \quad \mathcal{W}_r = \{ (x, y) | F_{xy} (r) > f^{-1} (r) \},
\]
is a uniformity base which generates a uniformity on \(X\) called \(\mathcal{U}_f\) [6, p.46, Th. 1.3.39].

We define the \(t\)-norm generated by \(f\) by:
\[
T_f (a, b) = f^{(-1)} \left( f(a) + f(b) \right)
\]
where \(f^{(-1)}\) is the quasi-inverse of \(f\), namely
\[
f^{(-1)} (x) = \begin{cases} 
    f^{-1} (x), & x \leq f(0), \\
    0, & x > f(0) .
\end{cases}
\]

It is well known and easy to see that \(f \circ f^{(-1)} (x) \leq x, \ \forall x \in [0, \infty] \) and \(f^{(-1)} \circ f (a) = a, \ \forall a \in [0, 1]\).

In the next section of this note we’ll construct generalized metrics on Menger spaces, related to some ideas which have appeared in [11] and [4], and using some properties of the probabilistic \(f\)-metric structures.

In the last section, using this generalized metrics, we’ll obtain a fixed point theorem on complete Menger spaces and we’ll give some consequences. We’ll give also, a fixed point alternative in complete Menger spaces.

The notations and the notions not given here are standard and follow [1], [8].

2 – A generalized metric on probabilistic \(f\)-metric structures

Let \(f : [0, 1] \rightarrow [0, 1]\) a continuous and strictly decreasing function, such that \(f(1) = 0\).

Lemma 2.1. We consider a Menger space \((X, \mathcal{F}, T)\), where \(T \geq T_f\). For each \(k > 0\) let us define
\[
d_k (x, y) := \sup_{s > 0} \left( \int_0^\infty \frac{f \circ F_{xy} (t)}{t} dt \right)^k .
\]
and
\[ \rho_k(x, y) := \left( d_k(x, y) \right)^{\frac{1}{k + 1}}. \]

Then \( \rho_k \) is a generalized metric on \( X \).

**Proof:** It is clear that \( \rho_k \) is symmetric and \( \rho_k(x, x) = 0 \).

If \( \rho_k(x, y) = 0 \), then for each \( s > 0 \),
\[ \int \frac{f \circ F_{xy}(t)}{t} dt = 0 \]
which implies
\[ \int \frac{f \circ F_{xy}(u)}{u} du = 0, \quad \forall t > s > 0. \]

Since \( \frac{f \circ F_{xy}(u)}{u} \geq \frac{f \circ F_{xy}(t)}{t} \geq 0 \) for each \( u \in (s, t) \), then
\[ 0 = \int \frac{f \circ F_{xy}(u)}{u} du \geq \frac{f \circ F_{xy}(t)}{t} (t-s), \quad \forall t > s > 0, \]
which implies \( f \circ F_{xy}(t) = 0, \forall t > 0 \). Since \( f \) is a strictly decreasing function and \( f(1) = 0 \) then \( F_{xy}(t) = 1, \forall t > 0 \), that is \( x = y \).

Because \((X, F, T)\) is a Menger space and \( T \geq T_f \), we have
\[ F_{xy}(u+v) \geq T \left( F_{xz}(u), F_{zy}(v) \right) \geq T_f \left( F_{xz}(u), F_{zy}(v) \right), \quad \forall x, y, z \in X, \forall u, v \in \mathbb{R}. \]

Let us take \( u = \alpha t \) and \( v = \beta t \), where \( \alpha, \beta \in (0, 1), \alpha + \beta = 1 \). Then
\[ F_{xy}(t) \geq f^{(-1)} \left( f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t) \right), \quad \forall x, y, z \in X, \forall t > 0, \forall \alpha, \beta \in (0, 1), \alpha + \beta = 1, \]
and so
\[ f \circ F_{xy}(t) \leq (f \circ f^{(-1)}) \left( f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t) \right) \leq f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t), \quad \forall x, y, z \in X, \forall t > 0, \forall \alpha, \beta \in (0, 1), \alpha + \beta = 1. \]

