PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 1 – 1998

EXTENDING AND LIFTING SOME LINEAR TOPOLOGICAL STRUCTURES

Le Mau Hai and Pham Hien Bang

Abstract: In this paper we investigate some results on extending and lifting some linear topological structures and applications to the extension and lift problems of continuous linear and holomorphic maps.

0 – Introduction

The aim of this paper is to investigate the extending and lifting some linear topological structures. The obtained results will be applied to the extension and lift problems of continuous linear maps and holomorphic maps in locally convex spaces. In the case of Banach spaces the extension and lift problems of continuous linear maps have been investigated by several authors (see [4], [5], ...). Palamodov [10] and Gejler [3] have investigated these problems for locally convex spaces, in particular, for Fréchet spaces. In [3] Gejler has proved that a Fréchet (resp. Montel–Fréchet; nuclear Fréchet) space F has the lifting property for the class of all Fréchet (resp. of all Montel–Fréchet, nuclear Fréchet) spaces if and only if F is finite dimensional. A version of the theorem of Gejler for Lipschitz maps, uniformly continuous maps and locally uniformly continuous maps has been proved in [7]. Recently Vogt has obtained very important results for the lifting problem of continuous linear maps in the class of nuclear Fréchet spaces (see Vogt [14], [15]).

In Section 2 we investigate extending and lifting Schwartz and s-nuclear structures in locally convex spaces. Examples on applications of the obtained results to extension and lift problems of continuous linear maps and holomorphic maps are presented in Section 3.

Received: May 18, 1996; *Revised*: October 17, 1996.

1 – Preliminaries

In this section we fix notations and formulate some well-known facts needed through this paper.

Given Banach spaces A, B and a continuous linear map T from A to B. For each $n \in \mathbb{N}$ let

$$\alpha_n(T) = \inf\Big\{\|T - S\|\colon\ S\colon\ A \to B \text{ is continuous linear and } \dim \operatorname{Im} S \le n\Big\} .$$

$\alpha_n(T)$ is called the n-approximation number of T (see [10]). T is said to be map of type ℓ_p if

$$\rho_p(T) = \Big(\sum_{n\ge1} \alpha_n(T)\Big)^{1/p} < \infty .$$

Let $L_p(A,B)$ denote the set of maps of type ℓ_p from A to B. A map $T\colon\ A \to B$ is called s-nuclear if $T \in L_p(A,B)$ for every $p > 0$. It is known [10] that

(1.1) $\rho_p(\lambda T) = |\lambda|\, \rho_p(T)$;

(1.2) $\rho_p(S+T) \le \sigma_p([\rho_p(S) + \rho_p(T)])$, where

$$\sigma_p = \begin{cases} 2 & \text{if } p \ge 1, \\ 2^{(\frac{2}{p})-1} & \text{for } 0 < p < 1 \ ; \end{cases}$$

(1.3) $L_p(A,B)$ is complete, i.e. if $\{T_n\} \subset L_P(A,B)$ and $\rho_p(T_n - T_m) \to 0$, then there exists $T \in L_p(A,B)$ such that $\rho_p(T_n - T) \to 0$.

(1.4) $\rho_p(ST) \le \rho_p(S)\, \rho_p(T)$ for $T \in L_p(A,B)$, $S \in L_p(B,C)$;

(1.5) A continuous linear map $T\colon\ A \to B$ is s-nuclear if and only if there exists sequences $\{\lambda_n\} \subset \mathbb{R}$, $\{u'_n\} \subset A'$ and $\{b_n\} \subset B$ such that

$$(1.6)\qquad \begin{cases} T(u) = \displaystyle\sum_{n\ge1} \lambda_n\, u'_n(u)\, v_n & \text{for } u \in A, \\ \sup\Big\{\|u'_n\| + \|v_n\|\Big\} < \infty, & \\ \displaystyle\sum_{n\ge1} |\lambda_n|^p < \infty & \text{for every } p > 0 \ , \end{cases}$$

where A' denotes the Banach space of all continuous linear functionals on A equipped with the sup-norm on the unit ball of A.

Now let E be a locally convex space. By $\mathcal{U}(E)$ we denote the set of all balanced convex neighbourhoods of zero in E. For each $U \in \mathcal{U}(E)$ let q_U denote the Minkowski functional of U on E. The completion of $E/q_U := E/\operatorname{Ker} q_U$

equipped with the norm q_U is denoted by $E(U)$ and $\pi(U)\colon E \to E(U)$ denotes the canonical map from E into $E(U)$.

For $U, V \in \mathcal{U}(E)$ with $V \subset U$ let $\omega(V,U)\colon E(V) \to E(U)$ denote the canonical map. A locally convex space E is called a Schwartz (resp. s-nuclear) space if for each $U \in \mathcal{U}(E)$ there is $V \in \mathcal{U}(E)$ such that $V \subset U$ and $\omega(V,U)$ is compact (resp. s-nuclear).

The space E is said to be a DF-space if E has a fundamental sequence of bounded sets and if every bounded countable union of equicontinuous subsets of E' is equicontinuous, where E' denotes the strongly dual space of E.

