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REGULARITY UP TO THE BOUNDARY OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR CERTAIN DEGENERATE ELLIPTIC EQUATIONS

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Abstract: We prove, through a more careful analysis of its symbols, the ellipticity of certain pseudodifferential operators. We use this result and the concatenation technique to prove the regularity up to the boundary of solutions of the Dirichlet problem for certain classes of second-order degenerate elliptic equations.

I – Introduction

This work is concerned with the regularity up to the boundary of solutions of the Dirichlet problem for certain classes of second-order degenerate elliptic equations in the plane.

On this subject, Baouendi [2] showed, by using a priori estimates, that the solutions of the Dirichlet problem for certain degenerate elliptic operators are smooth up to the boundary. The degeneracy considered has constant rate at the boundary.

Bergamasco, Gerszonowicz and Petronilho [3], by using the theory of Fourier integral operators and pseudodifferential operators, showed that the method of transfer to the boundary via the associated heat equations, as introduced by Treves [8], [9] in the case of nondegenerate equations, can be used to prove regularity for a class of equations that degenerate at the boundary. More precisely they considered the cases when the operators are of the form:

$$P = \Big(\partial_t + it^k\, a(t,x)\,\partial_x + b(t,x)\Big)\,\Big(\partial_t - it^l\,\tilde{a}(t,x)\,\partial_x + \tilde{b}\,(t,x)\Big)$$

with $a.\tilde{a} > 0$ for $t = 0$; $a, \tilde{a}, b, \tilde{b}$ smooth functions; k, l nonnegative integers

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or

$$P = \Big(\partial_t + it^k\, a(t,x)\,\partial_x\Big)\,\Big(\partial_t - it^l\,\tilde{a}(t,x)\,\partial_x\Big)$$

where $k, l\ \geq 4$ are integers; $a.\tilde{a} > 0$ for $t > 0$ (the boundary given by $t = 0$).

Bergamasco and Petronilho [4], by using only the theory of pseudodifferential operators, showed the same type of result when the operators are of the form:

$$\text{(H1)} \qquad P = \Big(\partial_t + it^k\, a(t,x)\,\partial_x + b(t,x)\Big)\,\Big(\partial_t - it^l\,\tilde{a}(t,x)\,\partial_x + \tilde{b}(t,x)\Big)$$

where $k, l\ \geq 4$ are integers; $a.\tilde{a} > 0$ for $t > 0$; $b, \tilde{b}$ smooth functions

or

$$\text{(H2)} \qquad P = \Big(\partial_t + i\,\psi(t)\,a(t,x)\,\partial_x + b(t,x)\Big)\,\Big(\partial_t - i\,\tilde{\psi}(t)\,\tilde{a}(t,x)\,\partial_x + \tilde{b}(t,x)\Big)$$

with $a.\tilde{a} > 0$ for $t = 0$; $\psi, \tilde{\psi}$ smooth functions, $\psi, \tilde{\psi} \geq 0$; $b, \tilde{b}$ as before.

In [3] and [4], except in the elliptic case, it is **fundamental** the fact that the coefficient of ∂_x, in each factor, has one zero of a certain order in $t = 0$. This order depends on the class of operators in study and may be infinite. In case of nondegenerate equations see e.g. [1], [5].

In this work we prove the regularity up to boundary of solutions of the Dirichlet problem for a class of operators where the **coefficient of** ∂_x, in each factor, **need not have one zero in** $\mathbf{t = 0}$, which allows us to cover examples which are not contained in [2], [3] and [4], see examples after Theorem 2.

We prove also that the desired regularity holds for a class of operators which are perturbations (of order ≤ 1) of operators that are factored as a product of two factors of order one (see Theorem 3).

In section II we prove, through a more careful analysis of its symbols, the ellipticity of certain pseudodifferential operators (see Theorem 1), an essential point in the proof of Theorem 2.

In section III, by using the result of section II and following the lines of the proof of Theorem 1 of [4], we prove Theorem 2.

In section IV, by using the concatenations of [6] we prove Theorem 3.

We call attention to the fact that the method of concatenations has already been used to prove local solvability, hypoellipticity and uniqueness of solutions of the Cauchy problem. We do not know any reference that uses this method to prove the hypoellipticity of boundary value problem as we have done here.

