PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 1 – 1998

PROJECTIVE LIMITS OF BANACH VECTOR BUNDLES

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Abstract: In this paper we exhibit a Fréchet vector bundle structure on projective limits of Banach vector bundles by considering as structure group, in place of the pathological general linear group of a Fréchet space, an appropriate "enlarged" topological group. Moreover, we study some geometric properties of the bundles under consideration and prove that certain results of the theory of connections, known so far for Banach bundles, hold also in our framework. As an application of the methods employed here, we endow the set of infinite jets of the local sections of a Banach vector bundle with the structure of a Fréchet vector bundle.

Introduction

The study of Fréchet manifolds was of the interest of many authors (see e.g. H. Omori [7], [8]; M.E. Verona [12], [13]; J.A. Leslie [5], [6]). As a matter of fact, the peculiarities of the structure of such manifolds, which are direct reflections of the difficulties emerging from the study of Fréchet spaces, led to the consideration of certain types of Fréchet manifolds, such as the generalized manifolds of M.E. Verona, the inverse limits of Lie groups of H. Omori, etc.

However, we do not have yet an analogous study of Fréchet vector bundles (v.b's for short). This is mainly a result of the pathological structure of the general linear group $GL(\mathbb{E})$ of a Fréchet space $\mathbb{E}$. Indeed, $GL(\mathbb{E})$ is not even a topological group. Moreover, serious problems appear in the study of the geometric properties of a Fréchet v.b. For instance, for a linear connection there is not always a parallel displacement along a curve of the basis, since we cannot, in general, solve linear differential equations in Fréchet spaces.

Received: August 14, 1996; *Revised*: November 20, 1996.
1991 Mathematics Subject Classification: 58A05, 58A30, 58B25.

The purpose of this paper is to study a certain type of Fréchet v.b's: those obtained as projective limits of Banach v.b's. More precisely, we exhibit a Fréchet v.b structure on such a limit $\varprojlim E_i$ by replacing the pathological group $GL(\mathbb{E})$, where $\mathbb{E}$ is the Fréchet fiber type of $\varprojlim E_i$, by an appropriate enlarged topological group $H^0(\mathbb{E})$. Namely, if the Banach fiber types $\mathbb{E}_i$'s of E_i's form a projective system with corresponding connecting morphisms $\rho_{ji}\colon \mathbb{E}_j \to \mathbb{E}_i$ $(j \geq i)$, we define

$$H^0(\mathbb{E}) = \left\{ (f_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} \mathcal{L}is(\mathbb{E}_i)\colon \; \rho_{ji} \circ f_j = f_i \circ \rho_{ji} \; (j \geq i) \right\} .$$

$H^0(\mathbb{E})$ is a topological group since it coincides with the projective limit of the Banach–Lie groups

$$H_i^0(\mathbb{E}) = \left\{ (f_1, ..., f_i) \in \prod_{j=1}^{i} \mathcal{L}is(\mathbb{E}_j)\colon \; \rho_{jk} \circ f_j = f_k \circ \rho_{jk} \; (i \geq j \geq k) \right\} , \qquad i \in \mathbb{N} .$$

Then, $\varprojlim E_i$ is a Fréchet vector bundle with structure group $H^0(\mathbb{E})$. In other words, it is fully determined by a new type of $H^0(\mathbb{E})$-valued transition functions defined bellow. Conversely, we show that every Fréchet v.b with $H^0(\mathbb{E})$-valued transition functions can always be thought of as the limit of a projective system of Banach v.b's. We thus obtain a characterization of the bundles under consideration.

Concrete examples of Fréchet v.b's of the previous type are also included in this paper. In particular, we prove that the bundle $J^\infty(\pi)$ of ∞-jets of the local sections of a Banach v.b (E, π, B) is a projective limit of Banach v.b's. In this way, $J^\infty(\pi)$, which so far has been studied mainly from a topological point of view, is equipped with a Fréchet vector bundle structure.

In the second part of the paper we deal with some geometric properties of a limit v.b $E = \varprojlim E_i$. We show that the limit of a projective system of connections on E_i is a connection on E, which is characterized by a generalized type of Christoffel symbols. We also prove, for connections of the previous type, the existence of parallel displacement along any curve of the basis, a fact which is not always true in the general case of a Fréchet connection for the already mentioned reasons. Finally, for the sake of completeness, we present a study of the corresponding holonomy groups.

1 – Projective systems of Banach vector bundles

Considering a projective system $\{E_i; f_{ji}\}_{i,j\in\mathbb{N}}$ of Banach v.b's one observes that the corresponding limit $\varprojlim E_i$ does not always inherit the structure of E_i's, as in the case of projective systems of algebraic or topological structures. More precisely, serious difficulties arise in the study of the corresponding transition functions, since the latter take values in the general linear group of the Fréchet fiber type of $\varprojlim E_i$, which group does not admit a reasonable topology compatible with its algebraic structure. In this section we give necessary conditions ensuring that $\varprojlim E_i$ admits a vector bundle structure.