We divide the both members of inequality by \( t \), integrate from \( s \) to \( ms \) and multiply with \( s^k \), where \( s > 0, m > 1, k > 0 \). We obtain
\[ s^k \int_s^{ms} \frac{f \circ F_{xy}(t)}{t} dt \leq s^k \int_s^{ms} \frac{f \circ F_{xz}(at)}{t} dt + s^k \int_s^{ms} \frac{f \circ F_{zy}(bt)}{t} dt, \quad \forall m > 1, \forall s > 0. \]
We take $\alpha t = u$, respectively $\beta t = v$ in the first, respectively, the second term of the right side of the previous inequality and it follows that:

$$s\int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt \leq \frac{1}{\alpha^k} (\alpha s)^k \int_{\alpha s}^{\infty} \frac{f \circ F_{xz}(u)}{u} \, du + \frac{1}{\beta^k} (\beta s)^k \int_{\beta s}^{\infty} \frac{f \circ F_{zy}(v)}{v} \, dv$$

$$\leq \frac{1}{\alpha^k} (\alpha s)^k \int_{\alpha s}^{\infty} \frac{f \circ F_{xz}(u)}{u} \, du + \frac{1}{\beta^k} (\beta s)^k \int_{\beta s}^{\infty} \frac{f \circ F_{zy}(v)}{v} \, dv$$

$$\leq \frac{1}{\alpha^k} \sup_{s > 0} (\alpha s)^k \int_{\alpha s}^{\infty} \frac{f \circ F_{xz}(u)}{u} \, du + \frac{1}{\beta^k} \sup_{s > 0} (\beta s)^k \int_{\beta s}^{\infty} \frac{f \circ F_{zy}(v)}{v} \, dv,$$

$$\forall m > 1, \forall s > 0.$$ 

By making $m \to \infty$ and taking $\sup$ in the left side of the previous inequality and by observing that

$$\sup_{s > 0} (\alpha s)^k \int_{\alpha s}^{\infty} \frac{f \circ F_{xz}(u)}{u} \, du = \sup_{s > 0} \int_{s}^{\infty} \frac{f \circ F_{xz}(t)}{t} \, dt$$

and

$$\sup_{s > 0} (\beta s)^k \int_{\beta s}^{\infty} \frac{f \circ F_{zy}(v)}{v} \, dv = \sup_{s > 0} \int_{s}^{\infty} \frac{f \circ F_{zy}(t)}{t} \, dt,$$

we obtain that

$$\sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt \leq \frac{1}{\alpha^k} \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{xz}(u)}{u} \, du$$

$$+ \frac{1}{\beta^k} \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{zy}(v)}{v} \, dv.$$ 

(2.1)

Let us denote

$$\begin{cases} a = \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt, \\ b = \frac{1}{\alpha^k} \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{xz}(t)}{t} \, dt, \\ c = \frac{1}{\beta^k} \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{zy}(t)}{t} \, dt. \end{cases}$$
If $b = \infty$ or and $c = \infty$ it follows that $\rho_{k}(x, z) = b \frac{1}{k+1} = \infty$ or and $\rho_{k}(z, y) = c \frac{1}{k+1} = \infty$ and it is obvious that $\rho_{k}(x, y) \leq \infty = \rho_{k}(x, z) + \rho_{k}(z, y)$.

We suppose that $b < \infty$ and $c < \infty$. The inequality (2.1) becomes:

$$a \leq \frac{b}{\alpha^{k}} + \frac{c}{\beta^{k}} = \frac{b}{\alpha^{k}} + \frac{c}{(1 - \alpha)^{k}}, \quad \forall \alpha \in (0, 1),$$

which implies $a \leq \inf_{0 < \alpha < 1} \left( \frac{b}{\alpha^{k}} + \frac{c}{(1 - \alpha)^{k}} \right), \quad \forall \alpha \in (0, 1)$.

We define the function $g : (0, 1) \to \mathbb{R}_{+}, \quad g(\alpha) = \frac{b}{\alpha^{k}} + \frac{c}{(1 - \alpha)^{k}}$. We observe that $g$ has a minimum in $\alpha_{0} = \frac{b^{1}}{b^{1} + c^{1}} (g'(\alpha_{0}) = 0)$.

Therefore

$$a \leq \frac{b}{\alpha_{0}^{k}} + \frac{c}{(1 - \alpha_{0})^{k}} = (b^{1} + c^{1})^{k+1}$$

and it is clear that

$$\rho_{k}(x, y) = a \frac{1}{k+1} \leq b \frac{1}{k+1} + c \frac{1}{k+1} = \rho_{k}(x, z) + \rho_{k}(z, y).$$

**Lemma 2.2.** Let $(X, F, T)$ be a Menger space with $T \geq T_{f}$. Then $\mathcal{U}_{f} \subset \mathcal{U}_{\rho_{k}}$.