II – Ellipticity of certain pseudodifferential operators

Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin and let $T > 0$. We will now recall a definition and some properties of a class of symbols for pseudodifferential operators given in [3] and [4].

Let $\lambda(t,x)$ be a smooth nonnegative function in $[0,T] \times \overline{U}$.

Let $m \in \mathbb{R}$ and let $0 \le \delta < 1$; set $V_T = \{(t,s) \in \mathbb{R}^2 : \, 0 \le t \le s \le T\}$.

Definition. A function $h(t,s,x,\xi)$ belongs to the class $\mathcal{A}^{m,\lambda}_{1,\delta} = \mathcal{A}^{m,\lambda}_{1,\delta}(U)$ if h is smooth on $V_T \times U \times (\mathbb{R}^n|\{0\})$ and there exists $B > 0$ such that for every $\alpha, \beta \in \mathbb{Z}^n_+$, and every $\gamma, \epsilon \in \mathbb{Z}_+$, and every compact set K contained in $V_T \times U$, there exists a constant $c = c(\alpha,\beta,\gamma,\epsilon,K)$ such that

$$\left|\partial_t^\epsilon \partial_s^\gamma \partial_x^\alpha \partial_\xi^\beta h(t,s,x,\xi)\right| \le c(1+|\xi|)^{m+\epsilon+\gamma+\delta|\alpha|-|\beta|} \exp\Big(-B\,|\xi| \int_t^s \lambda(r,x)\,dr\Big)$$

for all $(t,s,x,\xi) \in K \times (\mathbb{R}^n|\{0\})$.

Properties.

1) $\partial_t^\epsilon \partial_s^\gamma \partial_x^\alpha \partial_\xi^\beta (\mathcal{A}^{m,\lambda}_{1,\delta}) \subset \mathcal{A}^{m+\epsilon+\gamma+\delta|\alpha|-|\beta|,\lambda}_{1,\delta}$;

2) $\mathcal{A}^{m,\lambda}_{1,\delta} \subset \mathcal{S}^m_{1,\delta}$, uniformly in $(t,s) \in V_T$;

3) If $h \in \mathcal{A}^{m,\lambda}_{1,\delta}$ and $g \in \mathcal{A}^{m',\lambda'}_{1,\delta'}$ then $h.g \in \mathcal{A}^{m+m',\lambda+\lambda'}_{1,\delta''}$, where $\delta'' = \max\{\delta,\delta'\}$.

We consider now $G(t,s)$ and $H(t,s)$ pseudodifferential operators of order zero in U, depending smoothly on $t \le s$ and $s \le t$, respectively, in $[0,T]$ (if T is small).

We assume that the symbols of G and H are given respectively by:

$$g(t,s,x,\xi) = \sum_{j=0}^{+\infty} g_j(t,s,x,\xi)$$

where $g_j \in \mathcal{A}^{-j,\tilde{a}}_{1,0}$, $j = 0,1,...$, with $\tilde{a}$ a smooth function in $[0,T] \times \overline{U}$ and $\tilde{a} > 0$ for $t = 0$

and

$$h(t,s,x,\xi) = \sum_{j=0}^{+\infty} h_j(t,s,x,\xi)$$

where $h_j \in \mathcal{A}^{-j/2,a}_{1,1/2}$, $j = 0,1,\ldots$, with a a smooth function in $[0,T] \times \overline{U}$ and $a > 0$ for $t > 0$ (in the definition of $\mathcal{A}^{m,\lambda}_{1,1/2}$, the class of symbols to which h_j belongs, $j = 0,1,\ldots$, we take $V_T = \{(t,s) \in \mathbb{R}^2 : \, 0 \le s \le t \le T\}$).

Further we assume that the principal parts of g and h are given respectively by:

$$g_0(t,s,x,\xi) = \exp\Big(-|\xi| \int_t^s \tilde{a}(r,x)\,dr\Big)\,, \qquad 0 \le t \le s \le T\ ,$$

and

$$h_0(t,s,x,\xi) = \exp\Big(-|\xi| \int_s^t a(r,x)\,dr\Big)\,, \qquad 0 \le s \le t \le T\ .$$

We can now state the

Theorem 1. *The operator*

$$N(x,D_x) = \int_0^T G(0,s)\,H(s,0)\,ds$$

is an elliptic pseudodifferential operator of order -1 on U.

