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HERMITEAN ULTRADISTRIBUTIONS

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Abstract: In this paper we introduce the hermitean ultradistributions by a duality argument. The method we use is exactly the same introduced by Schwartz for the distribution theory. Our space of hermitean ultradistributions contains all the Schwartz tempered distributions but is not related with the space $\mathcal{D}'$ of all distributions. In our space $\mathcal{G}'$ of hermitean ultradistributions the derivatives are linear continuous operators and the Fourier transform is a vectorial and topological isomorphism. The construction of $\mathcal{G}'$ is based on the construction of a space of functions $\mathcal{G}$, strictly included in the Schwartz space $\mathcal{S}$, but still dense in $\mathcal{S}$. This space $\mathcal{G}$ is an inductive limit of finite-dimensional vector spaces. Finally we give a sequential representation of our hermitean ultradistributions and we apply the theory to the series of multipoles used by physicists.

1 – Introduction

The distribution space $\mathcal{D}'$ introduced by Laurent Schwartz [13] as the strong dual of the space $\mathcal{D}$ of C^∞ functions from $\mathbb{R}^N$ into $\mathbb{C}$ with compact support generalizes the space L^1_{loc} of locally integrable functions. The derivative operators $\partial^\alpha\colon \mathcal{D} \to \mathcal{D}$ ($\alpha \in \mathbb{N}^N$) are linear continuous and can be extended by duality to $\mathcal{D}'$. In order to define the Fourier transform Schwartz constructed a space of C^∞ functions invariant for the classical Fourier transform: the Schwartz space $\mathcal{S}$ of C^∞ rapidly decreasing functions in $\mathbb{R}^N$. The Fourier transform is an isomorphism in $\mathcal{S}$ and can be extended, by duality, as an isomorphism in the space $\mathcal{S}'$ of tempered distributions.

In order to generalize the space $\mathcal{S}'$ and the Fourier transform defined there, it is natural to construct a space of C^∞ functions, continuously and densely

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embedded in $\mathcal{S}$, and such that the classical Fourier transform is an isomorphism in that space. Our space $\mathcal{G}$ verifies these requirements and is an inductive limit of finite-dimensional vector spaces. The construction of $\mathcal{G}$ is based on the same idea that led the first author to the construction of the spaces $\mathcal{U}_W$ (see Loura [6]). In the space $\mathcal{G}'$ — the strong dual of $\mathcal{G}$ — we introduce the derivative operators, the product by polynomials and the Fourier transform. We call $\mathcal{G}'$ the space of hermitean ultradistributions.

Our space $\mathcal{G}'$ contains the multipole series used by physicists, and in fact they are the real motivation for ultradistributions. These are series of the type $\sum_{\alpha\in\mathbb{N}^N} a_\alpha \partial^\alpha \delta_{\mathbf{a}}$, where the a_α are scalars verifying some decreasing property and $\delta_{\mathbf{a}}$ is the Dirac distribution at the point $\mathbf{a}\in\mathbb{R}^N$. Observe that such a series is not convergent in $\mathcal{D}'$ unless all but a finite number of a_α are equal to 0.

Other spaces of ultradistributions have been introduced by several authors. Using duality arguments we mention: the Gevrey classes of ultradistributions of Lions and Magenes [5], based on the Gevrey functions [1], the Komatsu [4] theory, based on classes of ultradifferentiable functions, and the duality theory of analytical functions due to Guelfand and Chilov [3]. All these spaces are technically much more difficult, and possibly less general, then ours. Using holomorphic representations Sato [11], [12] introduced the concept of hyperfunction. Using an axiomatic approach, in a similar way presented by Sebastião e Silva [16], [17] for the distributions, we mention Menezes [7] and Luisa Ribeiro [9], [10]. Using extension procedures for the Fourier transform combined with holomorphic representations, we mention Sebastião e Silva [15], [18] and Oliveira [8].

2 – The space $\mathcal{G}$

All the functions considered in this work are complex valued. We denote $\mathcal{P}$, or $\mathcal{P}(\mathbb{R}^N)$ if necessary, the space of all (complex) polynomials defined in $\mathbb{R}^N$ and, for each $m\in\mathbb{N}$, $\mathcal{P}_m$ is the subspace of $\mathcal{P}$ of polynomials of degree $\leq m$. We write

$$\mathcal{G}(\mathbb{R}^N)=\mathcal{G}=\left\{\varphi\in C(\mathbb{R}^N);\ \varphi(x)=p(x)\,e^{-\frac{|x|^2}{2}},\ p\in\mathcal{P}\right\}$$

and, for $m\in\mathbb{N}$,

$$\mathcal{G}_m(\mathbb{R}^N)=\mathcal{G}_m=\left\{\varphi\in\mathcal{G};\ \varphi(x)=p(x)\,e^{-\frac{|x|^2}{2}},\ p\in\mathcal{P}_m\right\}\ .$$

$\mathcal{G}$ is a vector subspace of $C^\infty(\mathbb{R}^N)$, $\mathcal{G}_m$ is an $m+1$-dimensional vector subspace of $\mathcal{G}$, $\mathcal{G}_m\subset\mathcal{G}_{m+1}$ and $\mathcal{G}$ is the union of all the $\mathcal{G}_m$. Let $\varphi\in\mathcal{G}_m$, $\alpha\in\mathbb{N}^N$ and

$q \in \mathcal{P}_k$ $(k \in \mathbb{N})$; then $\partial^\alpha \varphi \in \mathcal{G}_{m+|\alpha|}$ and $q\varphi \in \mathcal{G}_{m+k}$. This shows that $\mathcal{G}$ is a vector subspace of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in $\mathbb{R}^N$.

We introduce in $\mathcal{G}_m$ the norm of the uniform convergence in $\mathbb{R}^N$, $\|\varphi\|_m = \sup_{x \in \mathbb{R}^N} |\varphi(x)|$, but we point out that all the norms in $\mathcal{G}_m$ are equivalent since it is a finite dimensional vector space. Obviously $\mathcal{G}_m \underset{c}{\hookrightarrow} \mathcal{G}_{m+1}$ (we use $\hookrightarrow$ for continuous injection, $\underset{d}{\hookrightarrow}$ for continuous and dense injection and $\underset{c}{\hookrightarrow}$ for continuous and compact injection), $\mathcal{G}_m$ is closed in $\mathcal{G}_{m+1}$ and $\mathcal{G}_m$ has the topology induced by $\mathcal{G}_{m+1}$. For each $\alpha \in \mathbb{N}^N$, the derivative operator ∂^α is linear continuous from $\mathcal{G}_m$ into $\mathcal{G}_{m+|\alpha|}$. For each $q \in \mathcal{P}_k$, the product operator $P_q \colon \mathcal{G}_m \to \mathcal{G}_{m+k}$, defined by $P_q(\varphi) = q\varphi$, is linear continuous.

We introduce in $\mathcal{G}$ the inductive limit topology of the spaces $\mathcal{G}_m$. From the properties of inductive limits (see Grothendieck [2] and Sebastião e Silva [14]) we know that $\mathcal{G}$ is a locally convex complete Hausdorff topological vector space; it is a Montel and a Mackey (bornological) space.

