PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 4 – 1997

ON THE EXPLICIT SOLUTION OF THE LINEAR FIRST ORDER CAUCHY PROBLEM WITH DISTRIBUTIONAL COEFFICIENTS

C.O.R. Sarrico

Abstract: In [3] we have considered the n^{th} order linear Cauchy problem for a class of differential equations with distributional coefficients. We have extended the concept of solution of this problem and we have proved that these solutions are consistent with the classical solutions. Here we give necessary and sufficient conditions for existence, in this extended sense, of a solution of the problem $X' = UX + V$, $X(t_0) = a$, where $U \in C^\infty \oplus \mathcal{D'}^p_m$ ($\mathcal{D'}^p_m = \mathcal{D'}^p \cap \mathcal{D'}_m$, $\mathcal{D'}_m$ is the space of distributions with nowhere dense support, $\mathcal{D'}^p$ is the space of distributions of order $\le p$ in the Schwartz setting), $V \in \mathcal{D}'$, $a \in \mathbb{C}$ and $t_0 \in \mathbb{R}$. We also give an explicit and practical formula for computing this solution.

0 – Introduction

Let us introduce the following notation:

1) $\mathcal{D}$ is the space of indefinitely differentiable complex functions on $\mathbb{R}^N$ with compact support;

2) $\mathcal{D}'$ is the space of Schwartz distributions;

3) G is a group of unimodular transformations (i.e. linear transformations $h\colon \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$) which will be called the ruling group;

4) α denotes a function in $\mathcal{D}$ with $\int_{\mathbb{R}^N} \alpha = 1$ which is G-invariant and $\check{\alpha}$ the function such that $\check{\alpha}(t) = \alpha(-t)$ for all $t \in \mathbb{R}^N$;

5) $\mathcal{D'}^p$ ($p = 0, 1, 2, ..., \infty$) is the space of distributions of order $\le p$ in the sense of Schwartz;

6) $\mathcal{D'}_m$ denotes the space of distributions with nowhere dense support.

Received: September 5, 1996.

In [1], [2] we have defined a (G, α)-product of a distribution $T \in \mathcal{D}'^p$ by a distribution $S = \beta + f \in C^p \oplus \mathcal{D}'_m$ by the formula

$$T \underset{\alpha}{\cdot} S = T\beta + (T * \check{\alpha})\, S \ ,$$

where the products on the right-hand side are the classical ones. Such a product is indeed obtained by restricting a general product on $\mathcal{D}' \times \mathcal{D}'$ to the spaces $\mathcal{D}'^p \times (C^p \oplus \mathcal{D}'_m)$ in such a way that consistency with the classical products of $\mathcal{D}'^p$-distributions by C^p-functions is maintained.

Now, we consider the linear Cauchy problem of first order in dimension $N = 1$

$$P_a^V \equiv \begin{cases} X' = UX + V, \\ X(t_0) = a \ , \end{cases}$$

where $U = \gamma + T \in C^\infty \oplus \mathcal{D}'^p_m$, $\mathcal{D}'^p_m = \mathcal{D}'^p \cap \mathcal{D}'_m$, $V \in \mathcal{D}'$, $a \in \mathbb{C}$ and $t_0 \in \mathbb{R}$.

In the setting of classical Schwartz products (products of a $\mathcal{D}'^p$-distribution by a C^p-function), to solve the above problem we are forced to seek solutions in the narrow space C^p, we call such solutions classical solutions. Those solutions are clearly insufficient for applications in physical theories and we will just enlarge conveniently the concept of a solution of the Cauchy problem. To do so, we associate to the problem P_a^V the problem Q_a^V defined by

$$Q_a^V \equiv \begin{cases} X' = X\gamma + T \underset{\alpha}{\cdot} X + V, \\ X(t_0) = a \ , \end{cases}$$

where $X\gamma$ is taken in classical sense. The solution of Q_a^V will be called "w_α-solution" of P_a^V with respect to the ruling group G. They belong to the extended space $C^p \oplus \mathcal{D}'_m$ according to the following definition introduced in [3]:

0.1 Definition. We say that $X \in C^p \oplus \mathcal{D}'_m$ is a w_α-solution of P_a^V, with respect to the ruling group G, when there exists an open set $\Omega \in \mathbb{R}$, with $t_0 \in \Omega$, such that the restriction X_Ω of X to Ω is a C^p-function and X satisfies Q_a^V.