Definition 1.1. A projective system $\{(E_i, p_i, B); f_{ji}\}_{i,j\in\mathbb{N}}$ of Banach v.b's, over the same basis, with corresponding fibers of type $\mathbb{E}_i$, is said to be strong if the following conditions hold:

i) $\mathbb{E}_i$ $(i \in \mathbb{N})$ form a projective system with connecting morphisms ρ_{ji} $(j \geq i)$.

ii) For any $x \in B$, there exists a local trivialization (U, τ_i) of E_i, $i \in \mathbb{N}$, respectively, such that $x \in U$ and the following diagram is commutative, for each $j \geq i$:

$$\begin{array}{ccc} p_j^{-1}(U) & \xrightarrow{\tau_j} & U \times \mathbb{E}_j \\ \Big\downarrow f_{ji} & & \Big\downarrow \mathrm{id}_U \times \rho_{ji} \\ p_i^{-1}(U) & \xrightarrow{\tau_i} & U \times \mathbb{E}_i \end{array}$$

Proposition 1.2. *If $\{E_i; f_{ji}\}_{i,j\in\mathbb{N}}$ is a strong projective system of Banach vector bundles, as before, then the triplet*

$$\left(\varprojlim E_i,\, \varprojlim p_i,\, B\right)$$

is a Fréchet vector bundle.

Proof: Let $\mathbb{B}$ be the model of the manifold B. Then, $E := \varprojlim E_i$ is a differentiable manifold modelled on the Fréchet space $\mathbb{B} \times \varprojlim \mathbb{E}_i$. The differential structure of E is constructed as follows: Let $u = (u_i) \in E$. Then $p_1(u_1) = p_2(u_2) = \ldots =: x \in B$. If (U, τ_i), $i \in \mathbb{N}$, are the trivializations of Definition 1.1 through x and (U, ϕ) is a chart of B, then the charts $(V_i, \Phi_i) := (p_i^{-1}(U), (\phi \times \mathrm{id}_{\mathbb{E}_i}) \circ \tau_i)$ of E_i $(i \in \mathbb{N})$ form a projective system. The corresponding limit is a chart of E through u. The charts of the previous type define the desired differential structure of E.

Furthermore, the map $p := \varprojlim p_i$ can be defined, since $f_{ji} \circ p_j = p_i$ $(j \geq i)$, and reduces, with respect to the charts $\varprojlim \Phi_i$ and ϕ, to the first projection $\mathrm{pr}_1\colon \phi(U) \times \varprojlim \mathbb{E}_i \to \phi(U)$. Therefore (see [3]), p is smooth.

On the other hand, $\tau := \varprojlim \tau_i\colon\ p^{-1}(U) \to U \times \varprojlim \mathbb{E}_i$ is a diffeomorphism, since $(\phi \times \mathrm{id}_{\mathbb{E}}) \circ \tau = \varprojlim \Phi_i$. Moreover, $\mathrm{pr}_1 \circ \tau = p$ and $\tau_x := \mathrm{pr}_2 \circ \tau|_{p^{-1}(x)}$ is an isomorphism between the topological vector spaces $\varprojlim(p_i^{-1}(x))$, $\varprojlim \mathbb{E}_i$, since $\tau_x = \varprojlim(\tau_i)_x$.

To complete the proof it suffices to check the differentiability of the transition functions

$$T_{UV}\colon\ U \cap V \to \mathcal{L}(\mathbb{E})\colon \quad x \mapsto \tau_x \circ \sigma_x^{-1}\ ,$$

where $\mathbb{E} := \varprojlim \mathbb{E}_i$, for any $\tau = \varprojlim \tau_i$ and $\sigma = \varprojlim \sigma_i$, if $\{(U, \tau_i)\}_{i\in\mathbb{N}}$ and $\{(V, \sigma_i)\}_{i\in\mathbb{N}}$ are local trivializations of E_i as in Definition 1.1. To this end we define the space

$$H(\mathbb{E}) := \left\{ (f_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} \mathcal{L}(\mathbb{E}_i)\colon\ \rho_{ji} \circ f_j = f_i \circ \rho_{ji},\ \forall\, j \geq i \right\}\ .$$

$H(\mathbb{E})$ is a Fréchet space, as a closed subspace of $\prod_{i\in\mathbb{N}} \mathcal{L}(\mathbb{E}_i)$, and it is isomorphic to the limit of the projective system of Banach spaces $\{H_i(\mathbb{E}); r_{ji}\}_{i,j\in\mathbb{N}}$, where

$$H_i(\mathbb{E}) = \left\{ (f_1, ..., f_i)\colon\ \rho_{jk} \circ f_j = f_k \circ \rho_{jk},\ \forall\, i \geq j \geq k \right\}\ ,$$
$$r_{ji}(f_1, ..., f_j) = (f_1, ..., f_i), \quad j \geq i\ .$$

The identification $H(\mathbb{E}) \equiv \varprojlim H_i(\mathbb{E})$ is given by

$$(f_1, f_2, f_3, ...) \equiv \Big((f_1), (f_1, f_2), (f_1, f_2, f_3), ... \Big)\ .$$

