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NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Abstract: The paper presents an existence theorem of monotonic solutions for a nonlinear functional integral equation of convolution type by using Darbo fixed point theorem associated with the Hausdorff measure of noncompactness.

1 – Introduction

Nonlinear functional integral equations of convolution type arise very often in applications, especially in numerous branches of mathematical physics [12]. The equations of such a kind have been investigated in several papers [2], [9], where the equations in question have solutions in some function spaces. Also, Banas and Knap [5] discussed the solvability of the considered equations in the space of Lebesgue integrable functions by using the technique of measures of weak noncompactness and the fixed point theorem due to Emmanuele [8]. In spite of this approach gives more general result under less restrictive assumptions than those in [2], [9], but the weak continuity condition for an operator is not easy to be satisfied in general. In this paper, we try to overcome this difficulty by using Darbo fixed point theorem associated with the Hausdorff measure of noncompactness, which is a strong measure.

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2 – Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+$ be the interval $[0,\infty)$ and $L^1(I)$ be the space of Lebesgue integrable functions on a measurable subset I of $\mathbb{R}$, with the standard norm

$$\|x\|_{L^1(I)} = \int_I |x(t)|\,dt\ .$$

The space $L^1(\mathbb{R}_+)$ will be shortly denoted by L^1 and its norm $\|\cdot\|$, i.e.

$$\|x\| = \int_0^\infty |x(t)|\,dt\ .$$

One of the most important operator studied in nonlinear functional analysis is the so-called superposition operator [1]. Assume that a function $f(t,x) = f\colon I\times\mathbb{R}\to\mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x\in\mathbb{R}$ and continuous in x for almost all $t\in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$(Fx)(t) = f(t,x(t))\,, \qquad t\in I\ .$$

The operator F in such a way is called the superposition operator generated by the function f. We have the following theorem due to Appell and Zabrejko [1].

Theorem 1. *The superposition operator F maps continuously the space $L^1(I)$ into itself if and only if*

$$|f(t,x)| \le a(t) + b|x|\ ,$$

for all $t\in I$ and $x\in\mathbb{R}$, where $a(t)\in L^1(I)$ and $b\ge 0$.

Next, we will mention a desired theorem concerning the compactness in measure of a subset X of $L^1(I)$ (cf. [7]).

Theorem 2. *Let X be a bounded subset of $L^1(I)$ consisting of functions which are a.e. nondecreasing (or nonincreasing) on the interval I. Then X is compact in measure.*

Furthermore, we recall a few facts about the convolution operator (cf. [11]). Let $k\in L^1(\mathbb{R})$ be a given function. Then for any function $x\in L^1$, the integral

$$(Kx)(t) = \int_0^\infty k(t-s)\,x(s)\,ds$$

exists for almost every $t \in \mathbb{R}_+$. Moreover, the function $(Kx)(t)$ belongs to the space L^1. Thus K is a linear operator which maps the space L^1 into L^1 and K is also bounded since $\|Kx\| \leq \|K\|_{L^1(\mathbb{R})}\|x\|$, for every $x \in L^1$; so, it will be continuous. Hence the norm $\|K\|$ of the convolution operator is majored by $\|K\|_{L^1(\mathbb{R})}$.

In the sequel, we have the following theorem due to Krzyz [10].

Theorem 3. *Assume that $k(t,s) = k\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is measurable on $\mathbb{R}_+^2$ and such that the integral operator*

$$(Kx)(t) = \int_0^\infty k(t,s)\, x(s)\, ds\,, \qquad t \geq 0\ ,$$

maps L^1 into itself. Then K transforms the set of nonincreasing functions from L^1 into itself if and only if for any $A > 0$ the following implication is true

$$t_1 < t_2 \ \to \ \int_0^A k(t_1,s)\, ds \geq \int_0^A k(t_2,s)\, ds\ .$$

Finally, we give a short note on measures of noncompactness and fixed point theorem. Let E be an arbitrary Banach space with norm $\|\cdot\|$ and the zero element θ.

Let also X be a nonempty and bounded subset of E and B_r be a closed ball in E centered at θ and radius r.