Theorem 1. *Let E be a topological vector space and let $f \colon \mathcal{G} \to E$ be a mapping; then f is continuous iff its restriction to each $\mathcal{G}_m$ is continuous.*

Proof: The inductive limit topology is, in this case (see Sebastião e Silva [14]), the final topology. ∎

Corollary 1. *Let E be a locally convex topological vector space and let $T \colon \mathcal{G} \to E$ be a linear mapping; then T is continuous. In particular, the algebraic dual of $\mathcal{G}$ is equal to the topological dual.*

Corollary 2. *For each $\alpha \in \mathbb{N}^N$ and each $q \in \mathcal{P}$, the derivative operator $\partial^\alpha \colon \mathcal{G} \to \mathcal{G}$ and the product operator $P_q \colon \mathcal{G} \to \mathcal{G}$ are linear continuous.*

Corollary 3. *The injection of $\mathcal{G}$ into $\mathcal{S}$ is continuous.*

Our space $\mathcal{G}$ contains all the Hermite functions; this implies that $\mathcal{G}$ is sequentially dense in $\mathcal{S}$ (see Schwartz [13]). Thus we may write the inclusions

$$\mathcal{G}_m \underset{c}{\hookrightarrow} \mathcal{G}_k \underset{c}{\hookrightarrow} \mathcal{G} \underset{d}{\hookrightarrow} \mathcal{S} \qquad (m \le k) \ .$$

Let $\varphi \in \mathcal{S}$; we recall that the Fourier transform of φ, denoted by $\mathcal{F}\varphi$ or $\widehat{\varphi}$, is defined by (where $x.\xi = x_1\xi_1 + ... + x_N\xi_N$ with $x = (x_1, ..., x_N)$ and $\xi = (\xi_1, ..., \xi_N)$)

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^N} e^{-i\,x.\xi}\, \varphi(x)\, dx \ .$$

Theorem 2. *The Fourier operator $\mathcal{F}\colon \mathcal{G} \to \mathcal{G}$ is a vectorial and topological isomorphism.*

Proof: By Corollary 1 of Theorem 1 we just have to show that $\mathcal{F}(\mathcal{G}) \subset \mathcal{G}$. Let $\varphi \in \mathcal{G}$; then $\varphi(x) = p(x)\, e^{-\frac{|x|^2}{2}}$, with $p \in \mathcal{P}$; from the properties of the Fourier transform in $\mathcal{S}'$ we know that, if $p(x) = \sum_{|\alpha| \le m} c_\alpha\, x^\alpha$, then

$$\begin{aligned}
\mathcal{F}(p\, e^{-\frac{|x|^2}{2}}) &= \mathcal{F}p * \mathcal{F}e^{-\frac{|x|^2}{2}} \\
&= \Big(\sum_{|\alpha| \le m} c_\alpha\, i^{|\alpha|} (2\pi)^N\, \partial^\alpha \delta \Big) * (2\pi)^{\frac{N}{2}}\, e^{-\frac{|x|^2}{2}} \\
&= q(x)\, e^{-\frac{|x|^2}{2}} ,
\end{aligned}$$

where $*$ stands for the convolution and $q \in \mathcal{P}$. ∎

3 – The space $\mathcal{G}'$

We define the space $\mathcal{G}'$ of hermitean ultradistributions as the strong dual of the space $\mathcal{G}$; we denote $\langle T, \varphi \rangle$ (or $\langle T, \varphi \rangle_N$ if necessary) the duality product between $T \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$. From the properties of $\mathcal{G}$ we see that $\mathcal{G}'$ is a locally convex complete Hausdorff topological space; it is also a Montel space. From $\mathcal{G} \underset{d}{\hookrightarrow} \mathcal{S}$ we see that $\mathcal{S}' \hookrightarrow \mathcal{G}'$ and that $\langle T, \varphi \rangle = \langle T, \varphi \rangle_{\mathcal{S}',\mathcal{S}}$ for all $T \in \mathcal{S}'$ and all $\varphi \in \mathcal{G}$.

Theorem 3. *The inclusion $\mathcal{G} \underset{d}{\hookrightarrow} \mathcal{G}'$ is verified.*

Proof: To prove that $\mathcal{G}$ is dense in $\mathcal{G}'$ we have to show that every $T \in \mathcal{G}''$ that is null on $\mathcal{G}$ is identically null. Let $T \in \mathcal{G}''$; the space $\mathcal{G}$ is reflexive, because it is a Montel space, so $T \in \mathcal{G}$, that is $T = p\, e^{-\frac{|x|^2}{2}}$ with $p \in \mathcal{P}$; now, if T is null on $\mathcal{G}$, we have $\int_{\mathbb{R}^N} T\, \varphi = 0$ for all $\varphi \in \mathcal{G}$ and this implies $T = 0$.

Theorem 3 shows that we can write the following inclusions:

$$\mathcal{G} \underset{d}{\hookrightarrow} \mathcal{S} \underset{d}{\hookrightarrow} \mathcal{E} \quad \text{and} \quad \mathcal{E}' \underset{d}{\hookrightarrow} \mathcal{S}' \underset{d}{\hookrightarrow} \mathcal{G}' ,$$

where $\mathcal{E}$ is the vector space $C^\infty(\mathbb{R}^N)$ equipped with the topology of uniform convergence of the function and each one of its derivatives in the compact subsets of $\mathbb{R}^N$. We see that our space of hermitean ultradistributions generalizes quite naturally the space of tempered distributions. We remark that the distribution space $\mathcal{D}'$ and our space $\mathcal{G}'$ are not included in one another.

4 – Derivatives, products by polynomials, Fourier transform and tensorial product

Let $\alpha \in \mathbb{N}^N$; we define the derivative operator $\partial^\alpha : \mathcal{G}' \to \mathcal{G}'$ by $\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$, $T \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$. The operator ∂^α is linear continuous and generalizes the usual derivative operator in $\mathcal{S}'$.

Let $p \in \mathcal{P}$; we define the product operator $T \to pT$ from $\mathcal{G}'$ into $\mathcal{G}'$ by $\langle pT, \varphi \rangle = \langle T, p\varphi \rangle$. It is a linear continuous operator that generalizes the usual product by polynomials in $\mathcal{S}'$. The Leibniz formulas for the derivatives of the product remain valid:

$$\partial^\alpha (pT) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta p \, \partial^{\alpha-\beta} T \; ;$$

$$p \, \partial^\alpha T = \sum_{\beta \le \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} ((\partial^\beta p) \, T) \; .$$

Next we define the tensorial product; a point $x \in \mathbb{R}^{N+M}$ is written in the form (x_1, x_2), with $x_1 \in \mathbb{R}^N$ and $x_2 \in \mathbb{R}^M$. Let $\varphi \in \mathcal{G}(\mathbb{R}^{N+M})$; then there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} \varphi(x_1, x_2) &= \sum_{\substack{\alpha \in \mathbb{N}^{N+M} \\ |\alpha| \le k}} a_\alpha \, x_1^{\alpha_1} \, x_2^{\alpha_2} \, e^{-\frac{|x_1|^2}{2}} \, e^{-\frac{|x_2|^2}{2}} \\ &= \sum_{\substack{\alpha \in \mathbb{N}^{N+M} \\ |\alpha| \le k}} a_\alpha \, \varphi_{\alpha,1}(x_1) \, \varphi_{\alpha,2}(x_2) \; , \end{aligned} \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2)$, and $\varphi_{\alpha,1} \in \mathcal{G}(\mathbb{R}^N)$ and $\varphi_{\alpha,2} \in \mathcal{G}(\mathbb{R}^M)$ are defined by

$$\varphi_{\alpha,1}(x_1) = x_1^{\alpha_1} \, e^{-\frac{|x_1|^2}{2}} \; , \tag{2}$$

$$\varphi_{\alpha,2}(x_2) = x_2^{\alpha_2} \, e^{-\frac{|x_2|^2}{2}} \; . \tag{3}$$

The function φ uniquely determine the functions $\varphi_{\alpha,1}$ and $\varphi_{\alpha,2}$.