If we define

$$(T^*_{UV})_i\colon\ U \cap V \to H_i(\mathbb{E})\colon \quad x \mapsto (\tau_{1x} \circ \sigma_{1x}^{-1}, ..., \tau_{ix} \circ \sigma_{ix}^{-1})\ , \qquad i \in \mathbb{N}\ ,$$

then $\{(T^*_{UV})_i\}_{i\in\mathbb{N}}$ is a projective system of smooth mappings and $\varprojlim_{i\in\mathbb{N}} (T^*_{UV})_i\colon U \cap V \to H(\mathbb{E})$ is also smooth in the sense of J. Leslie ([5], [6]) (cf. also [3]). Moreover $T_{UV} = \varepsilon \circ \varprojlim_{i\in\mathbb{N}} (T^*_{UV})_i$, where ε is the continuous linear map

$$\varepsilon\colon\ H(\mathbb{E}) \to \mathcal{L}(\mathbb{E})\colon\ (g_i) \mapsto \varprojlim g_i\ .$$

Therefore, T_{UV} is smooth and the proof is complete. ∎

Remarks. **i**) The space $\mathcal{L}(\mathbb{E})$ is not necessarily Fréchet. It is merely a Hausdorff locally convex topological vector space whose topology is determined by the uniform convergence on the bounded subsets of $\mathbb{E}$.

ii) From the proof of the previous Proposition, it is clear that the space $H(\mathbb{E})$ and the functions $\{T^*_{UV} := \varprojlim_{i\in\mathbb{N}}(T^*_{UV})_i\}$ play an important role. T^*_{UV} will be called *PLB-transition functions*. As a matter of fact, one observes that the structure group of the v.b $E = \varprojlim E_i$ is not any more the pathological general linear group $GL(\mathbb{E})$ (cf. comments in the Introduction), since it is in fact replaced by $H^0(\mathbb{E}) := H(\mathbb{E}) \cap \prod_{j\in\mathbb{N}} \mathcal{L}is(\mathbb{E}_j)$ which is a topological group. Indeed $H^0(\mathbb{E})$ coincides with the projective limit of the Banach Lie groups $H^0_i(\mathbb{E}) := H_i(\mathbb{E}) \cap \prod_{j=1}^{i} \mathcal{L}is(\mathbb{E}_j)$, $i \in \mathbb{N}$. As we shall prove in the sequel, $\{T^*_{UV}\}$ and $H^0(\mathbb{E})$ fully characterize those Fréchet v.b's which can be obtained as projective limits of Banach v.b's.

iii) The bundle $E = \varprojlim E_i$ is said to be a *PLB-vector bundle* and the local trivializations $(U, \varprojlim \tau_i)$ PLB-trivializations.

Corollary 1.3. *Let $(\varprojlim E_i, \varprojlim p_i, B)$ be a PLB-vector bundle. The following conditions hold true:*

i) *If $\{f_i\}_{i\in\mathbb{N}}$ are the canonical projections of $E = \varprojlim E_i$, then (f_i, id_B) : $E \to E_i$ are v.b-morphisms.*

ii) *If $\Gamma(E)$ (resp. $\Gamma(E_i)$) is the module of the sections of E (resp. E_i), then*

$$\Gamma(E) \equiv \varprojlim \Gamma(E_i) \ .$$

Proof: **i**) The f_i's are smooth since, with respect to the trivializations $(U, \varprojlim \tau_i)$ and (U, τ_i), they reduce to $\mathrm{id}_U \times \rho_i$, where ρ_i $(i \in \mathbb{N})$ are the canonical projections of the fiber space $\varprojlim \mathbb{E}_i$. On the other hand, the projection p of E is the limit $\varprojlim p_i$ of the corresponding projections of E_i, hence $p_i \circ f_i = p$ $(i \in \mathbb{N})$. In addition, $f_i|_{p^{-1}(x)}$ is the continuous linear map $\tau_{ix}^{-1} \circ \rho_i \circ \tau_x$. The proof of i) is completed if we take into account that the mapping

$$G\colon\ U \to \mathcal{L}(\mathbb{E}, \mathbb{E}_i): \quad b \mapsto \tau_{ib} \circ f_i|_{p^{-1}(b)} \circ \tau_b^{-1}$$

is constant $(G(b) = \rho_i)$, hence smooth.

ii) The $C^\infty(B, \mathbb{R})$-modules $\{\Gamma(E_i)\}_{i\in\mathbb{N}}$ form a projective system with connecting morphisms

$$\Gamma(E_j) \to \Gamma(E_i)\colon\ \ \sigma \mapsto f_{ji} \circ \sigma \qquad (j \geq i) \ .$$

The desired identification is given by the module isomorphism

$$\Gamma(E) \to \varprojlim \Gamma(E_i)\colon\ \ \sigma \mapsto (f_i \circ \sigma)_{i\in\mathbb{N}} \ . \ \blacksquare$$

In the next Theorem we characterize PLB-vector bundles by means of PLB-transition functions.