Before proving Theorem 1 we prove the following

Lemma 1. *If $f \in \mathcal{A}^{m,\lambda}_{1,\delta}$ with $\lambda > 0$ in $[0,T] \times \overline{U}$ and if*

$$b(t,x,\xi) = \int_t^T f(t,s,x,\xi)\,ds\,, \qquad 0 \le t \le s \le T\ ,$$

then for every compact set K_1 contained in $[0,T]$ and every compact set K_2 contained in U, there exists a constant $c = c(K_1,K_2) > 0$ such that

$$|b(t,x,\xi)| \le c(1+|\xi|)^{m-1}$$

for all $(t,x,\xi) \in K_1 \times K_2 \times (\mathbb{R}^n|\{0\})$.

Proof: Since $(t,x) \in K_1 \times K_2$ and $f \in \mathcal{A}^{m,\lambda}_{1,\delta}$ we have that there exists a constant $c = c(K_1,K_2) > 0$ such that

$$|f(t,s,x,\xi)| \le c(1+|\xi|)^m \exp\Big(-B|\xi| \int_t^s \lambda(r,x)\,dr\Big)$$

for all $(t,s,x,\xi) \in K$, where

$$K = \Big((K_1 \times [0,T]) \cap V_T\Big) \times K_2 \times (\mathbb{R}^n|\{0\})\ .$$

Since $\lambda > 0$ in $[0,T] \times \overline{U}$ there is a constant c_1 such that

$$|f(t,s,x,\xi)| \le c(1+|\xi|)^m \exp\Big[-c_1\,|\xi|\,(s-t)\Big]$$

for all $(t,s,x,\xi) \in K$.

Thus,

$$|b(t,x,\xi)| \le c(1+|\xi|)^m \int_t^T \exp\Big[-c_1\,|\xi|\,(s-t)\Big]\,ds$$
$$\le c_2(1+|\xi|)^{m-1}$$

for all $(t,x,\xi) \in K_1 \times K_2 \times (\mathbb{R}^n|\{0\})$. ∎

We prove now the Theorem 1. We make use of the letter c to denote various constants.

We call attention to the fact that in this proof we need to take more care (as compared to that done in Proposition 5 of [4]) in the analysis of the symbol of $N(x,D_x)$.

Proof of Theorem 1: We observe that the symbol of N, $\sigma(N)$, is given by

$$\sigma(N)(x,\xi) = \int_0^T F(s,x,\xi)\,ds$$

where

$$F(s,x,\xi) - \sum_{|\alpha|\le M-1} \frac{1}{\alpha!}\,\partial_\xi^\alpha g(0,s,x,\xi)\,D_x^\alpha h(s,0,x,\xi) \;\in\; \mathcal{S}_{1,1/2}^{-M/2}\,. \tag{1}$$

Set

$$\Sigma_0(x,\xi) = \int_0^T g(0,s,x,\xi)\,h(s,0,x,\xi)\,ds + \int_0^T \sum_{|\alpha|=1} \partial_\xi^\alpha g\,D_x^\alpha h$$
$$+ \int_0^T \sum_{|\alpha|=2} \frac{1}{\alpha!}\,\partial_\xi^\alpha g\,D_x^\alpha h\;.$$

By (1) we have

$$\Big|\sigma(N)(x,\xi) - \Sigma_0(x,\xi)\Big| \le c(1+|\xi|)^{-3/2}\;. \tag{2}$$

Set

$$\Sigma_1(x,\xi) = \int_0^T g_0\,h_0 + \int_0^T g_0\,h_1 + \int_0^T g_0\,h_2 + \int_0^T g_1\,h_0$$
$$+ \int_0^T \sum_{|\alpha|=1} \partial_\xi^\alpha g_0\,D_x^\alpha h_0 + \int_0^T \sum_{|\alpha|=1} \partial_\xi^\alpha g_0\,D_x^\alpha h_1$$
$$+ \int_0^T \sum_{|\alpha|=2} \frac{1}{\alpha!}\,\partial_\xi^\alpha g_0\,D_x^\alpha h_0\;.$$