Note that $X(t_0)$ makes sense, and also that there are only two groups of unimodular transformations of $\mathbb{R}$: $G_1 = \{I\}$ and $G_2 = \{I, -I\}$ where I is the identity function on $\mathbb{R}$. This means that in $\mathbb{R}$ there are only two ruling groups and in the applications to classical mechanics we must always adopt the orthogonal group G_2 as ruling group, as we have done in all examples of [1], [3].

The consistency of w_α-solutions with the classical solutions and the uniqueness of the u_α-solutions were proved in [3] and granted by the following theorems:

0.2 Theorem. *If $X \in C^p$ is a classical solution of P_a^V then, for all $\alpha \in \mathcal{D}$, G-invariant with $\int_{\mathbb{R}} \alpha = 1$, X is a w_α-solution of P_a^V with respect to G.*

0.3 Theorem. *Given $\alpha \in \mathcal{D}$, G-invariant, if there exists a w_α-solution of P_a^V with respect to G, in $C^q \oplus \mathcal{D}'_m$, with $q = \max\{1, p\}$, then this solution is unique.*

Recall that, sometimes the w_α-solutions of P_a^V may not depend of α, as we have seen in examples 5.1 and 5.2 of [3].

1 – The explicit w_α-solution of P_a^V in $C^p \oplus \mathcal{D}'_m$

Concerning the existence of w_α-solutions X of P_a^V in $C^p \oplus \mathcal{D}'_m$ it is easy to see that

1.1 Proposition. *Let $\alpha \in \mathcal{D}$, G-invariant with $\int_{\mathbb{R}} \alpha = 1$. If P_a^V has a w_α-solution in $C^p \oplus \mathcal{D}'_m$ $(p \geq 1)$ with respect to G, then $V \in C^{p-1} \oplus \mathcal{D}'_m$.*

Thus, if $p \geq 1$ and $V \notin C^{p-1} \oplus \mathcal{D}'_m$, P_a^V is impossible in our generalized sense and also in the classical sense.

The following Lemma is important to reach the explicit w_α-solutions of P_a^V.

1.2 Lemma. *If $T \in \mathcal{D}'_m$ then $\operatorname{supp} T = \operatorname{supp} T'$.*

Proof: We always have $\operatorname{supp} T' \subset \operatorname{supp} T$. We will see that we also have $\operatorname{supp} T \subset \operatorname{supp} T'$. The case $T = 0$ is trivial. Suppose $T \neq 0$ and $\operatorname{supp} T \not\subset \operatorname{supp} T'$. Then $\operatorname{supp} T \cap (\mathbb{R} \backslash \operatorname{supp} T') \neq \emptyset$. As $\mathbb{R} \backslash \operatorname{supp} T'$ is an open set and $\operatorname{supp} T$ has common points with this set, we conclude that there is a non void open interval $I \subset \mathbb{R} \backslash \operatorname{supp} T'$ such that $T \neq 0$ in I. Thus, $T' = 0$ in I and so T is a constant in I. Since $T \in \mathcal{D}'_m$, we conclude that $T = 0$ in I, which is a contradiction. ∎

Remark. It is easy to see that this result cannot be extended to partial derivatives in dimension $N > 1$.

Now, we can prove

1.3 Theorem. *Let $p \geq 1$, $V = \eta + R \in C^{p-1} \oplus \mathcal{D}'_m$, $t_0 \notin \operatorname{supp} T$, $t_0 \notin \operatorname{supp} R$ and suppose that there exists $S, Q \in \mathcal{D}'_m$ such that $S' = T$ and $Q' = R$. Then, given $\alpha \in \mathcal{D}$, G-invariant, with $\int_{\mathbb{R}} \alpha = 1$, P_a^V has a w_α-solution X in $C^p \oplus \mathcal{D}'_m$*

with respect to G, if and only if there is $B \in \mathcal{D}'_m$ such that

$$B' = e^{-(\check{\alpha}*S)}\Big[T \underset{\alpha}{\cdot} S(C+a) + e^{-A}(T \underset{\alpha}{\cdot} Q + \gamma\, Q - \eta\, S)\Big] , \tag{1.1}$$

where A and C are respectively the usual solutions of the Cauchy problems

$$\begin{cases} A' = \gamma, \\ A(t_0) = 0 , \end{cases} \tag{1.2}$$

$$\begin{cases} C' = e^{-A}\,\eta \\ C(t_0) = 0 . \end{cases} \tag{1.3}$$

In this case

$$X = e^A (C+a)\,(1+S) + Q + e^{A+(S*\check{\alpha})}\, B . \tag{1.4}$$

Remark. Note that in (1.4) only the third term of the sum can possibly depend on α.