Let $T \in \mathcal{G}'(\mathbb{R}^N)$ and $S \in \mathcal{G}'(\mathbb{R}^M)$; we define T tensorial S as the element $T \otimes S$ of $\mathcal{G}'(\mathbb{R}^{N+M})$ given by

$$\langle T \otimes S, \, \varphi \rangle_{N+M} = \sum_{\substack{\alpha \in \mathbb{N}^{N+M} \\ |\alpha| \le k}} a_\alpha \, \langle T, \varphi_{\alpha,1} \rangle_N \, \langle S, \varphi_{\alpha,2} \rangle_M \; ,$$

where $\varphi \in \mathcal{G}(\mathbb{R}^{N+M})$, $\varphi_{\alpha,1} \in \mathcal{G}(\mathbb{R}^N)$ and $\varphi_{\alpha,2} \in \mathcal{G}(\mathbb{R}^M)$ are given respectively by (1), (2) and (3). The mapping $(T,S) \to T \otimes S$ is bilinear continuous from $\mathcal{G}'(\mathbb{R}^N) \times \mathcal{G}'(\mathbb{R}^M)$ into $\mathcal{G}'(\mathbb{R}^{N+M})$ and has the following properties:

$$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} T \otimes S = \partial_{x_1}^{\alpha_1} T \otimes \partial_{x_2}^{\alpha_2} S \ ;$$

$$\langle T \otimes S, \varphi \rangle_{N+M} = \Big\langle T, \langle S, \varphi(x_1, \cdot) \rangle_M \Big\rangle_N = \Big\langle S, \langle T, \varphi(\cdot, x_2) \rangle_N \Big\rangle_M \ ;$$

$$(p \otimes q)\,(T \otimes S) = (pT) \otimes (qS) \qquad (p \in \mathcal{P}(\mathbb{R}^N), \quad q \in \mathcal{P}(\mathbb{R}^M)) \ .$$

We define the Fourier transform $\mathcal{F}\colon \mathcal{G}' \to \mathcal{G}'$ by $\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle$, $T \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$. The operator $\mathcal{F}$ is a vectorial and topological isomorphism in $\mathcal{G}'$ and generalizes the usual Fourier transform in $\mathcal{S}'$. For $\alpha \in \mathbb{N}^N$ we have:

$$\mathcal{F}\partial^\alpha T = (i\xi)^\alpha\, \mathcal{F}T \ ;$$

$$\partial^\alpha \mathcal{F}T = \mathcal{F}(-ix)^\alpha T \ ;$$

$$\mathcal{F}(T \otimes S) = (\mathcal{F}T) \otimes (\mathcal{F}S) \ .$$

5 – Sequential representation

We present in this section a useful sequential representation of our hermitean ultradistributions. For notational simplicity we take only the one dimensional case, that is $N = 1$.

For each $m \in \mathbb{C}$ we define the mapping $\widetilde{h}_m\colon \mathcal{G}_m \to \mathbb{C}^{m+1}$ by

$$\widetilde{h}_m(\varphi) = (a_0, ..., a_m) \ ,$$

where $\varphi \in \mathcal{G}_m$, $\varphi(x) = \sum_{j=0}^m a_j\, x^j\, e^{-\frac{x^2}{2}}$. We see that $\widetilde{h}_m$ is a vectorial and topological isomorphism, and the same is true for the transposed mapping ${}^{\mathrm{t}}\widetilde{h}_m : (\mathbb{C}^{m+1})' \to \mathcal{G}'_m$. If we call $\Phi_m : \mathbb{C}^{m+1} \to (\mathbb{C}^{m+1})'$ the canonical anti-isomorphism between $\mathbb{C}^{m+1}$ and its dual, and if we write $h_m = {}^{\mathrm{t}}\widetilde{h}_m \circ \Phi_m$, we see that $h_m\colon \mathbb{C}^{m+1} \to \mathcal{G}'_m$ is a vectorial and topological anti-isomorphism. Thus, for each $T \in \mathcal{G}'_m$, there exists one and only one $(b_0, ..., b_m) \in \mathbb{C}^{m+1}$ such that

$$\Big\langle T, \sum_{j=0}^m a_j\, x^j\, e^{-\frac{x^2}{2}} \Big\rangle = \sum_{j=0}^m a_j\, \overline{b_j}$$

for all $(a_0, ..., a_m) \in \mathbb{C}^{m+1}$. Obviously $T_n \to 0$ in $\mathcal{G}'_m$ iff $h_m^{-1} T_n \to 0$ in $\mathbb{C}^{m+1}$.

We denote $\mathbb{C}^\infty$ the vector space of all sequences of complex numbers equipped with the topology of pointwise convergence. We define $h\colon \mathbb{C}^\infty \to \mathcal{G}'$ by

$$\langle h(b_n)_{n\in\mathbb{N}}, \varphi\rangle = \sum_{j=0}^{m} a_j\, \overline{b_j} \ , \tag{4}$$

where $\varphi(x) = \sum_{j=0}^{m} a_j\, x^j\, e^{-\frac{x^2}{2}}$.

Theorem 4. *The mapping $h\colon \mathbb{C}^\infty \to \mathcal{G}'$ defined by (4) is a vectorial and topological anti-isomorphism.*

Proof: As $\mathcal{G}$ is the inductive limit of the sequence of finite dimensional spaces $(\mathcal{G}_m)_{m\in\mathbb{N}}$, $\mathcal{G}'$ is vectorial and topologically isomorphic with the projective limit $\varprojlim(\mathcal{G}'_m, \rho_m)_{m\in\mathbb{N}}$ where $\rho_m\colon \mathcal{G}'_{m+1} \to \mathcal{G}'_m$ is the restriction operator (for $T \in \mathcal{G}'_{m+1}$, $\rho_m T$ is the restriction of T to $\mathcal{G}_m$). If, for $n \geq 1$, we define the mapping u_m : $\mathbb{C}^{m+1} \to \mathbb{C}^m$ by $u_m = h^{-1}_{m-1} \circ \rho_{m-1} \circ h_m$, we see that $u_m(b_0, ..., b_m) = (b_0, ..., b_{m-1})$. Consequently $\mathcal{G}'$ is vectorial and topologically isomorphic with the projective limit $\varprojlim(\mathbb{C}^m, u_m)_{m\in\mathbb{N}}$ that is canonically identified with $\mathbb{C}^\infty$.

In $\mathbb{C}^\infty$ we introduce the following product:

$$(\alpha_n)_{n\in\mathbb{N}} . (\beta_n)_{n\in\mathbb{N}} = (\alpha_n \beta_n)_{n\in\mathbb{N}} \ .$$

In this way $\mathbb{C}^\infty$ becomes a complex commutative algebra with unity; the product is continuous. Given an operator $S\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ and a sequence $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{C}^\infty$ we define the operator $(\alpha_n)_{n\in\mathbb{N}}.S$ in the usual way, that is $((\alpha_n)_{n\in\mathbb{N}}.S)(\beta_n)_{n\in\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}.(S(\beta_n)_{n\in\mathbb{N}})$. The sum of two operators and the product by scalars are standard.