The hypothesis about the symbols g and h implies that the following inequality holds:

$$\Big|\Sigma_0(x,\xi) - \Sigma_1(x,\xi)\Big| \leq c(1+|\xi|)^{-3/2} \; . \tag{3}$$

We observe that

$$\Big|\Sigma_1(x,\xi) - \int_0^T g_0\, h_0\Big| \leq \sum_{j=1}^{6} |L_j(x,\xi)|$$

where

$$L_j(x,\xi) = \int_0^T M_j(s,x,\xi)\, ds$$

with $M_j \in \mathcal{A}^{\epsilon,a+\tilde{a}}_{1,1/2}$; $j = 1,\ldots,6$, $(\epsilon = -1$ or $\epsilon = -1/2)$.

The properties of the class of symbols guarantee that

$$M_j \in \mathcal{A}^{\epsilon,a+\tilde{a}}_{1,1/2} \,, \qquad j = 1,\ldots,6 \;\text{ and }\; \epsilon = -1 \;\text{ or } \epsilon = -1/2 \; .$$

For each $j = 1,\ldots,6$ by using Lemma 1 with M_j in the place of f, $\lambda = a+\tilde{a}$ and $t = 0$, we obtain

$$|L_j(x,\xi)| = \Big|\int_0^T M_j(s,x,\xi)\, ds\Big| \leq c(1+|\xi|)^{\epsilon-1} \leq c(1+|\xi|)^{-3/2}$$

for $x \in \tilde{K} \subset U$, where $\tilde{K}$ is a compact set.

Thus,

$$\Big|\Sigma_1(x,\xi) - \int_0^T g_0\, h_0\Big| \leq c(1+|\xi|)^{-3/2} \; . \tag{4}$$

The inequalities (2), (3) and (4) show that

$$\sigma(N) = \int_0^T g_0\, h_0 \;\text{ modulo } \mathcal{S}^{-3/2}_{1,1/2} \; . \tag{5}$$

By using the definition of g_0 and h_0 (see p.3) we have

$$\begin{aligned}\int_0^T g_0(0,s,x,\xi)\, h_0(s,0,x,\xi)\, ds &= \int_0^T \exp\Big[-|\xi| \int_0^s (a+\tilde{a})\,(r,x)\, dr\Big]\, ds \\ &\leq \int_0^T \exp(-c\,|\xi|\, s)\, ds \\ &\leq c(1+|\xi|)^{-1} \; .\end{aligned} \tag{6}$$

In the penultimate inequality we use the fact that $a+\tilde{a} > 0$.

One has also that

$$\int_0^T g_0\, h_0\, ds \geq \int_0^T \exp(-c\,|\xi|\, s)\, ds \geq c(1+|\xi|)^{-1} \; . \tag{7}$$

Here we use the fact that $a + \tilde{a} \leq c$.

Thus, (5), (6) and (7) imply that $N(x, D_x)$ is elliptic of order -1. ∎

Remark. Theorem 1 admits a microlocal version, which will be used in the proof of Theorem 2; the proof of the microlocal version will be omitted.

III – Application to the Dirichlet problem for certain second-order degenerate elliptic differential equations

By using Theorem 1 and following the lines of the proof of Theorem 1 of [4] we prove a result on regularity up to the boundary of solutions of the Dirichlet problem for certain second-order degenerate elliptic equations in the plane.

We have included the proof for the sake of completeness.

Theorem 2. *Let $U \subset \mathbb{R}$ be a neighborhood of the origin and let $T > 0$. Let a, $\tilde{a}$, b, $\tilde{b}$ be smooth functions on $[0,T] \times \overline{U}$. We assume that a and $\tilde{a}$ are real, $a > 0$ for $t > 0$ and $\tilde{a} > 0$ for $t = 0$.*

Let

$$P = \Big(\partial_t + i\, a(t,x)\, \partial_x + b(t,x)\Big)\, \Big(\partial_t - i\, \tilde{a}(t,x)\, \partial_x + \tilde{b}(t,x)\Big) \; .$$

If $u \in C^\infty([0,T], D'(U))$ satisfies

$$(*) \qquad \begin{aligned} Pu &= f \in C^\infty([0,T] \times U) \; , \\ u_{|t=0} &= g \in C^\infty(U) \; , \end{aligned}$$

then $u \in C^\infty([0,T] \times U)$, perhaps after shrinking T, U.