**Proof:** It can be shown that $\inf_{a \in 1} \frac{T(a, a)}{T_{f}(a, a)} = 1$ and, using [6, p.41, Th. 1.3.22] we obtain that $(X, F)$ is a probabilistic $f$-metric structure. By using Remark 1.3 it suffices to show that

$$\forall \alpha \in (0, f(0)), \quad \exists \delta(\epsilon) : \quad \rho_{k}(x, y) < \delta \Rightarrow F_{xy}(\epsilon) > f^{-1}(\epsilon).$$

We observe that

$$\rho_{k}(x, y) < \delta \iff \sup_{s > 0} s^{k} \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt < \delta^{k+1}$$

$$\iff \forall s > 0, \quad s^{k} \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt < \delta^{k+1}$$

$$\Rightarrow \forall m > 1, \forall s > 0, \quad s^{k} \int_{s}^{ms} \frac{f \circ F_{xy}(t)}{t} \, dt < \delta^{k+1}.$$  

We take $s$ fixed, $s = \frac{\epsilon}{2}$ and $m = 2$. It follows

$$\left( \frac{\epsilon}{2} \right)^{k} \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{f \circ F_{xy}(t)}{t} \, dt < \delta^{k+1}.$$
But $t \leq \varepsilon \Rightarrow F_{xy}(t) \leq F_{xy}(\varepsilon) \Rightarrow f \circ F_{xy}(t) \geq f \circ F_{xy}(\varepsilon) \Rightarrow \frac{f \circ F_{xy}(t)}{t} \geq \frac{f \circ F_{xy}(\varepsilon)}{\varepsilon}$.

Therefore
\[
\left(\frac{\varepsilon}{2}\right)^k \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{f \circ F_{xy}(\varepsilon)}{\varepsilon} \, dt \leq \left(\frac{\varepsilon}{2}\right)^k \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{f \circ F_{xy}(t)}{t} \, dt < \delta^{k+1},
\]

which implies $\left(\frac{\varepsilon}{2}\right)^{k+1} \frac{f \circ F_{xy}(\varepsilon)}{\varepsilon} < \delta^{k+1}$. If we choose $\delta = \frac{\varepsilon}{2}$ we have $f \circ F_{xy}(\varepsilon) < \varepsilon$, which shows that the relation (2.2) is satisfied for $\delta(\varepsilon) = \frac{\varepsilon}{2}$.

**Lemma 2.3.** If $(X, F, T)$ is a complete Menger space under $T \geq T_f$, then $(X, \rho_k)$ is complete.

**Proof:** We suppose that $(x_n)$ is a $\rho_k$-Cauchy sequence, that is,
\[
\forall \varepsilon > 0, \quad \exists n_0(\varepsilon): \quad \forall n \geq n_0(\varepsilon), \forall p \geq 0 \Rightarrow \rho(x_n, x_{n+p}) < \varepsilon.
\]

From Lemma 2.2 we have that $(x_n)$ is a $U_F$-Cauchy sequence. Since $(X, F, T)$ is a complete Menger space, we obtain that $(x_n)$ is a $U_F$-convergent sequence, that is
\[
\exists x_0 \in X \text{ such that } \forall \varepsilon > 0, \exists n_1(\varepsilon): \forall n \geq n_1(\varepsilon) \Rightarrow F_{x_nx_0}(\varepsilon) > f^{-1}(\varepsilon).
\]

It remains to show that $(x_n)$ is a $\rho_k$-convergent sequence. From (2.3) we obtain that
\[
\varepsilon \geq \lim_{p \to \infty} \rho(x_n, x_{n+p}) = \lim_{p \to \infty} \sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{x_nx_{n+p}}(t)}{t} \, dt \geq \lim_{p \to \infty} s^k \int_{s}^{\infty} \frac{f \circ F_{x_nx_{n+p}}(t)}{t} \, dt, \quad \forall n \geq n_0(\varepsilon), \forall s > 0.
\]