For $k \in \mathbb{N}$ we define the operators $S_k\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ and $S_{-k}\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ by

$$S_0 = \text{identity} \ ;$$

$$S_1(\alpha_n)_{n\in\mathbb{N}} = (0, \alpha_0, ..., \alpha_n, ...) \ , \qquad S_{-1}(\alpha_n)_{n\in\mathbb{N}} = (\alpha_{n+1})_{n\in\mathbb{N}} \ ;$$

$$S_{k+1} = S_1\, S_k, \quad S_{-(k+1)} = S_{-1}\, S_{-k} \qquad (\text{for } k > 1) \ .$$

The formulas

$$S_{-k}\Big[(\alpha_n)_{n\in\mathbb{N}}.S_r\Big] = (\alpha_{n+k})_{n\in\mathbb{N}}.S_{r-k} \ , \tag{5}$$

$$S_k\Big[(\alpha_n)_{n\in\mathbb{N}}.S_m\Big] = (\alpha_{n-k})_{n\in\mathbb{N}}.S_{k+m} \ , \tag{6}$$

valid for all $k, m \in \mathbb{N}$ and $r \in \mathbb{Z}$, are easily verified. In formula (6) we adopt the convention that $\alpha_{n-k} = 0$ whenever $n - k < 0$.

The following result is a direct consequence of Theorem 4:

Corollary 1. *Let* $T \in \mathcal{G}'$; *then* $\overline{h^{-1}(T)} = (\langle T, x^n\, e^{-\frac{x^2}{2}}\rangle)_{n\in\mathbb{N}}$. *In particular, for all* $f \in L^1_{\mathrm{loc}} \cap \mathcal{S}'$,

$$h^{-1}(f) = \Big(\int_{\mathbb{R}} \overline{f(x)}\, x^n\, e^{-\frac{x^2}{2}}\, dx\Big)_{n\in\mathbb{N}} \ .$$

Examples: From $\langle \delta, e^{-\frac{x^2}{2}}\rangle = 1$ and $\langle \delta, x^n\, e^{-\frac{x^2}{2}}\rangle = 0$ $(n \in \mathbb{N}\backslash\{0\})$ we see that

$$h^{-1}(\delta) = (1,0,0,...,0,...) \ . \tag{7}$$

From $\langle \delta_1, x^n e^{-\frac{x^2}{2}}\rangle = \frac{1}{\sqrt{e}}$ $(n \in \mathbb{N})$, where δ_1 is the Dirac distribution at the point 1, we see that

$$h(1,1,...,1,...) = \sqrt{e}\,\delta_1 \ ;$$

$$h(a,a,...,a,...) = \overline{a}\,\sqrt{e}\,\delta_1 \quad \text{for all} \ \ a \in \mathbb{C} \ .$$

From $\langle \delta_{-1}, x^n e^{-\frac{x^2}{2}}\rangle = \frac{(-1)^n}{\sqrt{e}}$ $(n \in \mathbb{N})$, where δ_{-1} is the Dirac distribution at the point -1, we see that

$$h((-1)^n)_{n\in\mathbb{N}} = \sqrt{e}\,\delta_{-1} \ ;$$

$$h(a(-1)^n)_{n\in\mathbb{N}} = \overline{a}\,\sqrt{e}\,\delta_{-1} \quad \text{for all} \ \ a \in \mathbb{C} \ .$$

More generally, for $r \in \mathbb{R}$, we have $\langle \delta_r, x^n e^{-\frac{x^2}{2}}\rangle = e^{-\frac{r^2}{2}}\, r^n$ $(n \in \mathbb{N})$, where δ_r is the Dirac distribution at the point r; then

$$h(a\,r^n)_{n\in\mathbb{N}} = \overline{a}\, e^{\frac{r^2}{2}}\,\delta_r \quad \text{for all} \ \ a \in \mathbb{C} \ .$$

For $k \in \mathbb{N}$, $k > 0$, the symbol $k!!$ means the product of all odd positive integers less than or equal to k; we also adopt the convention $0!! = 1$. We introduce once and for all the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defined by

$$\gamma_0 = \sqrt{2\pi}\,, \qquad \gamma_n = (n-1)!!\ \sqrt{2\pi}\ \frac{1+(-1)^n}{2} \quad (\text{for } n \in \mathbb{N}\backslash\{0\}) \ ,$$

that is $(\sqrt{2\pi},\ 0,\ \sqrt{2\pi},\ 0,\ 3\sqrt{2\pi},\ 0,\ 15\sqrt{2\pi},\ 0,\ 105\sqrt{2\pi},\ ...)$.

Corollary 2. *We have* $h^{-1}(1) = (\gamma_n)_{n\in\mathbb{N}}$.

Proof: From $\int_{\mathbb{R}}(x^k e^{-\frac{x^2}{2}})'\,dx = 0$ we see that, for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} x^{k+2}\, e^{-\frac{x^2}{2}}\, dx = (k+1) \int_{\mathbb{R}} x^k\, e^{-\frac{x^2}{2}}\, dx \ ;$$

since $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$, this implies $\int_{\mathbb{R}} x^{2k}\, e^{-\frac{x^2}{2}}\, dx = (2k-1)!!\,\sqrt{2\pi}$. Obviously $\int_{\mathbb{R}} x^{2k+1}\, e^{-\frac{x^2}{2}}\, dx = 0$. ∎

Theorem 5. *Let $T \in \mathcal{G}'$; then $h^{-1}(xT) = S_{-1} h^{-1}(T)$. We also have:*

$$h^{-1}\Big(\sum_{k=0}^{m} c_k\, x^k\Big) = \sum_{k=0}^{m} \overline{c_k} (\gamma_{n+k})_{n\in\mathbb{N}} \ ; \tag{8}$$

$$h^{-1}\Big(\sum_{k=0}^{m} c_k\, x^k\, T\Big) = \sum_{k=0}^{m} \overline{c_k}\, S_{-k}\, h^{-1}(T) \ , \tag{9}$$

where $c_k \in \mathbb{C}$ for $k = 0, ..., m$.

Proof: Let $(\alpha_n)_{n\in\mathbb{N}} = h^{-1}(xT)$ and $(\beta_n)_{n\in\mathbb{N}} = h^{-1}(T)$; we have

$$\overline{\alpha_n} = \langle xT,\, x^n\, e^{-\frac{x^2}{2}}\rangle = \langle T,\, x^{n+1}\, e^{-\frac{x^2}{2}}\rangle = \overline{\beta_{n+1}} \ ,$$

thus $(\alpha_n)_{n\in\mathbb{N}} = S_{-1}(\beta_n)_{n\in\mathbb{N}}$. This equality and the definition of S_{-k} imply (9). Equality (8) is a consequence of (9) and Corollary 2 of Theorem 4. ∎

Theorem 6. *Let $T \in \mathcal{G}'$; then $h^{-1}(\partial T) = [S_{-1} - (n)_{n\in\mathbb{N}}.S_1]\, h^{-1}(T)$.*

Proof: Write $(\alpha_n)_{n\in\mathbb{N}} = h^{-1}(\partial T)$ and $(\beta_n)_{n\in\mathbb{N}} = h^{-1}(T)$; then, for $n \in \mathbb{N}\backslash\{0\}$, we have

$$\overline{\alpha_n} = \langle \partial T,\, x^n\, e^{-\frac{x^2}{2}}\rangle = \langle T,\, x^{n+1}\, e^{-\frac{x^2}{2}}\rangle - \langle T,\, n x^{n-1}\, e^{-\frac{x^2}{2}}\rangle = \overline{\beta_{n+1}} - n\,\overline{\beta_{n-1}} \ .$$

For $n = 0$ we have $\overline{\alpha_0} = \overline{\beta_1}$ and this implies that

$$\alpha_0 = \beta_1\,, \qquad \alpha_n = \beta_{n+1} - n\,\beta_{n-1} \ \ \forall\, n \in \mathbb{N}\backslash\{0\} \ . \tag{10}$$

Corollary 1. *Let $T \in \mathcal{G}'$ and $m \in \mathbb{N}\backslash\{0\}$. Then $\partial^m T = 0$ iff $T \in \mathcal{P}_{m-1}$.*

Proof: We have to show that, if $\partial T = 0$, T is a constant. Write $(\beta_n)_{n\in\mathbb{N}} = h^{-1}(T)$; then

$$\Big[S_{-1} - (n)_{n\in\mathbb{N}}.S_1\Big](\beta_n)_{n\in\mathbb{N}} = (0, 0, ..., 0, ...)$$

and this implies $(\beta_n)_{n\in\mathbb{N}} = \frac{\beta_0}{\sqrt{2\pi}}(\gamma_n)_{n\in\mathbb{N}}$, that is $T = \frac{\overline{\beta_0}}{\sqrt{2\pi}}$.

