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NON VANISHING CONJUGACY CLASSES FOR AN IRREDUCIBLE CHARACTER OF S_n

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Abstract: An irreducible character of the symmetric group S_n is a triangular character if it is associated to a partition of the form $(m, m-1, ..., 2, 1)$. We prove that an irreducible character χ is triangular if and only if it vanishes on all conjugacy classes whose cycle decomposition contains at least one transposition.

Furthermore if the character χ is not triangular and $\chi \neq [2,2]$, there is a class where a transposition and a cycle of length one occur, for which χ does not vanish.

1 – Introduction

Let S_n be the symmetric group of degree n, and χ be an irreducible complex character of S_n. There is a natural one-to-one correspondence between the irreducible characters of S_n and the nonincreasing partitions of n. So if $m_1 \geq m_2 \geq ... \geq m_t \geq 1$ and $m_1+m_2+...+m_t = n$, denote by $[m_1, m_2, ..., m_t]$ the irreducible character of S_n which is associated to the partition $(m_1, m_2, ..., m_t)$. Such a partition can be pictured by t left-justified rows of boxes, where the number of boxes in the i^{th} row is m_i, and which is called a Young frame.

If $\alpha \in S_n$, let $\alpha = \alpha_1 \alpha_2 ... \alpha_r$ be its cycle decomposition. Let k_i be the length of the cycle α_i. Clearly $k_i \geq 1$ and $\sum_i k_i = n$. We will identify the conjugacy class of the permutation α with the sequence $(k_1, k_2, ..., k_r)$, where the k_i appear in an arbitrary order. Collecting parts of equal length repetitions may be indicated by the use of superscripts. As for example we represent by $(k_1, 1^t)$ the class $(k_1, 1, ..., 1)$, where the 1 appears $t = n - k_1$ times. The class $(k_1, 1^t)$ where $t = 0$ is just the class (n).

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It is important to know in some problems, if a nonlinear character vanishes in a particular conjugacy class of S_n. For example it is known that χ is self associated if and only if χ is zero on all odd permutations. In [2] some classes are described where the character does not vanish. In [3, Corollary 2.4.9] a necessary condition is given for the classes where the value of a character is not zero.

2 – Triangular characters

An irreducible character χ of S_n is said to be a triangular character, if $\chi = [m, m-1, ..., 1]$, where $m \geq 1$.

Of course, for each n, the symmetric group S_n admits at most one triangular character. And this happens if and only if n is a triangular number.

The main theorems give a complete description of the triangular characters, as being the only irreducible characters that vanish on all the conjugacy classes whose cycle decomposition contains a transposition.

Theorem 2.1. *Let $\chi = [m_1, ..., m_t]$ be an irreducible nontriangular character of S_n, where $n > 1$. Then*

a) *There exists a permutation σ containing a transposition in its cycle decomposition and such that $\chi(\sigma) \neq 0$.*

b) *There exists a permutation σ_1 containing a transposition in its cycle decomposition and such that if σ_1 belongs to a class $(k_1, ..., k_u, 2^r, 1^t)$ and σ_2 belongs to $(k_1, ..., k_u, 2^{r-1}, 1^{t+2})$, then one of the following conditions holds:*

- **i)** $|\chi(\sigma_1)| = |\chi(\sigma_2)| \neq 0$;
- **ii)** $\chi(\sigma_1) \neq 0$ *and* $\chi(\sigma_2) = 0$;
- **iii)** $\chi(\sigma_1) = 0$ *and* $\chi(\sigma_2) \neq 0$.

Theorem 2.2. *Let χ be an irreducible nontriangular character of S_n, where $n > 2$ and $\chi \neq [2, 2]$ (if $n = 4$). There is a permutation π whose cycle decomposition contains at least one transposition and a cycle of length one and such that $\chi(\pi) \neq 0$.*

Triangular characters are self associated characters; thus they vanish on all odd permutations. Moreover they vanish on all classes whose cycle decomposition contains a cycle of even length, which is easy to prove by applying the

Murnagham–Nakayama Rule [1]. In particular they vanish on all the conjugacy classes whose cycle decomposition contain at least one cycle of length two.