Proof: By using the integrating factors $\exp B(t,x)$ and $\exp \tilde{B}(t,x)$ where $B(t,x) = \int_0^t b(s,x)\, ds$ and $\tilde{B}(t,x) = \int_0^t \tilde{b}(s,x)\, ds$ a simple computation shows that u is a solution of

$$\begin{aligned} Pu &= f \\ u_{|t=0} &= g \end{aligned}$$

if and only if $v = \exp(B) \cdot u$ is a solution of

$$\begin{aligned} \tilde{P}v &= h \\ v_{|t=0} &= g \end{aligned}$$

where

$$h = \exp(B) \cdot f \,, \qquad \tilde{P} = (\partial_t + i\,a\,\partial_x + A)\,Q\,(\partial_t - i\,\tilde{a}\,\partial_x + \tilde{A}) \ ,$$

with $Q = \exp(B - \tilde{B})$, $A = ac$ and $\tilde{A} = \tilde{a}\,\tilde{c}$ for smooth functions c, $\tilde{c}$.

Thus, the hypoellipticity of the problem $(*)$ is equivalent to the hypoellipticity of the following problem

$$(**) \qquad \begin{aligned} \tilde{P}u &= f \in C^\infty([0,T] \times U) \ , \\ u_{|t=0} &= g \in C^\infty(U) \ . \end{aligned}$$

The proof of hypoellipticity of $(**)$ will be made by showing that the wave-front set of u is empty, i.e., $WF(u) = \emptyset$. For this, it suffices to show that

$$WF(u) \cap (U \times \mathbb{R}^+) = \emptyset \ , \tag{8}$$

$$WF(u) \cap (U \times \mathbb{R}^-) = \emptyset \ . \tag{9}$$

We will start with the proof of (8).

Let

$$\begin{aligned} L_+ &= \partial_t + i\,a\,\partial_x + A \ , \\ L_- &= \partial_t - i\,\tilde{a}\,\partial_x + \tilde{A} \ . \end{aligned}$$

The backward (resp. forward) Cauchy problem for L_+ (resp. L_-) has a microlocal parametrix which we denote $G_+(t,s)$ (resp. $G_-(t,s)$) (see construction in [3]; see also [2] for the case in which L_+ and L_- are pseudodifferential operators); this means that

$$\begin{aligned} L_+G_+(t,s) &\sim 0 \,, \qquad 0 \le t \le s \le T \ , \\ G_+ &\sim I \,, \qquad t = s \ , \end{aligned}$$

and

$$\begin{aligned} L_-G_-(t,s) &\sim 0 \,, \qquad 0 \le s \le t \le T \ , \\ G_- &\sim I \,, \qquad t = s \ , \end{aligned}$$

where $A \sim 0$ denotes the fact that A is a C^∞ function of (t,s) valued in the space of regularizing operators in the space variables.

Let $u \in C^\infty([0,T], D'(U))$ satisfy $(**)$. We uncouple $(**)$:

$$L_+v = f \in C^\infty([0,T] \times U) \ , \tag{10}$$

$$QL_-u = v \quad \text{and} \quad u_{|t=0} = g \in C^\infty(U) \ . \tag{11}$$

By using the fact that G_+ and G_- are both regularizing for $t \neq s$, we deduce from (10) (denoting by $\approx$ the equivalence modulo smooth functions on $[0,T] \times U$)

$$v(t,.) \approx G_+(t,T)\, v(T,.) - \int_t^T G_+(t,s)\, f(s,.)\, ds \tag{12}$$

for $0 \leq t \leq T$, and therefore

$$v \approx 0 \quad \text{(i.e. it is smooth for } t < T) \ . \tag{13}$$

Now (11) implies

$$u(t,.) \approx G_-(t,0)\, g + \int_0^t G_-(t,s)\, Q^{-1}\, v(s,.)\, ds$$

for $0 \leq t \leq T$.