The Hausdorff measure of noncompactness $\chi(X)$ [4] is defined as

$$\chi(X) = \inf\Big\{r > 0\colon \text{there exists a finite subset } Y \text{ of } E \text{ such that } x \subset Y + B_r\Big\}\ .$$

Another measure was defined in the space L^1 [3]. For any $\varepsilon > 0$, let

$$c(X) = \lim_{\varepsilon\to 0}\Big\{\sup_{x\in X}\Big\{\sup\Big[\int_D |x(t)|\, dt,\ D \subset \mathbb{R}_+,\ \operatorname{meas} D \leq \varepsilon\Big]\Big\}\Big\}$$

and

$$d(X) = \lim_{T\to\infty}\Big\{\sup\Big[\int_T^\infty |x(t)|\, dt\colon\ x \in X\Big]\Big\}\ ,$$

where $\operatorname{meas} D$ denotes the Lebesgue measure of a subset D.

Put

$$\gamma(X) = c(X) + d(X)\ .$$

Then we have the following theorem [3], which connects between the two measures $\chi(x)$ and $\gamma(x)$.

Theorem 4. *Let X be a nonempty, bounded and compact in measure subset of L^1. Then*

$$\chi(X) \le \gamma(X) \le 2\,\chi(X)\ .$$

As an application of measures of noncompactness, we recall the fixed point theorem due to Darbo [6].

Theorem 5. *Let Q be a nonempty, bounded, closed and convex subset of E and let $H\colon\ Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0,1)$ such that*

$$\mu(HX) \le k\,\mu(X)\ ,$$

for any nonempty subset X of C. Then H has at least one fixed point in the set Q.

3 – Main result

Now the following nonlinear integral equation of convolution type will be investigated

$$\chi(t) = f\left(t, \int_0^\infty k(t-s)\,x(\varphi(s))\,ds\right), \qquad t \ge 0\ . \tag{1}$$

For further purposes the operator

$$(Hx)(t) = f\left(t, \int_0^\infty k(t-s)\,x(\varphi(s))\,ds\right)$$

will be written as the product $Hx = F\,Kx(\varphi)$ of the convolution operator

$$(Kx)(t) = \int_0^\infty k(t-s)\,x(s)\,ds$$

and the superposition operator

$$(Fx)(t) = f(t, x(t))\ .$$

Thus equation (1) becomes

$$x = Hx = F\,Kx(\varphi)\ . \tag{2}$$

We shall treat the equation (1) under the following assumptions which are listed below.

i) $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there are a function $a \in L^1$ and a constant $b \geq 0$ such that

$$|f(t,x)| \leq a(t) + b|x| \ ,$$

for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, $f(t,x) \geq 0$ for $x \geq 0$ and f is assumed to be nonincreasing in the first variable and nondecreasing in the second one;

ii) the function $k\colon \mathbb{R} \to \mathbb{R}_+$ belongs to the space $L^1(\mathbb{R})$ and for any $A > 0$ and for all $t_1, t_2 \in \mathbb{R}_+$, the following condition is satisfied

$$t_1 < t_2 \ \to \ \int_0^A k(t_1 - s)\, ds \geq \int_0^A k(t_2 - s)\, ds \ ;$$

iii) $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, absolutely continuous and there is a constant $B > 0$ with the property $\varphi'(t) \geq B$ for almost all $t \in \mathbb{R}_+$;

iv) $\frac{b\|K\|}{B} < 1$.

Then we can prove the following theorem.

Theorem 6. *Let the assumptions i)–iv) be satisfied. Then the equation (1) has at least one solution $x \in L^1$ being a.e. nonincreasing on $\mathbb{R}_+$.*

Proof: First of all observe that for a given $x \in L^1$ the function Hx belongs to L^1, which is a consequence of the assumptions i)–iii).

Additionally, using (2) we get

$$\begin{aligned}
\|Hx\| &= \|F\,Kx(\varphi)\| \\
&\leq \int_0^\infty \Big[a(t) + b\Big|\int_0^\infty k(t-s)\, x(\varphi(s))\, ds\Big|\Big]\, dt \\
&\leq \|a\| + b\|Kx(\varphi)\| \\
&\leq \|a\| + b\|K\|\, \|x(\varphi)\| \\
&\leq \|a\| + b\|K\| \int_0^\infty |x(\varphi(s))|\, ds \\
&\leq \|a\| + \frac{b\|K\|}{B} \int_0^\infty |x(\varphi(s))|\, \varphi'(s)\, ds \\
&\leq \|a\| + \frac{b\|K\|}{B}\, \|x\| \ .
\end{aligned}$$

From this estimate and iv) we infer that the operator H maps the ball B_r into itself, where

$$r = \frac{\|a\|}{1 - b\|k\|\, B^{-1}} \ .$$

Further, let Q_r stand for the subset of B_r consisting of all functions which are a.e. positive and nonincreasing on $\mathbb{R}_+$. Note that Q_r is nonempty, bounded, closed and convex subset of L^1. Moreover, in view of Theorem 2 the set Q_r is compact in measure.