Proof: Let $X = \beta + f \in C^p \oplus \mathcal{D}'_m$ be the w_α-solution of P^V_a. This means that there is an open set $\Omega \subset \mathbb{R}$ such that

1) $t_0 \in \Omega$;

2) X_Ω is a C^p-function (i.e. $f_\Omega = 0$);

3) $\beta' + f' = (\beta+f)\,\gamma + T \underset{\alpha}{\cdot} (\beta+f) + \eta + R$;

4) $\beta(t_0) = a$.

Condition 3 is equivalent to

$$\beta' - \beta\gamma - \eta = -f' + f\gamma + T\beta + (T * \check{\alpha}) f + R$$

where the left-hand side is a C^{p-1}-function and the right-hand side is a $\mathcal{D}'_m$ distribution. Thus, each side of this equality equals the zero function and so, 3 is equivalent to

$$\begin{cases} \beta' - \beta\gamma = \eta, \\ f' - f[\gamma + (T*\check{\alpha})] = T\beta + R . \end{cases}$$

Hence, 1), 2), 3), 4) are equivalent to

a) $\begin{cases} \beta' - \beta\gamma = \eta \\ \beta(t_0) = a \end{cases}$;

b) $f' - f[\gamma + (T*\check{\alpha})] = T\beta + R$;

c) $f_\Omega = 0$;

d) $t_0 \in \Omega$.

The solution of a), a usual linear Cauchy problem of the first order is given by

$$\beta = (C + a)\, e^{A} \ ,$$

where A and C are respectively defined by (1.2) and (1.3). Putting $F = A+(S*\check{\alpha})$ we have $F' = \gamma + (T * \check{\alpha})$ and we can multiply both sides of b) by e^{-F} to obtain successively:

$$(e^{-F} f)' = e^{-F}\, T\, \beta + e^{-F} R \ ,$$
$$(e^{-F} f)' = e^{-F} S'\beta + e^{-F} Q' = (e^{-F}\beta\, S)' - (e^{-F}\beta)'\, S + (e^{-F} Q)' - (e^{-F})'\, Q \ ,$$
$$(e^{-F} f)' = (e^{-F}\beta\, S + e^{-F} Q)' + e^{-F} F'\beta\, S - e^{-F}\beta' S + e^{-F} F' Q \ .$$

If there is $B \in \mathcal{D}'_m$ such that

$$B' = e^{-F} F'\beta\, S - e^{-F}\beta' S + e^{-F} F' Q \tag{1.5}$$

we can compute f because

$$(e^{-F} f)' = (e^{-F}\beta\, S + e^{-F} Q + B)' \ .$$

Thus, $e^{-F} f = e^{-F}\beta S + e^{-F} Q + B +$ constant and constant $= 0$ as $e^{-F} f - e^{-F}\beta S - e^{-F} Q - B \in \mathcal{D}'_m$. So, we have

$$f = \beta\, S + Q + e^{F} B$$

which is consistent with c) and d) because, by Lemma 1.2,

$$\operatorname{supp} B = \operatorname{supp} B' \subset (\operatorname{supp} S \cup \operatorname{supp} Q) \ ,$$
$$\operatorname{supp} f \subset (\operatorname{supp} S \cup \operatorname{supp} Q) = \operatorname{supp} S' \cup \operatorname{supp} Q' = \operatorname{supp} T \cup \operatorname{supp} R \ ,$$

$t_0 \notin \operatorname{supp} T$ and $t_0 \notin \operatorname{supp} R$.

Then, we have for X the following expression

$$X = \beta + f = \beta + \beta S + Q + e^{F} B = e^{A}(C + a)\,(1 + S) + Q + e^{A+(S*\check{\alpha})} B \ .$$