Since G_- is a pseudodifferential operator and $g \in C^\infty$ the pseudolocal property of G_- and (13) imply that u is C^∞ in $[0,T) \times U$ as desired.

Our goal will now be to prove (9).

For this we consider the problems

$$\begin{aligned} L_+ v &= f \\ v_{|t=0} &= v_0 \end{aligned}$$

where v_0 is the trace $v_{|t=0}$ and

$$\begin{aligned} QL_- u &= v \\ u_{|t=T} &= u_T \ . \end{aligned}$$

Here, also there exist (see [3]) pseudodifferential operators $H(t,s)$ and $G(t,s)$ such that

$$\begin{aligned} L_+ H(t,s) \sim 0\,, &\quad 0 \leq s \leq t \leq T \ , \\ H \sim I\,, &\quad t = s \ , \end{aligned}$$

and

$$\begin{aligned} QL_- G(t,s) \sim 0\,, &\quad 0 \leq s \leq t \leq T \\ G \sim I\,, &\quad t = s \ . \end{aligned}$$

As in the proof of (8), we deduce

$$v(t,.) \approx H(t,0)\, v_0 + \int_0^t H(t,s)\, f(s,.)\, ds \ . \tag{14}$$

Thus,

$$v(t,.) \approx H(t,0)\, v_0 \ . \tag{15}$$

Analogously, we prove

$$u(t,.) \approx G(t,T)\, u(T,.) - \int_t^T G(t,s)\, Q^{-1}\, v(s,.)\, ds\ .$$

Thus,

$$u(t,.) \approx -\int_t^T G(t,s)\, Q^{-1} v(s,.)\, ds\ . \tag{16}$$

By using (15) and (16) we have

$$u(t,.) \approx \int_t^T G(t,s)\, Q^{-1}\, H(s,0)\, v_0\, ds\ . \tag{17}$$

When $t = 0$ the initial condition $u_{|t=0} = g \in C^\infty(U)$ shows that

$$N(x,D_x)\, v_0 = \int_0^T G(0,s)\, Q^{-1}(s,x)\, H(s,0)\, v_0(.)\, ds \tag{18}$$

is smooth.

The construction of G and H (see [4]) guarantees that they are as in the microlocal version of Theorem 1 and therefore N is elliptic of order -1 in U.

The only difference between (18) and Theorem 1 is the presence of the non-vanishing factor Q^{-1} in $N(x,D_x)$; which does not entail any essential modification in the proof.

Since Nv_0 is smooth and N is elliptic we conclude that v_0 is smooth and therefore (15) implies $v \approx 0$.

Thus (16) implies $u \approx 0$, as desired. ∎

We present now some examples of operators for which the regularity of solutions of the Dirichlet problem is studied through our results and has not been previously studied by [2], [3], [4].

Examples:

1) $P = \Big(\partial_t + i(t+x^2)\,\partial_x + b(t,x)\Big)\Big(\partial_t - i\,\partial_x + \tilde{b}(t,x)\Big)$
where b, $\tilde{b}$ are smooth functions.

2) $P = \Big(\partial_t + i(\phi(t)+\psi(x))\,\partial_x + b(t,x)\Big)\Big(\partial_t - i(1+t)\,\partial_x + \tilde{b}(t,x)\Big)$
where $b, \tilde{b}$ are as in (1), ϕ is flat in $t = 0$ and $\phi > 0$ for $t > 0$; ψ is flat in $x = 0$ and $\psi \geq 0$.

3) $P = \Big(\partial_t + i\,\phi(t,x)\,\partial_x\Big)(\partial_t - i\partial_x)$
where ϕ is flat in $(t,x) = (0,0)$ and $\phi > 0$ for $t > 0$.

IV – Concatenation and regularity

We shall use the concatenations of [6] to prove the regularity up to the boundary of solutions of the Dirichlet problem for a class of operators which are perturbations (of order ≤ 1) of a subclass of operators studied by [3], [4].