Theorem 1.4. *Let (E, p, B) be a Fréchet v.b with fiber type $\mathbb{E}$ where the basis B is a Banach manifold. Let also $\{U_a\}_{a\in\mathcal{A}}$ be an open covering of B and $\{T_{ab}: U_a \cap U_b \to \mathcal{L}(\mathbb{E})\}_{a,b\in\mathcal{A}}$ the corresponding transition functions of E. Then, E is the limit of a strong projective system of Banach v.b's if and only if there exists a family of smooth maps $\{T^*_{ab}: U_a \cap U_b \to H(\mathbb{E})\}_{a,b\in\mathcal{A}}$ such that $T^*_{ab}(x) \circ T^*_{bc}(x) = T^*_{ac}(x)$, $T^*_{aa}(x) = (\mathrm{id}_{\mathbb{E}_i})_{i\in\mathbb{N}}$, for any $x \in U_a \cap U_b \cap U_c$, and*

$$T_{ab} = \varepsilon \circ T^*_{ab}\,, \qquad a, b \in \mathcal{A}\ .$$

Proof: If E is a PLB-v.b, then its transition functions satisfy the properties of the statement, as we saw in Proposition 1.2.

To prove the converse we proceed as follows: Since $\mathbb{E}$ is a Fréchet space, we know ([9]) that there exists a projective system of Banach spaces $\{\mathbb{E}_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$ such that $\mathbb{E} \equiv \varprojlim \mathbb{E}_i$. We define, for any $k \in \mathbb{N}$, the C^∞-mapping

$$T^k_{ab} := \mathrm{pr}_k \circ T^*_{ab}: \quad U_a \cap U_b \to \mathcal{L}(\mathbb{E}_k)\ ,$$

where pr_k denotes the projection to the k-factor. Then, $\{T^k_{ab}\}_{a,b\in\mathcal{A}}$ is a smooth cocycle determining, up to isomorphism, a Banach v.b over B, with total space E_k the quotient $\bigcup_{a\in\mathcal{A}}(\{a\} \times U_a \times \mathbb{E}_k)/\sim_k$, if "$\sim_k$" is the equivalent relation given by

$$(a, x, e) \sim_k (b, x', e') \quad \Longleftrightarrow \quad x = x' \text{ and } (T^k_{ba}(x))(e) = e'\ .$$

The corresponding projection is given by $p_k([(a, x, e)]_k) = x$, if $[(a, x, e)]_k$ denotes the equivalent class of (a, x, e). Setting, for any $j \geq i$,

$$f_{ji}: E_j \to E_i: \quad [(a, x, e)]_j \to [(a, x, \rho_{ji}(e))]_i\ ,$$

we obtain the projective system of v.b's $\{E_i; f_{ji}\}_{i,j\in\mathbb{N}}$ which is strong, since each E_i is of fiber type $\mathbb{E}_i$ and the trivializations $\{(U_a, \tau^i_a)\}_{i\in\mathbb{N},\ a\in\mathcal{A}}$, with

$$\tau^i_a: p_i^{-1}(U_a) \to U_a \times \mathbb{E}_i: \quad [(b, x, e)]_i \to \Big(x, (T^i_{ab}(x))(e)\Big)\ ,$$

satisfy condition ii) of Definition 1.1. As a result, the PLB-v.b $\varprojlim E_i$ can be defined. The corresponding "classical" transition functions are exactly the family $\{T_{ab}\}_{a,b\in\mathcal{A}}$, thus $E \equiv \varprojlim E_i$ and the proof is now complete. ∎

Remark. If we assume that the bundle E is c^r-differentiable ($r < +\infty$), then T^i_{ab} $(a, b \in \mathcal{A})$ and E_i are c^{r-1} in the usual sense of differentiability cite but c^r

in the sense of J. Leslie ([5], [6]). Therefore, the bundle $\varprojlim E_i$ is c^r-differentiable and the identification $E \equiv \varprojlim E_i$ holds in this case too.

Examples: **i**) Every Banach vector bundle E is a PLB-v.b as the limit of the trivial projective system $\{E; \mathrm{id}_E\}$.

ii) It is known ([1]) that the space $J^k(\pi)$ of k-jets of the local sections of a Banach v.b $\lambda = (E, \pi, B)$ admits a v.b structure over B. However, jets of infinite class $J^\infty(\pi) := \varprojlim_{k \in \mathbb{N}} J^k(\pi)$ have been, so far, studied mainly from a topological point of view. In addition, F. Takens ([10]) endows $J^\infty(\pi)$ with a differential structure specifying the $\mathbb{R}$-valued C^∞-mappings of $J^\infty(\pi)$. We shall prove here that $J^\infty(\pi)$ is a PLB-v.b, hence a Fréchet v.b, with a differential structure which is larger than this of Takens.