Next, take $x \in Q_r$, then $x(\varphi)$ is a.e. positive and nonincreasing on $\mathbb{R}_+$ and consequently $Kx(\varphi)$ is also of the same type in virtue of the assumption ii) and Theorem 3. Further, the assumption i) permits us to deduce that $Hx = F\,Kx(\varphi)$ is also a.e. positive and nonincreasing on $\mathbb{R}_+$. This fact, together with the assertion $H\colon B_r \to B_r$ gives that H is a self-mapping of the set Q_r.

Since the operator K is continuous (cf. section 2) and F is continuous in view of Theorem 1, we conclude that H maps continuously Q_r into Q_r.

From now on assume that X is a nonempty subset of Q_r and $\varepsilon > 0$ is fixed. Then for an arbitrary $x \in X$ and for a set $D \subset \mathbb{R}_+$, $\operatorname{meas} D \le \varepsilon$ we obtain

$$\begin{aligned}\int_D |(Hx)(t)|\, dt &\le \int_D \Big[a(t) + b\Big|\int_0^\infty k(t-s)\, x(\varphi(s))\, ds\Big|\Big]\, dt \\ &= \|a\|_{L^1(D)} + b\|Kx(\varphi)\|_{L^1(D)} \ .\end{aligned}$$

Further, keeping in mind that the operator K transforms the space $L^1(D)$ into itself and is continuous, we derive

$$\int_D |(Hx)(t)|\, dt \le \|a\|_{L^1(D)} + b\|K\|_D\, \|x(\varphi)\|_{L^1(D)} \ ,$$

where the symbol $\|K\|_D$ denotes the norm of the operator $K\colon\, L^1(D) \to L^1(D)$.

Consequently, we get

$$\int_D |(Hx)(t)|\, dt \le \|a\|_{L^1(D)} + \frac{b\|K\|_D}{B} \int_D |x(\varphi(t))|\, \varphi'(t)\, dt \ .$$

Now, applying the theorem on integration by substitution for Lebesgue integral we may write the last estimate as

$$\int_D |(Hx)(t)|\, dt \le \|a\|_{L^1(D)} + \frac{b\|K\|_D}{B} \int_{\varphi(D)} |\chi(t)|\, dt \ .$$

Hence, taking into account the obvious equality

$$\lim_{\varepsilon\to 0}\Big\{\sup\Big[\int_D a(t)\, dt\colon\ D \subset \mathbb{R}_+,\ \operatorname{meas} D \le \varepsilon\Big]\Big\} = 0$$

and the absolute continuity of the function φ, we obtain

$$c(HX) \le \frac{b\|K\|}{B}\, c(X) \;, \tag{3}$$

where the quantity $c(X)$ was defined before in section 2.

Furthermore, fixing $T > 0$ we arrive at the following estimate

$$\begin{aligned}\int_T^\infty |(Hx)(t)|\, dt &\le \int_T^\infty a(t)\, dt + \frac{b\|K\|}{B} \int_T^\infty |x(\varphi(t))|\, \varphi'(t)\, dt \\ &= \int_T^\infty a(t)\, dt + \frac{b\|K\|}{B} \int_{\varphi(T)}^\infty |x(t)|\, dt \;.\end{aligned}$$

Since $\lim_{T\to\infty} \phi(T) = \infty$ the above inequality yields

$$d(HX) \le \frac{b\|K\|}{B}\, d(X) \;, \tag{4}$$

where d has been defined before in section 2.

Hence, combining (3) and (4) we get

$$\gamma(HX) \le \frac{b\|K\|}{B}\, \gamma(X) \;,$$

where γ denotes the measure of noncompactness defined also in section 2.

The above obtained inequality together with the properties of the operator H and the set Q_r established before and in conjunction with Theorem 4, enable us to apply Theorem 5. This completes the proof. ∎

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