PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 4 – 1997

AN APPROXIMATION PROCEDURE FOR FIXED POINTS OF STRONGLY LIPSCHITZ OPERATORS

Ram U. Verma

Abstract: Based on a modified iterative algorithm, fixed points of the operators of the form $S = T + U$ on nonempty closed convex subsets of Hilbert spaces are approximated. Here T is strongly Lipschitz and Lipschitz continuous and U is Lipschitz continuous.

1 – Introduction

Recently, Wittmann [5, Theorem 2] approximated fixed points of nonexpansive mappings T on nonempty closed convex subsets of Hilbert spaces by employing an iterative procedure

$$x_n = (1 - a_n)\, x_0 + a_n\, T x_{n-1} \quad \text{for} \;\; n \geq 1 \;, \tag{1}$$

where $\{a_n\}$ is an increasing sequence in $[0, 1)$ such that

$$\lim_{n\to\infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - a_n) = \infty \;. \tag{2}$$

This result, for example, applies to $a_n = 1 - n^{-a}$ with $0 < a \leq 1$, and improves a theorem of Halpern [1, Theorem 3] which does not apply to the case $a_n = 1 - 1/n$. Furthermore, (2) is not just sufficient, but also necessary for the convergence of $\{x_n\}$ for all T [1, Theorem 2].

Here we are concerned with the approximation of fixed points of operators of the form $S = T + U$, where T is strongly Lipschitz and Lipschitz continuous and U is Lipschitz continuous on a nonempty closed convex subset K of a real Hilbert

Received: July 17, 1996.

space H by using the following modified iterative procedure in a more general setting

$$(3) \qquad x_{n+1} = (1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t(T+U)\,x_n\Big] \quad \text{for} \ \ n\geq 0 \ ,$$

where $t > 0$ is arbitrary and the sequence $\{x_n\}$ lies in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \geq 0$.

For $U = 0$ in (3), we find the iterative algorithm

$$(4) \qquad x_{n+1} = (1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t\,T x_n\Big] \quad \text{for all} \ \ n\geq 0 \ .$$

For $t = 1$ in (4), we arrive at

$$(5) \qquad x_{n+1} = (1-a_n)\,x_n + a_n\,T x_n \quad \text{for all} \ \ n\geq 0 \ .$$

2 – Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Definition 2.1 An operator $T\colon H \to H$ is said to be strongly Lipschitz if, for all u, v in H, there exists a real number $r \geq 0$ such that

$$(6) \qquad \langle Tu - Tv,\, u-v\rangle \leq -r\|u-v\|^2 \ .$$

The operator T is called Lipschitz continuous if there exists a real number $s > 0$ such that

$$(7) \qquad \|Tu - Tv\| \leq s\|u-v\| \quad \text{for all} \ \ u,v \ \ \text{in } H \ .$$

It is easily seen that (7) implies that

$$(8) \qquad \langle Tu - Tv,\, u-v\rangle \leq s\|u-v\|^2 \quad \text{for all} \ \ u,v \ \ \text{in } H \ .$$

Definition 2.2. An operator $T\colon H \to H$ is said to be hemicontinuous if, for all u, v in H, the function

$$(9) \qquad t \to \Big\langle T(tu + (1-t)\,v),\, u-v\Big\rangle \quad \text{for} \ \ 0 \leq t \leq 1 \ ,$$

is continuous.

To this end, let us consider an example of strongly Lipschitz operator where the real number r in inequality (6) is slightly relaxed.

Example 2.1 [6]**:** Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T\colon K \to K$ be hemicontinuous on K such that, for a real number $r > -1$ and for all u, v in K,

$$\langle Tu - Tv,\, u - v\rangle \leq -r\|u - v\|^2 \ . \tag{10}$$

Then T has a unique fixed point in K.

3 – The fixed point theorem

In this section we consider the approximation of fixed points of a combination of strongly Lipschitz and Lipschitz continuous operators.

Theorem 3.1. *Let H be a real Hilbert space and let K be a non-empty closed convex subset of H. Let $T\colon K \to K$ be strongly Lipschitz and Lipschitz continuous with respective real numbers $r \geq 0$ and $s \geq 1$, and let $U\colon K \to K$ be Lipschitz continuous with a real number $m > 0$. Let F be a nonempty set of fixed points of $S = T + U$, and let $\{a_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \geq 0$. Then, for any x_0 in K, the sequence $\{x_n\}$ generated by the iterative algorithm (3) for*

$$0 \leq k = \left[\left((1-t)^2 - 2t(1-t)\,r + t^2\,s^2\right)^{1/2} + t\,m\right] < 1 \tag{11}$$

for all t such that $0 < t < 2(1 + r - m)/(1 + 2r + s^2 - m^2)$ and $1 + r - m > 0$, converges to a fixed point of $S = T + U$.