At this point, to complete the proof it is enough to note that (1.5) is equivalent to (1.1). Indeed:

$$e^{-F} F'\beta\, S - e^{-F}\beta' S + e^{-F} F' Q = e^{-F}\,(F'\beta\, S - \beta' S + F' Q) =$$

$$= e^{-A} \cdot e^{-(S*\check{\alpha})} \Big[\Big(\gamma + (T * \check{\alpha}) \Big) (C+a)\, e^{A}\, S$$

$$- \Big(C' e^{A} + (C+a)\, e^{A}\, \gamma \Big)\, S + \Big(\gamma + (T * \check{\alpha}) \Big) Q \Big]$$

$$= e^{-(S*\check{\alpha})} \Big[\gamma (C+a) S + (T * \check{\alpha})\, (C+a)\, S - C' S$$

$$- (C+a)\, \gamma\, S + e^{-A}\, \gamma\, Q + e^{-A} (T * \check{\alpha})\, Q \Big]$$

$$= e^{-(S*\check{\alpha})} \Big[(T * \check{\alpha})\, S (C+a) - e^{-A}\, \eta\, S + e^{-A} (T * \check{\alpha})\, Q + e^{-A}\, \gamma\, Q \Big]$$

$$= e^{-(S*\check{\alpha})} \Big[(T \underset{\alpha}{\cdot} S)\, (C+a) + e^{-A} (T \underset{\alpha}{\cdot} Q) + e^{-A}\, \gamma\, Q - e^{-A}\, \eta\, S \Big]$$

$$= e^{-(S*\check{\alpha})} \Big[(T \underset{\alpha}{\cdot} S)\, (C+a) - e^{-A} (T \underset{\alpha}{\cdot} Q + \gamma\, Q - \eta\, S) \Big] \ . \ \blacksquare$$

Applying Theorem 1.3 we can easily solve differential equations of first order.

Example: Let us consider the P_1^1 problem

$$P_1^1 \equiv \begin{cases} X' - \delta' X = 1, \\ X(-1) = 1 \ ; \end{cases}$$

we have $\gamma = 0$, $T = \delta' \in \mathcal{D}'^1$, $p = 1$, $\eta = 1$, $R = 0$, $A = 0$, $C = t+1$, $S = \delta$, $Q = 0$, $a = 1$, $t_0 = -1$ and

$$\begin{aligned} B' &= e^{-\check{\alpha}} \Big[(\delta' * \check{\alpha})\, \delta(t+2) - \delta \Big] = e^{-\check{\alpha}} \Big[(\check{\alpha})'\, \delta(t+2) - \delta \Big] \\ &= e^{-\check{\alpha}(0)} (-2\alpha'(0) - 1)\, \delta \ . \end{aligned}$$

Hence, $B = e^{-\check{\alpha}(0)} (-2\alpha'(0) - 1)\, H + \text{constant}$, where H is a Heaviside function. Thus, $B \in \mathcal{D}'_m$ if and only if $2\alpha'(0) + 1 = 0$ and constant $= 0$ which means that P_1^1 has w_α-solution for all $\alpha \in \mathcal{D}$ such that $\alpha'(0) = -\frac{1}{2}$, if we adopt the ruling group G_1. In this case the unique solution is

$$X(t) = (t+2)\, (1+\delta) = t + 2 + 2\, \delta(t) \ .$$

From this it follows in particular that P_1^1 has no classical solutions. Note also that this solution does not depend explicitly on the α function.

Clearly, if we adopt the ruling group G_2, there are no w_α-solutions because there are no even functions α such that $\alpha'(0) = -\frac{1}{2}$.

Correction. In [3], 5.4, pag. 389, the preceding Cauchy problem was considered instead of this one (as ought to be)

$$P_1^0 \equiv \begin{cases} X' - \delta''' X = 0, \\ X(-1) = 1 \ , \end{cases}$$

the solution of which is $X = 1 + \delta'' - 2\alpha^{\mathrm{IV}}(0)\,\delta$ (applying directly Theorem 1.3 or the definition 0.1) adopting the ruling group G_2 as we have done in [3].

ACKNOWLEDGEMENT – I wish to thank Prof. A. Vaz Ferreira of Bologna University for helpful discussions during my visit to this university, where the present research, partially supported by JNICT, project PRAXIS/2/2.1/MAT/125/94, has been performed.

I would also like to thank Prof. Owen Brison for assistance with the English.

REFERENCES

[1] Sarrico, C.O.R. – About a family of distributional products important in the applications, *Portugaliae Math.,* 45(3) (1988), 295–316.

[2] Sarrico, C.O.R. – Distributional products with invariance for the action of unimodular groups, *Riv. Mat. Univ. Parma,* (5)4 (1995), 79–99.

[3] Sarrico, C.O.R. – The linear Cauchy problem for a class of differential equations with distributional coefficients, *Portugaliae Math.,* 52(4) (1995), 379–390.

C.O.R. Sarrico,
Centro de Matemática e Aplicações Fundamentais,
Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex – PORTUGAL