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REMARKS ON NAGUMO'S CONDITION

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Abstract: For quasilinear elliptic equations it is commonly assumed that nonlinearity grows not faster than quadratically in the gradient of solution. This assumption is necessary in order to get an a priori bound on the gradient of solution, which is a part of most existence proofs. However, it appears that this condition is not essential for existence of solution, only different techniques of proof are needed in its absence.

1 – Introduction

Consider a quasilinear equation on an interval (a,b)

$$y'' + f(x,y,y') = 0 \quad \text{for} \ \ x \in (a,b) \ , \tag{1.1}$$

with general separated boundary conditions at $x = a, b$. To prove that (1.1) has solutions, one needs to estimate $|y'|$ on $[a,b]$. To this end the following "natural" condition (going back to work of M. Nagumo in the thirties and S.N. Bernstein some twenty years earlier) is imposed: for all $x \in [a,b]$ and $y' \in \mathbb{R}$ we have

$$|f(x,y,y')| \leq c(1+|y'|^2) \quad \text{with} \ \ c = c(y) \ . \tag{1.2}$$

To see that (1.2) implies boundness of $|y'|$, assuming boundness of $|y|$, notice that on any interval of monotonicity of $y(x)$ we can represent $x = x(y)$, and then $y' = p(y)$, and $y'' = p\,p'$. Let x_0 be the extremum point of $y(x)$, which is closest to x, and let $y_0 = y(x_0)$. Setting $q = p^2$ we express, using (1.1) and (1.2),

$$\frac{1}{2}\,\frac{dq}{dy} \leq c(1+q) \,, \qquad q(y_0) = 0 \ , \tag{1.3}$$

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from which the boundness of $q(y)$ (and hence of $p(y)$) follows, provided y lies in a bounded interval. Notice that this argument is independent of boundary conditions. If, on the other hand, the Nagumo's condition is violated, then $|y'|$ need not be bounded. For example, the problem (adapted from Kamke [9])

$$y'' + (1 + y'^2)^{\frac{3}{2}} = 0 \quad \text{on } (0, 2)\,, \qquad y(0) = y(2) = 0$$

has as a solution the upper half of the circle $(x-1)^2 + y^2 = 1$, with infinite derivatives at $x = 0$ and $x = 2$.

Even though in general one cannot estimate $|y'|$ in the absence of the Nagumo's condition, the argument above shows the possibility of generalization. Without Nagumo's condition the right hand side of (1.3) is superlinear in q. However, if y is restricted to a relatively small interval, $q = q(y) = y^2$ has no "time" to blow up. If one gets such a bound on $|y'|$ a priori, then existence of solutions follows by the well-known results, see e.g., [4].

Existence of solutions without Nagumo's condition was studied earlier, see e.g., A. Constantin [2], A. Tineo [11], A. Granas et al [8], and a number of other references given in those papers. We give a simplified approach and extensions of several of these results, and in the last section we consider a PDE case on a ball in $\mathbb{R}^n$. For simplicity we shall restrict ourselves to the Dirichlet boundary conditions, although the same approach applies equally well to any separated boundary conditions.

Throughout the paper we consider only classical solutions, which in case of (1.1) implies that $y \in C^2(a,b) \cap C[a,b]$.

2 – Boundary value problems in one dimension

We begin with a special class of equations (for $u = u(x)$) with $f(x) \in C^1(\mathbb{R})$

$$u'' + f(u') = 0 \quad \text{on } (0, l)\,, \qquad u(0) = u(l) = 0\,. \tag{2.1}$$

It turns out that solvability of (2.1) can be decided by analyzing the following initial value problem

$$y'' + f(y') = 0\,, \qquad y(0) = y'(0) = 0\,. \tag{2.2}$$

For example, if $f(0) = 0$, then by uniqueness for initial value problems, $y \equiv 0$ is the unique solution (2.2). By applying the same uniqueness theorem at any extreme point of $u(x)$, we conclude that $u \equiv 0$ is the only solution of (2.1). We may assume therefore that $f(0) \neq 0$. We shall assume for definiteness that

$$f(0) > 0 \ , \tag{2.3}$$

with the other case, $f(0) < 0$, being reducible to it by a change of variables $u \to -u$. Notice that solution of (2.2) is then negative near $x = 0$.