By using the Fatou’s lemma and the continuity of $f$ we obtain:
\[
\varepsilon \geq \lim_{p \to \infty} s^k \int_{s}^{\infty} \frac{f \circ F_{x_nx_{n+p}}(t)}{t} \, dt \geq s^k \int_{s}^{\infty} \lim_{p \to \infty} \frac{f \circ F_{x_nx_{n+p}}(t)}{t} \, dt = \frac{1}{s} \int_{s}^{\infty} f \left(\lim_{p \to \infty} F_{x_nx_{n+p}}(t)\right) \, dt, \quad \forall n \geq n_0(\varepsilon), \forall s > 0.
\]
It can be proved that \( \lim_{p \to \infty} F_{x_n x_{n+p}}(t) = F_{x_n x_0}(t) \) (actually we’ll use only the fact that \( \lim_{p \to \infty} F_{x_n x_{n+p}}(t) \geq F_{x_n x_0}(t) \)) and the previous relation becomes

\[
\epsilon \geq s^k \int_0^\infty \frac{\int f \circ F_{x_n x_0}(t)}{t} \, dt, \quad \forall n \geq n_0(\epsilon), \quad \forall s > 0,
\]

which implies

\[
\rho_k(x_n, x_0) = \sup_{s>0} s^k \int_0^\infty \frac{\int f \circ F_{x_n x_0}(t)}{t} \, dt \leq \epsilon, \quad \forall n \geq n_0(\epsilon).
\]

Thus the lemma is proved. ■

### 3 – A fixed point theorem and some consequences

It is well-known that a mapping \( A : X \to X \) (where \( (X, F) \) is a PM-space) is called \( s \)-contraction if there exists \( L \in (0, 1) \) such that \( F_{AxAy}(Lt) \geq F_{xy}(t) \) for all \( t \in \mathbb{R} \), for all \( x, y \in X \).

**Lemma 3.1.** If \( (X, F) \) is a probabilistic \( f \)-metric structure and \( A \) is an \( s \)-contraction then \( A \) is, for each \( k > 0 \), a strict contraction in \( (X, \rho_k) \).

**Proof:** Since \( F_{AxAy}(Lt) \geq F_{xy}(t) \) for some \( L \in (0, 1) \), and every real \( t \) then we have

\[
s^k \int_0^\infty \frac{\int f \circ F_{AxAy}(Lt)}{t} \, dt \leq \epsilon \int_0^\infty \frac{\int f \circ F_{xy}(t)}{t} \, dt.
\]

If we make \( Lt = u \) in the left side, then we obtain

\[
\frac{1}{L^k} \int_{sL}^\infty \frac{\int f \circ F_{AxAy}(u)}{u} \, du \leq s^k \int_0^\infty \frac{\int f \circ F_{xy}(t)}{t} \, dt
\]

\[
\leq \sup_{s>0} s^k \int_0^\infty \frac{\int f \circ F_{xy}(t)}{t} \, dt = d_k(x, y), \quad \forall s > 0.
\]

Therefore, if we take sup in the first member of the above inequality, then we obtain that

\[
\frac{1}{L^k} d_k(Ax, Ay) \leq d_k(x, y) \quad \text{and it is clear that}
\]

\[
(3.1) \quad \rho_k(Ax, Ay) \leq L_1 \rho_k(x, y) \quad \text{where} \quad L_1 = L \frac{k}{k+1} \in (0, 1)
\]

and the lemma is proved. ■

Now, we can prove our main result:
Theorem 3.2. Let \((X, \mathcal{F}, T)\) be a complete Menger space with \(T \geq T_1\). If there exists some \(k > 0\) such that for every pair \((x, y) \in X\) one has
\[
\sup_{s > 0} s^k \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt < \infty,
\]
then every \(s\)-contraction on \(X\) has a unique fixed point.

Proof: The relation (3.2) shows that \(\rho_k\) is a metric. From Lemma 3.1 we obtain that \(A\) is a strict contraction in \((X, \rho_k)\). Let \(x \in X\) be an arbitrary point. From (3.1) we have that \((A^i x)\) is a \(\mathcal{U}_{\rho_k}\)-Cauchy sequence. By using the Lemma 2.3, we observe that \((A^i x)\) is a \(\rho_k\)-convergent sequence to \(x_0\). It is easy to see that \(x_0\) is the unique fixed point of \(A\).

Corollary 3.3 (cf. [10]). Let \((X, \mathcal{F}, T)\) be a complete Menger space under \(T \geq T_1\), where \(f(0) < \infty\) and suppose that for each pair \((x, y) \in X^2\) there exists \(t_{xy}\) for which \(F_{xy}(t_{xy}) = 1\). Then every \(s\)-contraction on \(X\) has a unique fixed point.