We observe at first that the fiber type of $J^k(\pi)$ is the Banach space $P^k(\mathbb{B}, \mathbb{E}) := \mathbb{E} \times \mathcal{L}(\mathbb{B}, \mathbb{E}) \times \ldots \times \mathcal{L}^k_s(\mathbb{B}, \mathbb{E})$, where $\mathbb{E}$ is the fiber type of λ, $\mathbb{B}$ the model of B and $\mathcal{L}^k_s(\mathbb{B}, \mathbb{E})$ the space of symmetric continuous k-linear maps of $\mathbb{B}^k$ to $\mathbb{E}$. $\{P^k(\mathbb{B}, \mathbb{E})\}_{k \in \mathbb{N}}$ form a projective system with connecting mappings the natural projections. The corresponding limit coincides with the Fréchet space $P^\infty(\mathbb{B}, \mathbb{E}) = \mathbb{E} \times \mathcal{L}(\mathbb{B}, \mathbb{E}) \times \mathcal{L}^2_s(\mathbb{B}, \mathbb{E}) \times \ldots$.

We define the v.b-morphisms

$$f_{\ell k} \colon J^\ell(\pi) \to J^k(\pi) \colon \; j^\ell_b\, \xi \mapsto j^k_b\, \xi \qquad (\ell \geq k)$$

and, for an arbitrarily chosen $x \in B$, we consider the trivializations $\{(U, \sigma_k)\}_{k \in \mathbb{N}}$ of $J^k(\pi)$, where

$$\sigma_k(j^k_x \xi) = \Big(x, \xi_\alpha(\alpha_0(x)),\, D\xi_\alpha(\alpha_0(x)),\, \ldots,\, D^k \xi_\alpha(\alpha_0(x))\Big) ,$$

if (U, α, α_0) is a vector chart of λ and $\xi_\alpha := \mathrm{pr}_2 \circ \alpha \circ \xi \circ \alpha_0^{-1}$. Then $\sigma_k \circ f_{\ell k} = (\mathrm{id} \times \rho_{\ell k}) \circ \sigma_\ell$ holds, for any $\ell \geq k$. As a result, the properties of Definition 1.1 are satisfied and $J^\infty(\pi)$ is a (Fréchet) PLB-v.b. The corresponding differential structure defined on $J^\infty(\pi)$ is larger than Takens's structure. Indeed the latter is determined by the condition

$$g \in C^\infty(J^\infty(\pi), \mathbb{R}) \iff$$

$$\iff \exists\, k \in \mathbb{N}, \;\; U_k \subseteq J^k(\pi) \text{ open and } g_k \in C^\infty(U_k, \mathbb{R}) \text{ such that } g|_{f_k^{-1}(U_k)} = g_k \circ f_k ,$$

where $f_k \colon J^\infty(\pi) \to J^k(\pi)$ are the canonical projections. This condition is satisfied if and only if g is the limit of the projective system of smooth mappings

$\{g_k \circ f_{ik}\}_{i\geq k}$. Therefore, Takens's $\mathbb{R}$-valued smooth maps on $J^\infty(\pi)$, being projective limits of smooth maps, are also smooth in the sense of J. Leslie (see [5], [6] and [3]). Taking into account that a smooth map (in Leslie's sense) is not necessarily a projective limit, we get the assertion.

iii) We know (cf. e.g. [1]) that if B is a differential manifold modelled on the Banach space $\mathbb{B}$ and $\mathbb{E}$ another Banach space, then the vector bundle $L(TB,\mathbb{E}) := \bigcup_{x\in B} \mathcal{L}(T_xB,\mathbb{E})$, with fiber type $\mathcal{L}(\mathbb{B},\mathbb{E})$, can be defined over B. However, we do not have an analogous result if we replace $\mathbb{E}$ by a Fréchet space $\mathbb{F}$, since the classical proof fails. In particular, certain problems arise if someone tries to check the differentiability of transition functions, since they take values in the (non Fréchet) topological vector space $\mathcal{L}(\mathcal{L}(\mathbb{B},\mathbb{F}),\mathcal{L}(\mathbb{B},\mathbb{F}))$. Here we shall prove that $L(TB,\mathbb{F})$ is indeed, a Fréchet vector bundle using the theory of PLB-v.b's. To this end we consider a projective system of Banach spaces $\{\mathbb{E}_i\,;\rho_{ji}\}_{i,j\in\mathbb{N}}$ such that $\varprojlim \mathbb{E}_i \equiv \mathbb{F}$. Then the family $\{L(TB,\mathbb{E}_i); f_{ji}\}$ is a projective system of Banach v.b's, where

$$f_{ji}\colon\, L(TB,\mathbb{E}_j) \to L(TB,\mathbb{E}_i)\colon \quad f \mapsto \rho_{ji}\circ f \ .$$

Moreover, condition i) of Definition 1.1 holds since $\varprojlim \mathcal{L}(\mathbb{B},\mathbb{E}_i)$ exists and coincides with $\mathcal{L}(\mathbb{B},\mathbb{F})$. Condition ii) is also satisfied, for an arbitrarily chosen $x\in B$, by the trivializations

$$\sigma_i\colon\, p_i^{-1}(U) \to U\times\mathcal{L}(\mathbb{B},\mathbb{E}_i)\colon\;\; g\mapsto (p_i(g),\, g\circ\overline{\phi}^{-1})\,, \qquad i\in\mathbb{N}\ ,$$

where p_i is the projection of $L(TB,\mathbb{E}_i)$, (U,ϕ) a chart of B through x and $\overline{\phi}:\ T_xB\to\mathbb{B}$ the corresponding isomorphism. Therefore, we can define the PLB-v.b $\varprojlim L(TB,\mathbb{E}_i)$ which is isomorphic to $L(TB,\mathbb{F})$ by means of

$$(\rho_i\circ f)_{i\in\mathbb{N}} \equiv f\ ,$$

where ρ_i are the canonical projections of $\mathbb{F}$.