Lemma 2.1. *Assume that a bounded nontrivial solution of (2.2) exists on an interval $[-l_1, l_2]$, with some $l_1, l_2 \geq 0$, and assume for definiteness that*

$$|y(-l_1)| \geq |y(l_2)| \ .$$

Define $l_3 > 0$ by

$$y(-l_3) = y(l_2) \ .$$

Then the boundary value problem (2.1) is solvable for any $l \leq l_2 + l_3$. If, on the other hand, $y(x) \to -\infty$ when $x \to -l_1$ and when $x \to l_2$, then the problem (2.1) is solvable for any $l < l_1 + l_2$, and no solution exists for $l \geq l_1 + l_2$.

Proof: We begin by noticing that (2.3) implies that

$$y'(x) < 0 \ \text{ for } x > 0 \quad \text{ and } \quad y'(x) > 0 \ \text{ for } x < 0 \ . \tag{2.4}$$

Indeed, by (2.3) $y(x)$ is concave for small $x > 0$, and hence it is decreasing there. If $y'(x_0) = 0$ for some $x_0 > 0$, then using (2.3) again, $y''(x_0) < 0$, a contradiction, since x_0 is not a point of local maximum, and hence $y(x)$ is decreasing at x_0. If $y(x)$ is a solution of (2.2), then $u(x) = y(x) + c$ with a suitable c, satisfying $0 < c \leq |y(l_2)|$, will be a solution of (2.1) for any $l \leq l_2 + l_3$, and the lemma follows. (Keep in mind that if $u(x)$ solves the equation in (2.2), so does $u(x - \alpha)$ for any constant α.) The last assertion follows similarly. ∎

As a corollary we obtain the following result from [8].

Corollary 1 ([8]). *Assume there exist two constants $\alpha < 0 < \beta$, such that*

$$f(\alpha) = f(\beta) = 0 \ . \tag{2.5}$$

Then the problem (2.1) is solvable for any l.

Proof: We only need to show that the problem (2.2) is solvable on $(-\infty, \infty)$. By local existence theorem (2.2) is solvable on $(0, x_0)$, for some $x_0 > 0$. Letting as before, $y' = p(y)$, we rewrite (2.1)

$$\frac{dp}{dy} = \frac{f(p)}{p} \ , \qquad p(y_0) = p_0 \ , \tag{2.6}$$

where, in view of (2.3), $y_0 = y(x_0) < 0$, and $p_0 = y'(x_0)$ satisfies $\alpha < p_0 < 0$. From (2.6) we see that $p'(x_0) < 0$. As we increase x past x_0, both $y(x)$ and $p = y'(x)$ are decreasing, and from (2.6) we see that p never gets smaller than α (α is a fixed point of (2.6)), which implies that $y(x)$ does not become infinite at a finite x, and $\alpha < y'(x) < 0$. Hence, solution of (2.2) exists for all x, concluding the proof. (Here $y(x) \to -\infty$ as $x \to \infty$.) ∎

We illustrate our lemma with two examples.

Example 1: The problem

$$-y'' = 1 + {y'}^2 \,, \qquad y(0) = y(l) = 0$$

has a unique positive solution for $l < \pi$, and no solutions for $l \geq \pi$.

Indeed, solution of

$$-y'' = 1 + {y'}^2 \,, \qquad y(0) = y'(0) = 0$$

is $y = \ln \cos x$, and the Lemma 2.1 applies.

Example 2: Consider the problem

$$-y'' = 1 + {y'}^4 \,, \qquad y(0) = y(l) = 0 \; . \tag{2.7}$$

Let $\bar{l} = 2 \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\tan 2x}}$. Then for $l \leq \bar{l}$ the problem (2.7) has a unique positive solution, while for $l > \bar{l}$ it has no solution.