From the previous remarks and [3, Corollary 2.4.9] we conclude the following result.

Proposition 2.3. *Let* $\chi = [m, m-1, ..., 1]$ *be a triangular character of* S_n. *If* $k_1 \geq ... \geq k_r$ *and* χ *does not vanish on the class* $(k_1, ..., k_r)$, *then* $k_1, ..., k_r$ *are odd integers and* $(k_1, ..., k_r) \preceq (2m-1, 2m-5, ...)$.

This is not a sufficient condition, since we have that $(3^2, 1^4) \preceq (7, 3)$, but the character $[4, 3, 2, 1]$ of S_{10} vanishes on the class $(3^2, 1^4)$.

However it is easy to get the converse for cycles.

Proposition 2.4. *Let* $\chi = [m, m-1, ..., 1]$ *be a triangular character of* S_n, *and* $\sigma \in S_n$ *a cycle of length* p. *Then* $\chi(\sigma) \neq 0$, *if and only if* p *is odd and* $p \leq 2m-1$.

3 – Proofs

The proofs of the propositions are essentially based on the Murnagham–Nakayama rule [1] for the Young frames.

Recall that the boundary of a frame is the set of boxes whose right edge, bottom edge, or bottom right vertex belong to the geometric boundary of the frame. A regular boundary part of a frame is a set of successive boundary boxes whose deletion leads to another frame.

For the application of the Murnagham–Nakayama Rule we can take advantage of the fact that the order of the cycles of a permutation is arbitrary. So if the diagram does not contain regular parts of a certain length, which corresponds to the order of a cycle of a permutation, we can start from this cycle to prove that the required character vanishes on this permutation. It is also convenient to take the cycles of length one last; if there remains a frame which one must reduce by cutting out single spaces, the number of ways in which this can be done is equal to the degree of the representation corresponding to the diagram obtained.

Consider the sequence $(p_1, ..., p_{s+1})$ $(s \geq 0)$, that is obtained in the following way: Let p_1 be the number of boxes of the boundary of the frame associated with χ. If $p_1 \neq n$, after deleting these p_1 boxes, we obtain another frame. Let p_2 be the number of boxes of the boundary of this new frame. One continues in this way until there is nothing left.

Let χ' be the character that corresponds to the diagram obtained, if we remove successively $p_1, ..., p_s$ boxes. Note that if $s = 0$, then $\chi' = \chi$, and if $s > 0$ we denote by χ'' the diagram obtained after removing $p_1, ..., p_{s-1}$ boxes.

Let $t \in \{1, ..., s+1\}$. There is just one way of removing $p_1, ..., p_t$ boxes of the diagram χ. If we denote by λ the character associated to the diagram obtained, and if σ belongs to the conjugacy class $(p_1, ..., p_t, k_1, ..., k_r)$ and σ' to the class $(k_1, ..., k_r)$, for some integers k_i, such that $\sum p_i + \sum k_i = n$, then $|\chi(\sigma)| = |\lambda(\sigma')|$.

Recall that two characters of S_n are associated if their corresponding frames are reflections about the main diagonal of each other. The reflection of $\epsilon = [1, 1, ..., 1]$, the alternating character, is $[n]$ the principal character. If χ_1 and χ_2 are associated, then $\chi_1(\sigma) = \epsilon(\sigma)\,\chi_2(\sigma)$, and thus $|\chi_1(\sigma)| = |\chi_2(\sigma)|$. So whenever two characters are associated to conjugate diagrams, we will analyse just one of them.

Proof of Theorem 2.1: Let us classify all the characters χ which are not triangular in five classes and some subclasses depending on the frame χ'. In each case we will exhibit permutations σ and σ_1 satisfying the conditions a) and b), respectively. The permutations will be represented by the lengths of the cycles that appear in its cycle factorization $(k_1, k_2, ..., k_q)$, in the following order: When applying the Nakayama Rule k_1 is the number of boxes we remove from the diagram in the first place, and so on.