Corollary 2. *Every element $T \in \mathcal{G}'$ is primitivable and any two primitives of T differ by a constant.*

Proof: Write $(\beta_n)_{n\in\mathbb{N}} = h^{-1}(T)$, let α_0 be an arbitrary complex number, and define $(\alpha_n)_{n\in\mathbb{N}}$ by $\alpha_1 = \beta_0$ and $\alpha_{n+1} = \beta_n + n\,\alpha_{n-1}$ for all $n \in \mathbb{N}\backslash\{0\}$. By (10) we see that $\partial[h(\alpha_n)_{n\in\mathbb{N}}] = T$. ∎

Theorem 7. *Let $k \in \mathbb{N}$; then $h^{-1}(\delta^{(k)}) = ({}^k\alpha_n)_{n\in\mathbb{N}}$ where ${}^k\alpha_n$ is defined by*

$$
{}^k\alpha_n = \begin{cases} 0 & \text{for } n > k \ \text{ or } k-n \text{ odd,} \\ \dfrac{(-1)^k + (-1)^n}{2}\,(-1)^{\frac{k-n}{2}}\,\dfrac{k!}{\left(\frac{k-n}{2}\right)!\;2^{\frac{k-n}{2}}} & \text{for } n \le k \ \text{ and } k-n \text{ even .} \end{cases}
$$

Proof: The theorem is proved by induction on k. Equality $h^{-1}(\delta) = ({}^0\alpha_n)_{n\in\mathbb{N}}$ is a direct consequence of (7) and the definition of ${}^0\alpha_n$.

Assume the result true for k and let us prove it for $k+1$. From (10) we see that we have to prove the following equalities:

$$
{}^{k+1}\alpha_0 = {}^k\alpha_1 \;; \tag{11}
$$

$$
{}^{k+1}\alpha_n = {}^k\alpha_{n+1} - n\,{}^k\alpha_{n-1} \qquad \forall\, n \in \mathbb{N}\backslash\{0\} \;. \tag{12}
$$

We write ${}^{k+1}\alpha_n$

$$
{}^{k+1}\alpha_n = \begin{cases} 0 & \text{for } n > k+1 \\ & \quad \text{or } k{-}n{+}1 \text{ odd,} \\ \dfrac{(-1)^{k+1} + (-1)^n}{2}\,(-1)^{\frac{k-n+1}{2}}\,\dfrac{(k+1)!}{\left(\frac{k-n+1}{2}\right)!\;2^{\frac{k-n+1}{2}}} & \text{for } n \le k+1 \\ & \quad \text{and } k{-}n{+}1 \text{ even,} \end{cases} \tag{13}
$$

and we first prove equality (11). If $k=0$, then ${}^1\alpha_0 = 0 = {}^0\alpha_1$. If $k>0$ and k is even, then ${}^{k+1}\alpha_0 = 0 = {}^k\alpha_1$. If $k>0$ and k is odd, then

$$
\begin{aligned}
{}^{k+1}\alpha_0 &= \frac{(-1)^{k+1}+1}{2}\,(-1)^{\frac{k+1}{2}}\,\frac{(k+1)!}{\left(\frac{k+1}{2}\right)!\;2^{\frac{k+1}{2}}} \\
&= -\frac{(-1)^k - 1}{2}\,(-1)^{\frac{k-1}{2}+1}\,\frac{(k+1)\,k!}{\frac{k+1}{2}\left(\frac{k-1}{2}\right)!\;2\;2^{\frac{k-1}{2}}} \\
&= \frac{(-1)^k - 1}{2}\,(-1)^{\frac{k-1}{2}}\,\frac{k!}{\left(\frac{k-1}{2}\right)!\;2^{\frac{k-1}{2}}} = {}^k\alpha_1 \;.
\end{aligned}
$$

Now we prove (12). If $n > k+1$ or $k-n+1$ is odd, then ${}^k\alpha_{n+1} = {}^k\alpha_{n-1} = {}^{k+1}\alpha_n = 0$, and (12) is verified. If $n = k+1$ then ${}^k\alpha_{n+1} = 0$ and

$$-(k+1)\,{}^k\alpha_k = -\frac{(-1)^k + (-1)^k}{2}\,(-1)^0\,\frac{(k+1)\,k!}{0!\;2^0} = (-1)^{k+1}\,(k+1)! = {}^{k+1}\alpha_{k+1}\;.$$

Finally, if $n < k$ and $k-n+1$ is even, then

$$\begin{aligned}
{}^k\alpha_{n+1} - n\,{}^k\alpha_{n-1} &= \frac{(-1)^k + (-1)^{n+1}}{2}\,(-1)^{\frac{k-n-1}{2}}\,\frac{k!}{\left(\frac{k-n-1}{2}\right)!\;2^{\frac{k-n-1}{2}}} - \\
&\quad - \frac{(-1)^k + (-1)^{n-1}}{2}\,(-1)^{\frac{k-n+1}{2}}\,\frac{n\,k!}{\left(\frac{k-n+1}{2}\right)!\;2^{\frac{k-n+1}{2}}} = \\
&= -\frac{(-1)^k + (-1)^{n+1}}{2}\,(-1)^{\frac{k-n+1}{2}}\left(\frac{k!}{\left(\frac{k-n-1}{2}\right)!\;2^{\frac{k-n-1}{2}}} + \frac{n\,k!}{\left(\frac{k-n+1}{2}\right)!\;2^{\frac{k-n+1}{2}}}\right) \\
&= \frac{(-1)^{k+1} + (-1)^n}{2}\,(-1)^{\frac{k-n+1}{2}}\,\frac{k!\left(\frac{k-n+1}{2}\right)2 + n\,k!}{\left(\frac{k-n+1}{2}\right)!\;2^{\frac{k-n+1}{2}}} \\
&= \frac{(-1)^{k+1} + (-1)^n}{2}\,(-1)^{\frac{k-n+1}{2}}\,\frac{(k+1)!}{\left(\frac{k-n+1}{2}\right)!\;2^{\frac{k-n+1}{2}}} \\
&= {}^{k+1}\alpha_n\;.
\end{aligned}$$

In Table 1 we present Theorem 7 for the particular cases $k = 0,1,2,3,4,5,6,7$.