Indeed, integrating the problem

$$-y'' = 1 + {y'}^4 \,, \qquad y(0) = y'(0) = 0$$

by using the substitution $y' = p(y)$, we obtain, for $x > 0$,

$$y' = -\sqrt{-\tan 2y} \,, \qquad y(0) = 0 \; .$$

Integrating again, we see that as $x \to \frac{\bar{l}}{2}$, we have $y \to -\frac{\pi}{4}$ and $y' \to -\infty$.

Comparing these two examples, we see that situation is similar, i.e. a solution exists only when the length of the interval is less than a critical number. The difference is in how the solution "escapes" when l approaches the critical value. If Nagumo's condition is satisfied, this happens by maximum of solutions going to infinity. In the other case, the solution develops an infinite derivative at the end of the interval.

Next we consider a class of equations, where the nonlinearity has a special structure:

$$y'' + g(x, y, y') + h(x, y, y') = 0 \text{ on } (0,1)\,, \qquad y(0) = y(1) = 0\ . \tag{2.8}$$

In the following result we place no restriction on the growth of g in y'.

Theorem 2.1. *Assume that for all $(x, y, y') \in [0,1] \times [-M, M] \times \mathbb{R}$ the functions g and h are continuous, and we have*

$$y\, g(x, y, y') \le 0\ , \tag{2.9}$$

$$|h(x, y, y')| \le c(1 + y'^2)\,, \qquad c = c(M)\ . \tag{2.10}$$

Assume finally that any solution of (2.8) satisfies an a priori estimate

$$\max_{0 \le x \le 1} |y(x)| \le M\ . \tag{2.11}$$

Then the problem (2.8) has a solution.

Proof: We begin by deriving an a priori estimate of $|y'(x)|$. It suffices to estimate $|y'(x)|$ on every interval on which $y(x)$ is positive (or negative). So assume that $y(x)$ is positive on some subinterval $I \equiv [\alpha, \beta] \subseteq [0,1]$, and $y(\alpha) = y(\beta) = 0$. For any point $\overline{x} \in I$ we can find a point of local maximum $x_0 \in I$, so that $y(x)$ is either increasing or decreasing on $(\overline{x}, x_0)$. As before, on this interval we may express $x = x(y)$, and then $y' = p(y)$, $y'' = p'\,p$. Since by (2.9) g is negative on I, we obtain, using (2.10) and (2.11),

$$p'\,p \ge -h \ge -c_0(1 + p^2)\ , \tag{2.12}$$

with some constant c_0. Finally, setting $q = p^2$, and $v = y_0 - y$, where $y_0 = y(x_0)$, we obtain from (2.12)

$$\frac{1}{2}\frac{dq}{dv} \le c_0(1 + q)\,, \qquad q(0) = 0\ . \tag{2.13}$$

(If y is negative on I, then we obtain (2.12) with reversed sign. Setting $q = p^2$ and $v = y - y_0$, we again obtain (2.13).) It follows that q is bounded by a constant c_1 on the interval $[0, y_0 - y(\overline{x})]$, since the length of this interval is bounded by M. Hence $|y'|$ is bounded by $\sqrt{c_1}$ for $x \in [\overline{x}, x_0]$. In particular, $|y'(\overline{x})| \le \sqrt{c_1}$. Since $\overline{x}$ was arbitrary, we have a bound for $|y'|$ on I, and the proof follows by the well-known results, see [4] or [7]. ∎

Example: Consider the problem ($M > 0$ a positive constant)

$$y'' - y\, e^{y'} + (M - y)\,(1 + y'^2) = 0 \ \text{ on } (0,1)\,, \qquad y(0) = y(1) = 0\ . \tag{2.14}$$

By the maximum (and minimum) principle any solution of (2.14) satisfies $0 < y < M$, and so the theorem applies, implying existence of solutions.