Note that $\chi' = [m', 1^r]$, for some $m' \geq 1$ and $r \geq 0$. We may assume that $m' \geq r+1$, since the characters $[m', 1^r]$ and $[r+1, 1^{m'-1}]$ are associated. So consider the following classes:

(**1**) $m' \geq 2$, $r \geq 1$ and $m' + r > 3$.

(**2**) $m' > 1$ and $r = 0$.

(**3**) $m' = 2$ and $r = 1$ and χ'' is not a triangular character.

(**4**) $m' = 1$ and $r = 0$ and χ'' is not a triangular character.

(**5**) $\chi'' = [4, 3, 2, 1]$ or $\chi'' = [3, 2, 1]$.

Let us now list permutations satisfying in each case the conditions a) and b).

(**1**) **a**) If r is even and $m' = r+1$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_s, 2^{m'-1}, 1)$; we have $\chi(\sigma_2) = 0$.

b) If r is odd and $m' = r+1$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_s, 2^{m'-2}, 1^3)$; we have $\chi(\sigma_2) = 0$.

c) If $m' > r+1$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_s, m'-1, 2, 1^{r-1})$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$.

(**2**) Take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_s, 2, 1^{m'-2})$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$.

(**3**) In this case we have $\chi'' = [3+v, 3, 2, 1^u]$, for some $u, v \geq 0$ and we can assume that $v \geq u$. Since χ'' is not a triangular character we can not have $u = v = 1$. So consider the following subclasses:

a) $u > 1$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, v+2, u+2, 2^2)$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$, if $v = u+2$, and $\chi(\sigma_2) = 0$, if $v \neq u+2$.

b) $u = v = 0$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, 3, 2^2, 1)$; we have $\chi(\sigma_2) = 0$.

c) $u \in \{0, 1\}$ and $v > u$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, 2, v+4, u+2)$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$.

(**4**) Now we have $\chi'' = [2+v, 2, 1^u]$ for some $u, v \geq 0$. Once more we may assume that $v \geq u$, and clearly we can not have $u = v = 1$.

a) $u > 2$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, v, u, 2^2)$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$, if $v = u+2$, and $\chi(\sigma_2) = 0$, if $v \neq u+2$.

b) **i**) $u = 2$, $v \neq 2$, $v \neq 4$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, v, 4, 2)$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$.

ii) $u = 2$, $v = 2$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, 2^4)$; we have $\chi(\sigma_2) = 0$.

iii) $u = 2$, $v = 4$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, 4, 2^3)$; we have $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 2$.

c) $u \in \{0, 1\}$, take $\sigma, \sigma_1 \in (p_1, p_2, ..., p_{s-1}, u, v+2, 2)$; we have $\chi(\sigma_2) = 0$, if $v = u = 0$ and $|\chi(\sigma_2)| = |\chi(\sigma_1)| = 1$ in the other cases.

(**5**) In this case the diagram χ is not triangular, but after removing some boundary parts of maximum length we obtain a triangular diagram.

Let t be an integer, such that if we omit the t regular boundary parts of lengths $p_1, ..., p_t$ we obtain a nontriangular diagram, which we will denote by χ''', and removing p_{t+1} boxes , we obtain a triangular character. There is an integer m'', such that $\chi''' = [m''+v+1, m''+1, m'', ..., 3, 2, 1^u]$, where $u, v \geq 0$. Again we assume that $v \geq u$ and we can not have $u = v = 1$.

Let $n' = 1+2+...+m''$. It is clear that if we take σ_1 in the class $(p_1, ..., p_{t+1}, 2, 1^{n'-2})$, we have that $\chi(\sigma_1) = 0$ and $\chi(\sigma_2) \neq 0$. Let us now exhibit a permutation satisfying condition a).

If $u = 0$, take $\sigma \in (p_1, ..., p_t, p_{t+1} - 1, 2, 1^{n'-1})$.