Theorem 3. *Let $U \subset \mathbb{R}$ be a neighborhood of the origin and let $T > 0$. Let*

$$P = P(c_j) = (\partial_t - i\,a\,t^k\,\partial_x)\,(\partial_t - i\,b\,t^k\,\partial_x) + i\,c_j\,t^{k-1}\,\partial_x$$

where

$$a \cdot b < 0\,, \qquad k = 1, 2, \dots\;,$$

$$c_j = j(k+1)\,(a-b)\,, \qquad j = 0, 1, 2, \dots\;.$$

If $u \in C^\infty([0,T], D'(U))$ satisfies

$$P(c_j)\,u = f \in C^\infty([0,T] \times U)$$
$$u_{|t=0} = g \in C^\infty(U)$$

then $u \in C^\infty([0,T) \times U)$.

Proof: The proof will be made by induction on j.

If $j = 0$ we have $c_0 = 0$ and therefore

$$P(0) = (\partial_t - i\,a\,t^k\,\partial_x)\,(\partial_t - i\,b\,t^k\,\partial_x)\;.$$

The Theorem 3 of [3] says that $P(0)$ has the desired regularity.

We now assume that $P(c_j)$ has the desired regularity and proceed to show that $P(c_{j+1})$ has the desired regularity.

Let $u \in C^\infty([0,T], D'(U))$ be such that

$$P(c_{j+1})\,u = f \in C^\infty([0,T] \times U) \tag{19}$$

with

$$u_{|t=0} = g \in C^\infty(U)\;. \tag{20}$$

Let

$$X = \partial_t - i\,a\,t^k\,\partial_x\;,$$

$$Y = \partial_t - i\,b\,t^k\,\partial_x\;.$$

By using the following concatenation (see [4]):

$$\left(tY - \frac{c_{j+1}}{a-b} + 2\right) P(c_{j+1}) = P\Big[c_{j+1} - (k+1)\,(a-b)\Big] \left(tY - \frac{c_{j+1}}{a-b}\right)$$

we have

$$\Big(tY - (j+1)\,(k+1) + 2\Big)\, P(c_{j+1}) = P(c_j)\,\Big(tY - (j+1)(k+1)\Big)$$

and therefore

$$\Big(tY - (j+1)\,(k+1) + 2\Big)\, P(c_{j+1})\, u = P(c_j)\Big(tY - (j+1)\,(k+1)\Big)\, u\ . \tag{21}$$

Since $P(c_{j+1})u \in C^\infty([0,T]\times U)$, (21) implies that

$$P(c_j)\Big(tY - (j+1)\,(k+1)\Big)\, u \ \in\ C^\infty([0,T]\times U)\ . \tag{22}$$

By hypothesis $u_{|t=0} \in C^\infty(U)$ and therefore

$$\Big(tY\, u - (j+1)\,(k+1)\, u\Big)_{|t=0} \ \in\ C^\infty(U)\ . \tag{23}$$

Thus, the induction hypothesis guarantees that

$$tY\, u - (j+1)\,(k+1)\, u \ \in\ C^\infty([0,T]\times U)\ . \tag{24}$$

We recall the following relation (see [4]):

$$\left(tX + \frac{c_{j+1}}{a-b} - 1\right) \left(tY - \frac{c_{j+1}}{a-b}\right) - t^2\, P(c_{j+1}) = \frac{c_{j+1}}{a-b}\left(1 - \frac{c_{j+1}}{a-b}\right)$$

that is,

$$\begin{aligned}\Big(tX + (j+1)\,(k+1) - 1\Big)\,\Big(tY - (j+1)\,(k+1)\Big) - t^2\, P(c_{j+1}) &= \\ = (j+1)\,(k+1)\,\Big[1 - (j+1)\,(k+1)\Big]\ .\end{aligned} \tag{25}$$

Then (19), (24) and (25) imply that

$$(j+1)\,(k+1)\,\Big[1 - (j+1)\,(k+1)\Big]\, u \ \in\ C^\infty([0,T]\times U)\ .$$

Since $k = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$ we have

$$(j+1)\,(k+1)\,\Big[1 - (j+1)\,(k+1)\Big] \neq 0$$

and therefore $u \in C^\infty([0,T]\times U)$. The proof is complete. ∎

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