The problem (2.8) was considered in a recent paper of A. Constantin [2]. By assuming that $h(x, y, y')$ has a sublinear growth in y and y', he derived a C^2 a priori estimate of solutions without assuming uniform boundness of solution, as in (2.11). We shall extend this result, but first we need a lemma. This lemma (which generalizes the Gronwall's lemma) is contained as a particular case in J. Bihary [1] (see also [3]). We present its proof for completeness.

Lemma 2.2. *Let the functions $u(t)$, $f(t) \in C[t_0, \infty)$ satisfy $u(t) \geq 0$, $f(t) \geq 0$ for all $t \geq t_0$, and let a constant m satisfy $0 < m < 1$, and*

$$u(t) \leq c + \int_{t_0}^{t} f(\tau)[u(\tau)]^m \, d\tau \qquad \text{for } \ t \geq t_0\ , \tag{2.15}$$

with a positive constant c. Then

$$u(t) \leq \left[c^{1-m} + (1 - m) \int_{t_0}^{t} f(\tau)\, d\tau\right]^{\frac{1}{1-m}} \qquad \text{for all } \ t \geq t_0\ . \tag{2.16}$$

Proof: Denote the right hand side of (2.15) by $w(t)$. Then $w(0) = c$ and $w' = f\, u^m$. Raising (2.15) to power m, we have

$$w' \leq f\, w^m\,, \qquad w(0) = c\ .$$

Integrating, we conclude that $w(t)$ is bounded by the right hand side of (2.16). Since by (2.15) $u \leq w$, we conclude the lemma. ∎

Theorem 2.2. *For the problem (2.8) we assume that for all $(x, y, y') \in [0,1] \times \mathbb{R} \times \mathbb{R}$ the functions g and h are continuous, the condition (2.9) holds and*

$$|h(x, y, y')| \leq c\Big(1 + |y|^{\alpha} + |y'|^{\beta}\Big)\ , \tag{2.17}$$

with some constants $0 \leq \alpha, \beta < 1$ and $c > 0$. Then the problem (2.8) has a classical solution.

Proof: Denote $M = \max_{0 \le x \le 1} |y(x)|$. This time the estimate of M is not given a priori, but is derived in course of the proof. Proceeding as in the proof of the previous theorem, we obtain in place of (2.13) the following inequality

$$\frac{1}{2}\frac{dq}{dv} \le c\Big(1 + M^\alpha + q^{\frac{\beta}{2}}\Big), \quad q(0) = 0 \, , \tag{2.18}$$

and we need to estimate $q = y'^2$ for $v \in (0, M]$, where M itself needs to be estimated. Integrating (2.18) from 0 to any $v \in (0, M]$,

$$q(v) \le 2c(M + M^{\alpha+1}) + 2c \int_0^v q^{\frac{\beta}{2}}\, dv \, . \tag{2.19}$$

By Lemma 2.3 we obtain from (2.19), setting $M_1 \equiv 2c(M + M^{\alpha+1})$,

$$p^2(v) = q(v) \le \Big[M_1^{1-\frac{\beta}{2}} + (2-\beta)\, c\, M\Big]^{\frac{1}{1-\frac{\beta}{2}}} \le c_2\Big(M_1 + c_1\, M^{\frac{1}{1-\frac{\beta}{2}}}\Big) \, ,$$

where $c_1 = [(2-\beta)\, c]^{\frac{1}{1-\frac{\beta}{2}}}$, and $c_2 > 0$ an appropriate constant. Hence for all $v \in (0, M]$

$$\begin{aligned} |p(v)| &\le \sqrt{c_2}\Big(M_1^{\frac{1}{2}} + c_1\, M^{\frac{1}{2-\beta}}\Big) \\ &\le \sqrt{2\, c\, c_2}\Big(M^{\frac{1}{2}} + M^{\frac{\alpha+1}{2}}\Big) + c_1\, \sqrt{c_2}\, M^{\frac{1}{2-\beta}} \, . \end{aligned} \tag{2.20}$$