	k	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$h^{-1}(\delta)$	0	$\frac{0!}{0!\,2^0}$	0	0	0	0	0	0	0	0
$h^{-1}(\delta')$	1	0	$\frac{-1!}{0!\,2^0}$	0	0	0	0	0	0	0
$h^{-1}(\delta'')$	2	$\frac{-2!}{1!\,2^1}$	0	$\frac{2!}{0!\,2^0}$	0	0	0	0	0	0
$h^{-1}(\delta''')$	3	0	$\frac{3!}{1!\,2^1}$	0	$\frac{-3!}{0!\,2^0}$	0	0	0	0	0
$h^{-1}(\partial^{(4)}\delta)$	4	$\frac{4!}{2!\,2^2}$	0	$\frac{-4!}{1!\,2^1}$	0	$\frac{4!}{0!\,2^0}$	0	0	0	0
$h^{-1}(\partial^{(5)}\delta)$	5	0	$\frac{-5!}{2!\,2^2}$	0	$\frac{5!}{1!\,2^1}$	0	$\frac{-5!}{0!\,2^0}$	0	0	0
$h^{-1}(\partial^{(6)}\delta)$	6	$\frac{-6!}{3!\,2^3}$	0	$\frac{6!}{2!\,2^2}$	0	$\frac{-6!}{1!\,2^1}$	0	$\frac{6!}{0!\,2^0}$	0	0
$h^{-1}(\partial^{(7)}\delta)$	7	0	$\frac{7!}{3!\,2^3}$	0	$\frac{-7!}{2!\,2^2}$	0	$\frac{7!}{1!\,2^1}$	0	$\frac{-7!}{0!\,2^0}$	0

Table 1

Corollary 1. *Let $(\tau_n)_{n\in\mathbb{N}} \in \mathbb{C}^\infty$; then the following statements are equivalent:*

(**i**) $\tau_n = 0$ *for all* $n > k$;

(**ii**) $h(\tau_n)_{n\in\mathbb{N}}$ *is of the form* $\sum_{j=0}^{k} c_j\, \delta^{(j)}$ *where* $c_j \in \mathbb{C}$ *for* $j = 0, ..., k$.

Proof: (i) is a direct consequence of (ii). To prove that (i) implies (ii) we observe that the square matrix $(\beta_{mn})_{0\le m,n\le k}$, where $\beta_{mn} = {}^n\alpha_m$, is upper triangular; this implies that the linear system

$$[\beta_m] \begin{bmatrix} x_0 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} \tau_0 \\ \vdots \\ \tau_k \end{bmatrix}$$

has one and only one solution.

Corollary 2. *The vector subspace of $\mathcal{G}'$ spanned by $\{\delta^{(k)}\}_{k\in\mathbb{N}}$ is dense in $\mathcal{G}'$.*

Proof: Let $T \in \mathcal{G}'$ and write $(\alpha_n)_{n\in\mathbb{N}} = h^{-1}(T)$. As the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is the limit in $\mathbb{C}^\infty$ of the sequence $((\alpha_0, ..., \alpha_n, 0, 0, ...))_{n\in\mathbb{N}}$, the result follows from Corollary 1. ∎

Corollary 3. *Let $k \in \mathbb{N}\backslash\{0\}$ and $T \in \mathcal{G}'$. Then $x^k T = 0$ iff $T = \sum_{j=0}^{k-1} c_j\, \delta^{(j)}$ with $c_j \in \mathbb{C}$ for $j = 0, ..., k-1$.*

Proof: Write $(\alpha_n)_{n\in\mathbb{N}} = h^{-1}(T)$; then $x^k T = 0$ iff $h^{-1}(x^k T) = (0)_{n\in\mathbb{N}}$, or otherwise iff $S_{-k}(x_n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$. But this is equivalent to $\alpha_{n+k} = 0$ for all $n \in \mathbb{N}$ and the result follows from Corollary 1. ∎

Corollary 4. *Let $k \in \mathbb{N}\backslash\{0\}$ and $S \in \mathcal{G}'$. The equation $x^k T = S$ in $\mathcal{G}'$ has always solutions and two of its solutions differ by a distribution of the type $\sum_{j=0}^{k-1} c_j\, \delta^{(j)}$ with $c_j \in \mathbb{C}$ for $j = 0, ..., k-1$.*

Examples: The pseudo-function $\mathrm{Pf}\,\frac{1}{x}$ is an element of $\mathcal{S}'$, consequently belongs to $\mathcal{G}'$. We have

$$\langle \mathrm{Pf}\,\frac{1}{x}, e^{-\frac{x^2}{2}} \rangle = \text{p.v.} \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2}}}{x}\, dx = 0 \;,$$

$$\langle \mathrm{Pf}\,\frac{1}{x}, x^n\, e^{-\frac{x^2}{2}} \rangle = \text{p.v.} \int_{\mathbb{R}} \frac{x^n\, e^{-\frac{x^2}{2}}}{x}\, dx = \gamma_{n-1} \qquad \forall n \in \mathbb{N}\backslash\{0\} \;,$$

where p.v. means the principal value of the integral. From these equalities we conclude that $h^{-1}(\mathrm{Pf}\,\frac{1}{x}) = S_1(\gamma_n)_{n\in\mathbb{N}}$.

Some remarks are needed. In Theorem 5 we saw that the multiplication by x in $\mathcal{G}'$ corresponds to the operator S_{-1} in $\mathbb{C}^\infty$. Nevertheless the operator S_1 and the product . in $\mathbb{C}^\infty$ do not have an easy representation in $\mathcal{G}'$.

The action of S_1 on $h^{-1}(T) = (\alpha_n)_{n\in\mathbb{N}}$ corresponds to the choice of a solution in equation $xR = T$, exactly that solution R_0 such that $h^{-1}(R_0) = (0, \alpha_0, \alpha_1, ..., \alpha_n, ...)$. For $T \in \mathcal{G}'$ we could be tempted to define the product $\operatorname{Pf}\frac{1}{x} \times T$ as the element of $\mathcal{G}'$ represented by $S_1(\alpha_n)_{n\in\mathbb{N}}$, that is

$$\operatorname{Pf}\frac{1}{x} \times T = h(S_1(\alpha_n)_{n\in\mathbb{N}}), \qquad \text{where } \ (\alpha_n)_{n\in\mathbb{N}} = h^{-1}(T)\ .$$

This product does not have good properties because

$$\begin{aligned} h^{-1}\Big(\operatorname{Pf}\frac{1}{x} \times 1\Big) &= S_1(\gamma_n)_{n\in\mathbb{N}}\ , \\ h^{-1}\Big(\operatorname{Pf}\frac{1}{x} \times x\Big) &= S_1(h^{-1}(x)) + \gamma_0\, h^{-1}(\delta)\ . \end{aligned}$$

Still more difficult is the interpretation of the product . in $\mathbb{C}^\infty$, but we think that these difficulties are related with the simplicity we can solve in $\mathbb{C}^\infty$ some of the non trivial problems in $\mathcal{G}'$.

Theorem 6 gives an immediate interpretation in $\mathcal{G}'$ of the mixed operator $(n)_{n\in\mathbb{N}}.S_1$: it is the operator $-(\partial - x)$. This implies that the equation $(\partial - x)T = 0$ in $\mathcal{G}'$ has only the trivial solution $T = 0$. As the function $e^{\frac{x^2}{2}}$ is a classical solution of that equation we conclude that we cannot identify, in a natural way, $e^{\frac{x^2}{2}}$ with an element of $\mathcal{G}'$.

Theorem 8. *The series $\sum_{k=0}^\infty c_k\, \delta^{(k)}$ converges in $\mathcal{G}'$ iff, for each $n \in \mathbb{N}$, the series*

$$\sum_{k=0}^\infty (-1)^{n+k}\ \frac{(n+2k)!}{k!\, 2^k}\ \overline{c_{n+2k}} \tag{14}$$

is convergent in $\mathbb{C}$.