Proof: Since for \(s \leq t_{xy}\) we have
\[
0 \leq s^k \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt = s^k \int_{s}^{t_{xy}} \frac{f \circ F_{xy}(t)}{t} \, dt \leq s^k \int_{s}^{t_{xy}} \frac{f \circ F_{xy}(t)}{t} \, dt \leq s^k \int_{s}^{t_{xy}} f \circ F_{xy}(t) \, dt \leq s^k \int_{s}^{t_{xy}} f(0) \left(\ln(t_{xy}) - \ln(s)\right) \, dt,
\]
and for \(s > t_{xy}\) we have \(s^k \int_{s}^{\infty} \frac{f \circ F_{xy}(t)}{t} \, dt = 0\) then (3.2) holds and we can apply the theorem.

Corollary 3.4 ([6]). Let \((X, \mathcal{F}, T)\) be a complete Menger space with \(T \geq T_1\) such that for some \(k > 0\) and every pair \((x, y) \in X\) one has
\[
\sup_{s > 0} s^k \int_{s}^{\infty} \frac{1 - F_{xy}(t)}{t} \, dt < \infty.
\]
Then every \(s\)-contraction on \(X\) has a unique fixed point.

Proof: We take \(f(t) = f_1(t) = 1 - t\) and we apply the theorem.
Corollary 3.5. Let \((X, F, T)\) be a complete Menger space under \(T \geq T_1\) and suppose that there exists \(k > 0\) such that every \(F_{xy}\) has a finite \(k\)-moment. Then every \(s\)-contraction on \(X\) has a unique fixed point.

Proof: It is well-known that \((\mu_k)_{xy}^k = \int_0^\infty t^{k-1}(1 - F_{xy}(t)) \, dt < \infty\). Therefore

\[
s^k \int_0^\infty \frac{1 - F_{xy}(t)}{t} \, dt \leq \int_0^\infty t^{k-1}(1 - F_{xy}(t)) \, dt = \int_0^\infty s^{k-1}(1 - F_{xy}(t)) \, dt \leq (\mu_k)_{xy}^k < \infty
\]

and the corollary follows. 

Remark 3.6. For \(k = 1\) it can be obtained a known result (see [11, Corollary 2.2]).

Generally from the fixed point alternative ([3]) we obtain the following

Theorem 3.7. Let \((X, F, T)\) be a complete Menger space under \(T \geq T_f\) and \(A\) an \(s\)-contraction. Then for each \(x \in X\) either,

i) there is some \(k > 0\) such that \((A^k x)\) is \(\rho_k\)-convergent to the unique fixed point of \(A\), or

ii) for all \(k > 0\), for all \(n \in \mathbb{N}\) and for all \(M > 0\) there exists \(s := s(k, n, M)\) such that

\[
s^k \int_0^\infty \frac{f \circ F_{A^nxA^n+1x}(t)}{t} \, dt > M.
\]

Proof: We suppose that ii) is not true:

\[
\exists k > 0, \ \exists n_0 > 0, \ \exists M > 0, \ \forall s > 0 \text{ such that } s^k \int_0^\infty \frac{f \circ F_{A^{n_0}xA^{n_0+1}x}(t)}{t} \, dt \leq M.
\]

So, we have for some \(k > 0\), \(\rho_k(A^{n_0}x, A^{n_0+1}x) < \infty\). It follows that

\[
\forall p > 0, \quad \rho_k(A^{n_0}x, A^{n_0+p}x) \leq \sum_{i=0}^{p-1} \rho_k(A^{n_0+p}x, A^{n_0+p+1}x) \leq \leq (1 + L_1 + L_2 + \ldots + L_1^{p-1}) \rho_k(A^{n_0}x, A^{n_0+1}x) = \frac{1 - L_1^p}{1 - L_1} \rho_k(A^{n_0}x, A^{n_0+1}x) \leq \frac{\rho_k(A^{n_0}x, A^{n_0+1}x)}{1 - L_1} < \infty,
\]
where \( L_1 := L^{k+1} < 1 \). Therefore, the sequence of successive approximations, \((A^ix)\) is a \(\rho_k\)-Cauchy sequence. From Lemma 2.3 we obtain that \((A^ix)\) is \(\rho_k\)-convergent and it is easy to see that the limit of the sequence \((A^ix)\) is the unique fixed point of \(A\).

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