2 – Connections on PLB-vector bundles

In this section we study projective systems of connections on a PLB-vector bundle $E=\varprojlim E_i$. In particular, we prove that the corresponding limits are connections on E, which are characterized by a generalized type of Christoffel symbols. We also prove, for a connection of the previous type, that the notion of parallel displacement along any curve of the basis can be defined, a fact not necessarily true in general since we cannot solve differential equations in Fréchet spaces. A short study of the corresponding holonomy groups completes the section.

We note that among the equivalent Definitions of connections, we adopt that of J. Vilms ([14]).

Proposition 2.1. *Let $(\varprojlim E_i, \varprojlim p_i, B)$ be a PLB-v.b and $\mathcal{C} = \{U\}$ an open covering of B, as in Definition 1.1. If $\{D_i\}_{i\in\mathbb{N}}$ is a projective system of connections on E_i, then the corresponding limit $D := \varprojlim D_i$ is a connection on $E = \varprojlim E_i$ with Cristoffel symbols $\{\Gamma_U\}$ given by*

$$\Gamma_U(x) = \varprojlim_{i\in\mathbb{N}} (\Gamma_U^i(x)) , \qquad x \in B ,$$

where $\{\Gamma_U^i\}$ are the corresponding symbols of D_i $(i \in \mathbb{N})$.

Proof: The family $\{TE_i; Tf_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach manifolds and $\varprojlim TE_i$ is a Fréchet manifold with smooth structure constructed by the differentials of the charts of E (see the proof of Proposition 1.2), i.e., by charts which are projective limits. Moreover, $TE \equiv \varprojlim TE_i$ by means of the diffeomorphism $\varprojlim Tf_i$, if f_i $(i \in \mathbb{N})$ are the canonical projections of E.

Since each D_i is a connection on E_i $(i \in \mathbb{N})$, there exist smooth maps

$$\omega_U^i \colon \phi(U) \times \mathbb{E}_i \to \mathcal{L}(\mathbb{B}, \mathbb{E}_i) ,$$

such that D_i reduces locally, with respect to the charts Φ_i of E_i and $T\Phi_i$ of TE_i, to

$$D_U^i(x,\lambda,y,\mu) = \Big(x,\, \mu + \omega_U^i(x,\lambda)\cdot y\Big) .$$

If $r_{ji} \colon \mathcal{L}(\mathbb{B},\mathbb{E}_j) \to \mathcal{L}(\mathbb{B},\mathbb{E}_i) \colon g \mapsto \rho_{ji}\circ g$ $(j \geq i)$ are the connecting morphisms of $\mathcal{L}(\mathbb{B},\mathbb{E}) \equiv \varprojlim \mathcal{L}(\mathbb{B},\mathbb{E}_i)$, then $r_{ji}\circ\omega_U^j = \omega_U^i \circ (\mathrm{id}_{\mathbb{B}} \times \rho_{ji})$ holds and the smooth mapping

$$\omega_U := \varprojlim \omega_U^i \colon \;\; \phi(U) \times \mathbb{E} \to \mathcal{L}(\mathbb{B},\mathbb{E})$$

can be defined. Furthermore, D reduces locally, with respect to the charts $\varprojlim \Phi_i$ and $\varprojlim T\Phi_i$, to $D_U(x,\lambda,y,\mu) = (x, \mu + \omega_U(x,\lambda)\cdot y)$, hence it is a connection on E. The corresponding Cristoffel symbols are given by

$$\Gamma_U(x) = \omega_U(x,\cdot) = \varprojlim_{i\in\mathbb{N}} (\omega_U^i(x,\cdot)) = \varprojlim_{i\in\mathbb{N}} \Gamma_U^i(x) \;. \;\blacksquare$$

Remarks. **i**) A connection $D = \varprojlim D_i$, as above, is said to be a PLB-connection.

ii) It is known that any connection on a Banach v.b is fully determined by the corresponding Christoffel symbols which satisfy an appropriate compatibility relation (cf. e.g. [2]). In the sequel we shall prove that there is an analogous correspondence between PLB-connections and a generalized type of Cristoffel symbols.