Proof: The series $\sum_{k=0}^\infty c_k\, \delta^{(k)}$ is convergent in $\mathcal{G}'$ iff, for each $n \in \mathbb{N}$, the n-coordinate of the series $\sum_{k=0}^\infty \overline{c_k}\, h^{-1}(\delta^{(k)})$ is a convergent sequence of complex numbers. But Theorem 7 shows that the n-coordinate of the previous series is the following series of complex numbers:

$$\sum_{\substack{k=n \\ k-n \text{ even}}}^\infty \overline{c_k} \left[\frac{(-1)^k + (-1)^n}{2}\, (-1)^{\frac{k-n}{2}}\ \frac{k!}{\left(\frac{k-n}{2}\right)!\ 2^{\frac{k-n}{2}}}\right]\ . \tag{15}$$

Writing $k-n=2m$ and using $\frac{(-1)^k+(-1)^n}{2} = (-1)^k\,\frac{1+(-1)^{k-n}}{2}$ we see that the series (15) coincides with the series (14).

Corollary 1. *If, for some $\varepsilon>0$, the sequence $((2+\varepsilon)^N\,N!\,|c_k|)_{k\in\mathbb{N}}$ is bounded (where $N=\frac{k}{2}$ if k is even and $N=\frac{k-1}{2}$ if k is odd), then the series $\sum_{k=0}^{\infty} c_k\,\delta^{(k)}$ is convergent in $\mathcal{G}'$.*

Proof: For $k\in\mathbb{N}$ we have $|c_k|\le \frac{A}{(2+\varepsilon)^N\,N!}$ with $A\in[0,+\infty[$. For n even, and writing $n=2m$ ($m\in\mathbb{N}$), it is enough to prove the convergency of the series

$$\sum_{k=0}^{\infty} \frac{(2m+2k)!}{k!\,2^k}\,|c_{2m+2k}|$$

for which we apply D'Alembert criteria to the series

$$\sum_{k=0}^{\infty} \frac{(2m+2k)!}{k!\,2^k}\,\frac{A}{(2+\varepsilon)^{m+k}\,(m+k)!}\ .$$

For n odd the proof is similar. ∎

Corollary 2. *If the sequence $(\sqrt[k]{k!\,|c_k|})_{k\in\mathbb{N}}$ is bounded, then the series $\sum_{k=0}^{\infty} c_k\,\delta^{(k)}$ is convergent in $\mathcal{G}'$.*

Corollary 2 shows that all the multipole series that are convergent in the sense of Sebastião e Silva and Silva Oliveira (see [8] and [18]) are also convergent in $\mathcal{G}'$.

Corollary 3. *For $a\in\mathbb{R}$ we have $\sum_{k=0}^{\infty} \frac{a^k}{k!}\,\delta^{(k)} = \delta_{-a}$.*

Proof: We know that

$$h^{-1}\Big(\sum_{k=0}^{\infty} \frac{a^k}{k!}\,\delta^{(k)}\Big) = \Big((-1)^n\,a^n \sum_{k=0}^{\infty} \frac{(-1)^k\,a^{2k}}{k!\,2^k}\Big)_{n\in\mathbb{N}}\ .$$

For $n\in\mathbb{N}$ we have:

$$\langle \delta_{-a},\, e^{-\frac{x^2}{2}}\rangle = e^{-\frac{a^2}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k\,a^{2k}}{k!\,2^k} \qquad \text{if}\ \ n=0\ ;$$

$$\langle \delta_{-a},\, x^n\,e^{-\frac{x^2}{2}}\rangle = (-1)^n\,a^n\,e^{-\frac{a^2}{2}} = (-1)^n\,a^n \sum_{k=0}^{\infty} \frac{(-1)^k\,a^{2k}}{k!\,2^k} \qquad \text{if}\ \ n>0\ .$$

The last corollary suggests the definition of complex translations of δ in the following way: for $z\in\mathbb{C}$, we define δ_z, usually denoted $\delta(x-z)$, by

$$\delta_z = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\,\delta^{(k)}\ . \tag{16}$$

For $n \in \mathbb{N}$, we shall denote by $P_n(x)$ the n^{th}-Hermite polynomial defined by

$$\partial^n e^{-\frac{x^2}{2}} = (-1)^n \, P_n(x) \, e^{-\frac{x^2}{2}} \; .$$

For $x \in \mathbb{R}$ we have

$$\begin{cases} P_0(x) = 1, \\ P_n(x) = \displaystyle\sum_{j=0}^{N} {}^n\theta_j \, x^{n-2j} \qquad \text{for} \;\; n > 0 \; , \end{cases}$$

where $N = \frac{n}{2}$ if n is even, $N = \frac{n-1}{2}$ if n is odd, and

$${}^n\theta_j = \frac{(-1)^j \, n!}{j! \, (n-2j)! \, 2^j}$$

for $n > 0$ and $j = 0, ..., N$.

Theorem 9. *Let $T \in \mathcal{G}'$ and write $(\tau_n)_{n\in\mathbb{N}} = h^{-1}(T)$. Then $h^{-1}(\mathcal{F}T) = (\beta_n)_{n\in\mathbb{N}}$, with*

$$\begin{cases} \beta_0 = \sqrt{2\pi} \, \tau_0, \\ \beta_n = \sqrt{2\pi} \, i^n \displaystyle\sum_{j=0}^{N} {}^n\theta_j \, \tau_{n-2j} \qquad \textit{for} \;\; n > 0 \; . \end{cases} \tag{17}$$

Proof: We present the proof for $n > 0$ (for $n = 0$ the proof is similar):

$$\begin{aligned} \langle \mathcal{F}T, \, x^n \, e^{-\frac{x^2}{2}} \rangle &= \Big\langle T, \, \frac{1}{2\pi} (\mathcal{F}x^n * \mathcal{F}e^{-\frac{x^2}{2}}) \Big\rangle \\ &= \Big\langle T, \, \sqrt{2\pi} \, i^n (\delta^{(n)} * e^{-\frac{x^2}{2}}) \Big\rangle = \sqrt{2\pi} (-i)^n \sum_{j=0}^{N} {}^n\theta_j \, \langle T, \, x^{n-2j} \, e^{-\frac{x^2}{2}} \rangle \\ &= \sqrt{2\pi} (-i)^n \sum_{j=0}^{N} {}^n\theta_j \, \tau_{n-2j} \; . \end{aligned}$$

We remark that, instead of formulas (17), the following formulas, where $k \in \mathbb{N}$, can be useful:

$$\begin{cases} \beta_{2k} = \sqrt{2\pi} \displaystyle\sum_{j=0}^{k} (-1)^j \, \frac{(2k)!}{(k-j)! \, (2j)! \, 2^{k-j}} \, \tau_{2j}, \\ \beta_{2k+1} = i \, \sqrt{2\pi} \displaystyle\sum_{j=0}^{k} (-1)^j \, \frac{(2k+1)!}{(k-j)! \, (2j+1)! \, 2^{k-j}} \, \tau_{2j+1} \; . \end{cases}$$

Corollary. *Let $T \in \mathcal{G}'$ and $(\beta_n)_{n\in\mathbb{N}} = h^{-1}(\mathcal{F}T)$. Then $h^{-1}(\mathcal{F}^{-1}T) = \frac{1}{\sqrt{2\pi}}\,((-1)^n\,\beta_n)_{n\in\mathbb{N}}$.*

6 – Final remarks

Given the series $\sum_{k=0}^{\infty} c_k\,x^k$ $(c_k \in \mathbb{C})$ we consider the multipole series $\sum_{k=0}^{\infty} \widehat{c}_k\,\delta^{(k)}$ $(\widehat{c}_k = 2\pi\,i^k\,c_k)$ obtained by a formal application of the Fourier transform to the first series. As the Fourier transform is a vectorial and topological isomorphism in $\mathcal{G}'$, we see that the power series is convergent (in $\mathcal{G}'$) iff the correspondent multipole series is convergent and Theorem 8 gives us a necessary and sufficient condition on the coefficients c_k for that convergence.