When $U = 0$ in Theorem 3.1 , we arrive at the following result.

Corollary 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T\colon K \to K$ be strongly Lipschitz and Lipschitz continuous with corresponding constants $r \geq 0$ and $s \geq 1$. Let $\{a_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \geq 0$. Then, for any element x_0 in K, the sequence $\{x_n\}$ generated by the iterative algorithm (4) for*

$$0 \leq k_0 = \left[(1-t)^2 - 2t(1-t)\,r + t^2\,s^2\right]^{1/2} < 1 \tag{12}$$

for all t such that $0 < t < 2(1 + r)/(1 + 2r + s^2)$, converges to a fixed point of T.

Proof of Theorem 3.1: For an element z in F, we have

$$
\begin{aligned}
(13) \qquad & \|x_{n+1}-z\| = \\
&= \Big\|(1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t(T+U)\,x_n\Big] - (1-a_n)\,z \\
&\quad - a_n\Big[(1-t)\,z + t(T+U)\,z\Big]\Big\| \\
&= \Big\|(1-a_n)\,(x_n-z) + a_n\Big[(1-t)\,(x_n-z) + t(Tx_n-Tz) + t(Ux_n-Uz)\Big]\Big\| \\
&\le (1-a_n)\,\|x_n-z\| + a_n\Big\|(1-t)\,(x_n-z)+t(Tx_n-Tz)\Big\| + a_n\,t\|Ux_n-Uz\| \;.
\end{aligned}
$$

Since T is strongly Lipschitz and Lipschitz continuous, this implies that

$$
\begin{aligned}
(14) \qquad & \Big\|(1-t)\,(x_n-z)+t(Tx_n-Tz)\Big\|^2 = \\
&= (1-t)^2\,\|x_n-z\|^2 + 2t(1-t)\,\langle Tx_n-Tz,\, x_n-z\rangle + t^2\|Tx_n-Tz\|^2 \\
&\le \Big[(1-t)^2 - 2t(1-t)\,r + t^2\,s^2\Big]\,\|x_n-z\|^2 \;.
\end{aligned}
$$

Applying (14) to (13) and using the Lipschitz continuity of U, it follows that

$$
\begin{aligned}
\|x_{n+1}-z\| &\le \Big\{(1-a_n) + \Big[\Big((1-t)^2 - 2t(1-t)\,r + t^2\,s^2\Big)^{1/2} + t\,m\Big]\,a_n\Big\}\,\|x_n-z\| \\
&= \Big[1-(1-k)\,a_n\Big]\,\|x_n-z\| \\
&\le \prod_{j=0}^{n}\Big[1-(1-k)\,a_j\Big]\,\|x_0-z\| \;,
\end{aligned}
$$

where $0 \le k = [((1-t)^2 - 2t(1-t)r + t^2s^2)^{1/2} + tm] < 1$ for all t such that $0 < t < 2(1+r-m)/(1+2r+s^2-m^2)$ for $1+r-m > 0$. Since $\sum_{j=0}^{\infty} a_j$ diverges and $k < 1$, this implies that $\lim_{n\to\infty}\prod_{j=0}^{n}[1-(1-k)\,a_j] = 0$, and consequently, $\{x_n\}$ converges to z, a fixed point of $S = T+U$. This completes the proof. ∎

REFERENCES

[1] Halpern, B. – Fixed points of nonexpansive maps, *Bull. Amer. Math. Soc.*, 73 (1967), 957–961.

[2] Kirk, W.A. and Sternfeld, Y. – The fixed point property for nonexpansive mappings in certain product spaces, *Houston J. Math.*, 10 (1984), 207–214.

[3] Reich, S. – Approximating fixed points of nonexpansive maps, *PanAmer. Math. J.*, 4(2) (1994), 23–28.

[**4**] Rhoades, B.E. – Fixed point iterations for certain nonlinear mappings, *J. Math. Anal. Appl.,* 183 (1994), 118–120.

[**5**] Wittmann, R. – Approximation of fixed points of nonexpansive mappings, *Arch. Math.,* 58 (1992), 486–491.

[**6**] Yao, J.C. – Applications of variational inequalities to nonlinear analysis, *Appl. Math. Lett.,* 4 (1991), 89–92.

Ram U. Verma,
International Publications,
12046 Coed Drive, Orlando, Florida 32826 – U.S.A.
and
Istituto per la Ricerca di Base, Division of Mathematics,
I-86075 Monteroduni (IS), Molise – ITALY