To begin with we note that if $\mathbb{B}$ is a Banach space and $\mathbb{E} = \varprojlim \mathbb{E}_i$ a (Fréchet) projective limit of Banach spaces, then, using similar techniques as in the proof of Proposition 1.2, we may define the Fréchet space

$$H(\mathbb{E}, \mathcal{L}(\mathbb{B}, \mathbb{E})) := \Big\{ (f_i)_{i\in\mathbb{N}} : \ f_i \in \mathcal{L}(\mathbb{E}_i, \mathcal{L}(\mathbb{B}, \mathbb{E}_i)) \ \text{ and } \ \varprojlim f_i \text{ exists} \Big\}$$

as well as the continuous linear mapping

$$\varepsilon : \ H(\mathbb{E}, \mathcal{L}(\mathbb{B}, \mathbb{E})) \to \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{B}, \mathbb{E})) : \quad (f_i)_{i\in\mathbb{N}} \mapsto \varprojlim f_i \ ,$$

since $\mathcal{L}(\mathbb{B}, \mathbb{E}) \equiv \varprojlim(\mathcal{L}(\mathbb{B}, \mathbb{E}_i))$.

Proposition 2.2. *Let $\{E_i; f_{ji}\}$ be a strong projective system of Banach v.b's, as in Definition 1.1, and $E = \varprojlim E_i$ the corresponding PLB-v.b with fiber type $\mathbb{E} = \varprojlim \mathbb{E}_i$. Let also $\mathbb{B}$ the space model of the basis B and $\{(U_a, \Phi_a, \phi_a)\}_{a\in\mathcal{A}}$ a family of vector charts of E, which are projective limits of the vector charts (U_a, Φ_a^i, ϕ_a) of E_i ($i \in \mathbb{N}$), such that $B = \bigcup_a U_a$. Assume that*

$$\Big\{ \Gamma_a^* : \ \phi_a(U_a) \to H(\mathbb{E}, \mathcal{L}(\mathbb{B}, \mathbb{E})) \Big\}_{a\in\mathcal{A}}$$

is a family of smooth mappings such that, for any $i \in \mathbb{N}$, the family

$$\Big\{ \Gamma_a^i := \mathrm{pr}_i \circ \Gamma_a^* : \quad \phi_a(U_a) \to \mathcal{L}(\mathbb{E}_i, \mathcal{L}(\mathbb{B}, \mathbb{E}_i)) \Big\}_{a\in\mathcal{A}}$$

satisfies the ordinary compatibility condition (see e.g. [2, Lemma 1.5, p.5]). Then there exists a PLB-connection on E with corresponding Christoffel symbols $\{\varepsilon \circ \Gamma_a^*\}_{a\in\mathcal{A}}$.

Proof: Let D_i be the connection of E_i constructed by $\{\Gamma_a^i\}_{a\in\mathcal{A}}$ $(i \in \mathbb{N})$. Then D_i locally, with respect to the charts (U_a, Φ_a^i, ϕ_a), has the form

$$D_a^i(x,\lambda,y,\mu) = \Big(x,\, \mu + \Gamma_a^i(x)(\lambda)\cdot y\Big) \ .$$

Since $(\Gamma_a^i(x))_{i\in\mathbb{N}} \in H(\mathbb{E}, \mathcal{L}(\mathbb{B},\mathbb{E}))$, we have that

$$r_{ji} \circ \Gamma_a^j(x) = \Gamma_a^i(x) \circ \rho_{ji} \qquad (j \geq i) \ .$$

Therefore, $\{D_a^i\}_{i\in\mathbb{N}}$ $(\forall\, a \in \mathcal{A})$ and $\{D_i\}_{i\in\mathbb{N}}$ form projective systems and the PLB-connection $D := \varprojlim D_i$ can be defined on E. The corresponding Christoffel symbols are given by $\Gamma_a(x) = \varprojlim_{i\in\mathbb{N}} \Gamma_a^i(x)$, $a \in \mathcal{A}$, hence $\Gamma_a = \varepsilon \circ \Gamma_a^*$. ∎

A direct application of Propositions 2.1, 2.2 is the next basic result illustrating the bijective correspondence between PLB-connections and the generalized symbols $\{\Gamma_a^*\}$ as above.

Theorem 2.3. *A connection D on a PLB-v.b E is a PLB-connection if and only if the corresponding Christoffel symbols $\{\Gamma_a\}$ can be factored in the form $\Gamma_a = \varepsilon \circ \Gamma_a^*$, where $\{\Gamma_a^*\}$ satisfy the properties of Proposition 2.2.* ∎

Remark. As we prove in [11], each PLB-connection of $E = \varprojlim E_i$ corresponds to a principal connection form of a generalized bundle of frames of E.