Next we show that some functions not belonging to $\mathcal{S}'$ can be identified with elements of our space $\mathcal{G}'$. Let $\mathcal{H}_e$ be the vector space of complex functions f defined on $\mathbb{R}$ that can be prolonged as entire functions on $\mathbb{C}$ with exponential growth, that is $|f(z)| \le \alpha\,e^{\beta|z|}$ (α and β real constants, $\alpha, \beta > 0$). We know (see [8], p. 81) that an entire function $f(z) = \sum_{k=0}^{\infty} a_k\,z^k$ is in $\mathcal{H}_e$ iff $\sqrt[k]{k!\,|c_k|} \le C$ for $k \ge 1$ (C real constant). The vector space $\mathcal{H}_e$ is closed for the usual product and derivatives.

Let $f \in \mathcal{H}_e$ with $f(x) = \sum_{k=0}^{\infty} c_k\,x^k$. Corollary 2 of Theorem 8 shows that the power series $\sum_{k=0}^{\infty} c_k\,x^k$ is convergent in $\mathcal{G}'$ and therefore we may identify the function f with the sum (in $\mathcal{G}'$) of the series $\sum_{k=0}^{\infty} c_k\,x^k$. There is a coherence problem because, if $f \in \mathcal{S}' \cap \mathcal{H}_e$, we have two identifications for f, the one just described and the one coming from the inclusion $\mathcal{S}' \cap \mathcal{G}'$. We show that eventually the two identifications lead to the same ultradistribution. For $f \in \mathcal{S}' \cap \mathcal{H}_e$ we have, for all $\varphi \in \mathcal{G}$,

$$\Big\langle \sum_{k=0}^{\infty} c_k\,x^k,\,\varphi \Big\rangle = \lim_{n\to+\infty} \int_{\mathbb{R}} \sum_{k=0}^{n} c_k\,x^k\,\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx\ ,$$

by Lebesgue dominated convergence theorem.

Remark. Let $f \in L^1_{\rm loc}(\mathbb{R})$ be such that $\int_{\mathbb{R}} f(x)\,x^k\,e^{-\frac{x^2}{2}}\,dx$ is convergent for all $k \in \mathbb{N}$; we could be tempted to identify f with the element $T_f \in \mathcal{G}'$ given by

$$\langle T_f, \varphi\rangle = \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx\ .$$

Unfortunatelly the linear mapping $f \to T_f$ just defined is not injective because there are $L^1_{\rm loc}$ functions $g \neq 0$ such that $\int_{\mathbb{R}} g(x)\,x^k\,dx = 0$ for all $k \in \mathbb{N}$.

Let $\alpha \in \mathbb{C}$; the function $e^{i\alpha x}$ is in $\mathcal{H}_e$ and we have $e^{i\alpha x} = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} x^k$. Applying the Fourier transform we get

$$\mathcal{F}(e^{i\alpha x}) = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} 2\pi \, i^k \, \delta^{(k)} = 2\pi \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} 2\pi \, i^k \, \delta^{(k)} = 2\pi \, \delta_\alpha \ .$$

We just derived the formula (18) that implies formulas (19) and (20):

$$\delta_\alpha = \frac{1}{2\pi} \mathcal{F}(e^{i\alpha x}) \ , \tag{18}$$

$$\mathcal{F}^{-1}\delta_\alpha = \frac{1}{2\pi} e^{i\alpha x} \ , \tag{19}$$

$$\mathcal{F}^{-1}\delta_\alpha^{(k)} = \frac{1}{2\pi} (-ix)^k \, e^{i\alpha x} \ . \tag{20}$$

Now we are able to study the series $\sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \delta_\beta^{(k)}$ with $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned} \mathcal{F}^{-1}\Big(\sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \delta_\beta^{(k)}\Big) &= \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \mathcal{F}^{-1}(\delta_\beta^{(k)}) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} (-ix)^k \, e^{i\beta x} \\ &= \frac{1}{2\pi} e^{i\alpha x} \, e^{i\beta x} = \frac{1}{2\pi} e^{i(\alpha+\beta)x} \ ; \end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \delta_\beta^{(k)} = \frac{1}{2\pi} \mathcal{F}(e^{i(\alpha+\beta)x}) = \delta_{\alpha+\beta} \ .$$

We just have found a multipole series development for the ultradistribution $\delta_{\alpha+\beta}$:

$$\delta_{\alpha+\beta} = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \delta_\beta^{(k)} \ . \tag{21}$$

Remark. We may define the convolution of δ_α and δ_β by $\delta_\alpha * \delta_\beta = \delta_{\alpha+\beta}$. It is easily seen that $\mathcal{F}(\delta_\alpha * \delta_\beta) = (\mathcal{F}\delta_\alpha)\,(\mathcal{F}\delta_\beta)$.

Examples: Write $(e_n)_{n\in\mathbb{N}} = h^{-1}(e^x)$. From $\partial e^x = e^x$ we see that $[S_{-1} - (n)_{n\in\mathbb{N}}.S_1]\,(e_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ and this implies that

$$e_1 = e_0 \, ; \quad e_2 = 2\, e_0 \, ; \quad e_n = e_{n-1} + (n-1)\, e_{n-2} \ \ \forall\, n \geq 3 \ .$$

This shows that $e_0 \neq 0$ but we still have to compute e_0. From $e^x\, e^{-x} = 1$ and from the fact that the series $\sum_{k=0}^{\infty} \frac{x^k}{k!} e^{-x}$ is convergent in $\mathcal{G}'$, with sum equal to 1, we see that

$$\lim_{m\to+\infty} h^{-1}\Big[\Big(\sum_{k=0}^{m} \frac{x^k}{k!}\Big) e^{-x}\Big] = (\gamma_n)_{n\in\mathbb{N}} \ .$$

As $h^{-1}(e^{-x}) = ((-1)^n e_n)_{n\in\mathbb{N}}$ we see by (9) that

$$\gamma_0 = \lim_{m\to+\infty} \sum_{k=0}^{m} \frac{(-1)^k\, e_k}{k!} \ .$$

Writing $(\beta_n)_{n\in\mathbb{N}} = \frac{1}{e_0}(e_n)_{n\in\mathbb{N}}$ we see that $e_0 = \sqrt{2\pi}(\sum_{k=0}^{\infty} \frac{(-1)^k\, \beta_k}{k!})^{-1}$.

For $T \in \mathcal{G}'$ we define T_s, usually denoted by $T(-x)$, by

$$\langle T_s,\, p e^{-\frac{x^2}{2}}\rangle = \langle T,\, p(-x)\, e^{-\frac{x^2}{2}}\rangle \ .$$

It is easily seen that, if $(\alpha_n)_{n\in\mathbb{N}} = h^{-1}(T)$, then $h^{-1}(T_s) = ((-1)^n\, \alpha_n)_{n\in\mathbb{N}}$. If we write T in the form $T = T_p + T_i$, where

$$T_p = \frac{T+T_s}{2} \quad \text{and} \quad T_i = \frac{T-T_s}{2} \ ,$$

we see that $h^{-1}(T_p) = (\frac{1+(-1)^n}{2}\, \alpha_n)_{n\in\mathbb{N}}$ and $h^{-1}(T_i) = (\frac{1-(-1)^n}{2}\, \alpha_n)_{n\in\mathbb{N}}$. In particular

$$h^{-1}(\operatorname{ch} x) = \left(\frac{1+(-1)^n}{2}\, e_n\right)_{n\in\mathbb{N}} \quad \text{and} \quad h^{-1}(\operatorname{sh} x) = \left(\frac{1-(-1)^n}{2}\, e_n\right)_{n\in\mathbb{N}} \ .$$

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