Suppose now that $u > 0$. If $u + v$ is odd, take $\sigma \in (p_1, ..., p_t, 2^{p_{t+1}/2}, 1^{n'})$; if $v = u$ and both are odd, take $\sigma \in (p_1, ..., p_t, p_{t+1} - (u+1), 2^2, 1^{n'+u-3})$ and in the remaining cases take $\sigma \in (p_1, ..., p_t, p_{t+1} - (u+1), 2, 1^{n'+u-1})$.

Proof of Theorem 2.2: We will consider the same classes and subclasses of the proof of Theorem 2.1. Whenever the permutation σ contains also a cycle of length one in its cycle decomposition, we will take $\pi = \sigma$. In the same way if the permutation σ_1 we exhibited in the above proof satisfies $|\chi(\sigma_1)| = |\chi(\sigma_2)| \neq 0$ and σ_2 contains in its cycle decomposition at least a transposition, we can take $\pi = \sigma_2$. The remaining cases are the following:

I) class (1), if $r = 1$ and $m' > 3$.

II) class (2), if $m' = 2$.

III) class (3) a), if $v \neq u + 2$.

IV) class (3) c), if $u = 1$.

V) class (4), if $u \geq 2$ and $v \neq u + 2$.

VI) class (4), if $u = 0$.

We will exhibit in each case a permutation satisfying the required conditions which can be verified using the Nakayama Rule once more.

I) Take $\pi \in (p_1, p_2, ..., p_s, 2, 1^{m'-1})$.

II) In this case $\chi'' = [3 + v, 3, 1^u]$.
If $v \geq u$ and $v \neq u + 1$, or $u = 0$ and $v = 1$, take $\pi \in (p_1, p_2, ..., p_{s-1}, v + 2, 2, 1^{u+2})$.
If $u > 0$ and $v = u + 1$, take $\pi \in (p_1, p_2, ..., p_{s-1}, v + 3, 2, 1^{u+1})$.
If $v < u$, take $\pi \in (p_1, p_2, ..., p_{s-1}, u + 3, 2, 1^{v+1})$.

III) If $v = u$, take $\pi \in (p_1, p_2, ..., p_{s-1}, v, u, 3, 2^2, 1)$, and if $v > u$ take $\pi \in (p_1, p_2, ..., p_{s-1}, v + 2, 2, 1^{u+4})$.

IV) Take $\pi \in (p_1, p_2, ..., p_{s-1}, 2, 1^{v+7})$.

V) If $v > u$ or $v = u$ and both are even, take $\pi \in (p_1, p_2, ..., p_{s-1}, v+2, 2, 1^u)$, and if $v = u$ and both are odd, take $\pi \in (p_1, p_2, ..., p_{s-1}, v + 2, 2^2, 1^{u-2})$.

VI) If $v > 0$, take $\pi \in (p_1, p_2, ..., p_{s-1}, 2, 1^{v+2})$. If $v = 0$, we must have $s > 1$, since $\chi \neq [2, 2]$. Let χ''' be the obtained diagram after removing successively $p_1, ..., p_{s-2}$ boxes. Then $\chi''' = [3 + v', 3, 3, 1^{u'}]$. Assuming that $v' \geq u'$, consider the several subcases:
If $u' = v' = 0$, take $\pi \in (p_1, p_2, ..., p_{s-2}, 2^2, 1^5)$.
If $u' = v' = 1$, take $\pi \in (p_1, p_2, ..., p_{s-2}, 3, 2^2, 1^4)$.
If $u' = v' = 2$, take $\pi \in (p_1, p_2, ..., p_{s-2}, 3, 2^2, 1^6)$.
If $u' = v' > 2$, take $\pi \in (p_1, p_2, ..., p_{s-2}, v' + 4, v', 2^2, 1)$.
If $v' > u'$, take $\pi \in (p_1, p_2, ..., p_{s-2}, v' + 4, 2, 1^{3+u'})$.

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