In the remaining of the paper we study the notion of parallel displacement as well as the holonomy groups of a PLB-connection. We note that in the general case of a connection on a Fréchet v.b we cannot define a parallel displacement along a curve of the basis, due to the lack of a general theory of solving differential equations in Fréchet spaces. However if we restrict our study to the case of a PLB-v.b ($E = \varprojlim E_i$, $p = \varprojlim p_i$, B) endowed with a linear PLB-connection $D = \varprojlim D_i$, we obtain the following

Proposition 2.4. *Let $\beta : [0,1] \to B$ be a smooth curve. Then, for any $u \in E_{\beta(0)} := p^{-1}(\beta(0))$, there is a unique parallel section of E, along β, satisfying the initial condition $(0,u)$.*

Proof: Regarding the fiber $E_{\beta(0)}$ we check that $E_{\beta(0)} = \varprojlim_{i\in\mathbb{N}} E_{i\beta(0)}$. Hence u has the form $u = (u_i)_{i\in\mathbb{N}}$, where $u_i \in E_{i\beta(0)}$ and $f_{ji}(u_j) = u_i$ $(j \geq i)$, if f_{ji} are the connecting morphisms of E. Let, for any $i \in \mathbb{N}$, $\xi_i \colon [0,1] \to E_i$ be the unique parallel section of the Banach v.b E_i, along β, such that $\xi_i(0) = u_i$. Then, if ∂ denotes the fundamental vector field of $\mathbb{R}$, we have that

$$\begin{aligned} &p_i \circ f_{ji} \circ \xi_j = p_j \circ \xi_j = \beta \ , \\ &D_i \circ T(f_{ji} \circ \xi_j) \circ \partial = f_{ji} \circ D_j \circ T\xi_j \circ \partial = f_{ji} \circ 0 = 0 \ , \\ &(f_{ji} \circ \xi_j)(0) = u_i \ . \end{aligned}$$

As a result $f_{ji} \circ \xi_j = \xi_i$, for any $j \geq i$, and the smooth mapping

$$\xi := \varprojlim \xi_i : \quad [0,1] \to E$$

can be defined. ξ is the desired section of E since

$$\begin{aligned} &p \circ \xi = p_i \circ \xi_i = \beta \ , \\ &D \circ T\xi \circ \partial = \varprojlim (D_i \circ T\xi_i \circ \partial) = 0 \ , \\ &\xi(0) = (u_i) = u \ . \end{aligned}$$

Furthermore, ξ is unique, since if ξ' is another parallel section along β such that $\xi'(0) = u$, then each $f_i \circ \xi'$ is a parallel section of E_i along β through u_i. Therefore, the uniqueness of ξ_i implies that $f_i \circ \xi = f_i \circ \xi'$ $(i \in \mathbb{N})$ or, equivalently, $\xi = \xi'$. ∎

A direct application of Proposition 2.4 is the next main result.

Theorem 2.5. *Along any curve β of the basis of a PLB vector bundle $E = \varprojlim E_i$ there exists a parallel displacement τ_β. More precisely,*

$$\tau_\beta = \varprojlim_{i \in \mathbb{N}} \tau^i_\beta \ ,$$

where τ^i_β is the parallel displacement along β on E_i $(i \in \mathbb{N})$. ∎

Let us assume now that the connecting morphisms $f_{ji} : E_j \to E_i$ $(j \geq i)$ of the PLB-v.b $(\varprojlim E_i, \varprojlim p_i, B)$ are *surjective*. Then, concerning the holonomy groups Φ_x, Φ^i_x of D, D_i respectively $(x \in B)$, the next Proposition is valid.

Proposition 2.6. *Φ_x is a subgroup of $\varprojlim_{i \in \mathbb{N}} \Phi^i_x$.*

Proof: For any $j \geq i$, we set

$$\sigma_{ji} : \Phi^j_x \to \Phi^i_x : \quad \tau^j_\alpha \mapsto \tau^i_\alpha \ .$$

σ_{ji} is well defined since $f_{ji} \circ \tau_\alpha^j = \tau_\alpha^i \circ f_{ji}$. Moreover σ_{ji} is a group morphism and $\sigma_{ik} \circ \sigma_{ji} = \sigma_{jk}$ holds, for any $j \geq i \geq k$. Thus, the projective system $\{\Phi_x^i; \sigma_{ji}\}_{i,j \in \mathbb{N}}$ of groups can be defined. Φ_x is a subgroup of the corresponding limit by means of the injective morphism $h = \varprojlim h_i$, where

$$h_i \colon \Phi_x \to \Phi_x^i \colon \quad \tau_\alpha \mapsto \tau_\alpha^i \ .$$

Indeed, h_i is well defined, since each canonical projection $f_i \colon E = \varprojlim E_i \to E_i$ is surjective, $\sigma_{ji} \circ h_j = h_i$ holds (for any $j \geq i$) and h is 1-1 since

$$\tau_\alpha \in \operatorname{Ker} h \iff \tau_\alpha^i = \mathrm{id}_{E_{ix}}, \ \forall\, i \in \mathbb{N} \iff \tau_\alpha = \mathrm{id}_{E_x} \ . \ \blacksquare$$

Remark. We note here that the projective limit $\varprojlim_{i \in \mathbb{N}} \Phi_x^i$ can be defined, as we have proven, despite the fact that each Φ_x^i is a subgroup of $GL(p_i^{-1}(x))$ and $\{GL(p_i^{-1}(x)),\ i \in \mathbb{N}\}$ do not form a projective system.

ACKNOWLEDGEMENT – I am indebted to Professor E. Vassiliou for his stimulating discussions and valuable suggestions concerning the present paper. Also, I would like to thank the Referee for his remarks leading to the essential improvement of the style of this note.

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