Since $M \le \max_{(0,M]} |p(v)|$ (recall that $p = y'(x)$), we see from (2.20)

$$M \le \sqrt{2\, c\, c_2}\Big(M^{\frac{1}{2}} + M^{\frac{\alpha+1}{2}}\Big) + c_1\, \sqrt{c_2}\, M^{\frac{1}{2-\beta}} \, . \tag{2.21}$$

From (2.21) we conclude a bound on M. Returning to (2.20), we obtain a bound on $|y'(x)|$. Together these bounds imply a C^1 a priori estimate, concluding the proof, as in the previous theorem. ∎

Remark. In R.E. Gaines and J.L. Mawhin [5] it was shown that conditions (2.9) and (2.17) imply existence of a super- and subsolution pair for the equation (2.8). However, to obtain an existence result they needed to restrict the growth of g in y' by a condition only slightly more general then Nagumo's. A similar condition on g appears in a recent paper by A. Constantin [2]. We do not place any restrictions on the growth of g in y'.

3 – A class of quasilinear problems on a ball

In this section we show by means of an example that a similar approach applies to PDE's. By B we denote the ball $|x| < R$ in $\mathbb{R}^n$. By Du we denote the gradient of $u(x)$.

Theorem 3.1. *Consider the problem*

$$\Delta u + (M - u)\,\psi(|Du|^2) = 0 \quad \textit{for } |x| < R\,, \qquad u = 0 \quad \textit{on } |x| = R\;, \tag{3.1}$$

with a positive constant M, and a positive function $\psi(t) \in C^\alpha[0,\infty)$, with some $\alpha \in (0,1)$. Assume that

$$2M^2 < \int_0^\infty \frac{dq}{\psi(q)}\;. \tag{3.2}$$

Then the problem (3.1) has a unique positive solution $u \in C^{2+\alpha}(B)$.

Proof: By the strong maximum principle it follows that

$$0 < u < M \qquad \text{for all} \;\; x \in B\;. \tag{3.3}$$

By the well-known theorem of B. Gidas, W.-M. Ni and L. Nirenberg [6, p.221] it follows that solution is radially symmetric with $u'(r) < 0$ for all $0 < r < 1$. The equation (3.1) then takes the form

$$\begin{gathered} u_{rr} + \frac{n-1}{r}\,u_r + (M-u)\,\psi(u_r^2) = 0, \qquad \text{for} \;\; r \in (0,1)\;, \\ u_r(0) = u(1) = 0\;. \end{gathered} \tag{3.4}$$

Since $u(r)$ is decreasing, we can solve for $r = r(u)$, and then we can express $u_r = p(u)$, and $u_{rr} = p\,p'$. Setting $q = p^2$, and $v = u(0) - u$, we obtain from (3.4)

$$\begin{gathered} \frac{1}{2}\,\frac{dq}{dv} = \frac{n-1}{r}\,u_r + (M-u)\,\psi(q) \le M\,\psi(q)\;, \\ q(0) = 0\,, \qquad 0 \le v \le M\;. \end{gathered} \tag{3.5}$$

For any $\overline{v} \in (0, M]$ we integrate (3.5)

$$\int_0^{q(\overline{v})} \frac{dq}{\psi(q)} \le 2M^2\;. \tag{3.6}$$

Comparing (3.2) and (3.6), we conclude a bound on $q(v) = |Du|^2$. Together with (3.3) this gives us a C^1 a priori bound. Existence of solutions follows by the Theorem 13.8 in [7]. Uniqueness follows easily using the maximum principle. ∎

Example: The problem ($M > 0$ is a constant)

$$\Delta u + (M - u)\,(1 + |Du|^4) = 0 \quad \text{for } |x| < R\,, \qquad u = 0 \quad \text{on } |x| = R\,,$$

has a unique positive solution, provided $M < \frac{\sqrt{\pi}}{2}$.

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