Given E, F locally convex spaces and D an open subset of E. A map $f: D \to F$ is called holomorphic if f is continuous and $f|_{D\cap H}$ is holomorphic for every finite dimensional subspace H of E. By $L_b(E,F)$ we denote the space of all continuous linear maps from E into F equipped with the topology of uniform convergence on bounded sets in E and by $E \mathbin{\widehat{\otimes}_\pi} F$ we denote the projective tensor product of E and F.

Finally $H(D,F)$ denotes the space of holomorphic maps from D into F equipped with the compact-open topology.

2 – Extending and lifting Schwartz and s-nuclear structures

We denote $\mathcal{S}$ and $\mathcal{N}_\varepsilon$ the classes of all Schwartz spaces and of all s-nuclear spaces respectively. The main result of this section is Theorem 2.2 on lifting Schwartz and s-nuclear structures. Before obtaining this result first we prove a result on extending Schwartz and s-nuclear structures. This is the following

2.1 Theorem. *Let $E \in \mathcal{S}$ (resp. $\mathcal{N}_\varepsilon$) be a subspace of a locally convex space F. Then there exists a space $\widetilde{E} \in \mathcal{S}$ (resp. $\mathcal{N}_\varepsilon$) and a continuous linear map $h\colon F \to \widetilde{E}$ such that $\widetilde{e} = h|_E$ is embedding.*

Proof: The case where $E \in \mathcal{N}_\varepsilon$ has been proved in [8]. For the case $E \in \mathcal{S}$ by using Lemma 2.1 of Bellenot [1] on the factorization of compact operators and, after that, repeating the proof presented in [8] we obtain the proof of Theorem 2.1. ∎

The result on lifting Schwartz and s-nuclear structures in locally convex spaces is the following

2.2 Theorem. *Let $j\colon F \to E$ be a continuous linear surjective map between Fréchet spaces. If $E \in \mathcal{S}$ (resp. $\mathcal{N}_\varepsilon$) then there exist a Fréchet space $\widetilde{E} \in \mathcal{S}$ (resp. $\mathcal{N}_\varepsilon$) and a continuous linear map $h\colon \widetilde{E} \to F$ such that jh is surjective.*

Proof of Theorem 2.2 in the case $E \in \mathcal{S}$

For this case we need the following lemma

2.3 Lemma. *Let j be a continuous linear map from a Fréchet space F onto a separable Fréchet space E. Then there exists a separable subspace G of F such that $j(G) = E$.*

Proof: Let $g\colon E \to F$ be a continuous linear map such that $jg = \mathrm{id}$. Such a map exists by the selection theorem of Michael. Then setting $G = \overline{\operatorname{span} g(E)}$ we get a required subspace of F. ∎

Now we prove Theorem 2.2 in the case where $E \in \mathcal{S}$. Since every separable Fréchet space is image of a Fréchet–Montel space [3], from Lemma 2.3 we can assume that F is a Fréchet–Montel space.

Let $\{U_n\}$ be a decreasing basis of balanced convex neighbourhoods of zero in F. Put $G = \operatorname{Ker} j$, then $\{V_n = U_n \cap G\}$ and $\{W_n = j(U_n)\}$ are decreasing basis of balanced convex neighbourhoods of zero in G and E respectively. For each n consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_n & \xrightarrow{j'_n} & F'_n & \xrightarrow{e'_n} & G'_n & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle '\alpha^n_{n+1}} & & \big\downarrow {\scriptstyle '\omega^n_{n+1}} & & \big\downarrow {\scriptstyle '\beta^n_{n+1}} & & \\ 0 & \longrightarrow & E'_{n+1} & \xrightarrow{j'_{n+1}} & F'_{n+1} & \xrightarrow{e'_{n+1}} & G'_{n+1} & \longrightarrow & 0 \end{array} \tag{2.1}$$

where $\alpha^n_{n+1} = \omega(W_{n+1}, W_n)$, $\omega^n_{n+1} = \omega(U_{n+1}, U_n)$, $\beta^n_{n+1} = \omega(V_{n+1}, V_n)$, e_n, j_n are maps induced by e, j respectively, $e\colon G \to F$ denotes the canonical embedding. Without loss of generality we may assume that $'\alpha^n_{n+1}$ is compact for every $n \geq 1$. Let $\widetilde{\alpha}^n_{n+1}\colon F'_n \to \ell_\infty(S_{n+1}) \supset E'_{n+1}$ be a compact linear extension of $'\alpha^n_{n+1}$ and let $\widetilde{\alpha}^n_{n+1}\colon F'_n \to \ell_\infty(S_{n+1}) \supset E'_{n+1}$ be a compact linear extension of $'\alpha^n_{n+1}$ and q^n_1 be an embedding of $\operatorname{Im} \widetilde{\alpha}^n_{n+1}$ into the Banach space $C_{[0,1]}$ of continuous functions on $[0,1]$. Since $C_{[0,1]}$ has a Schauder basis, by lemma 2.1 of Bellenot [1], $q^n_1\, \widetilde{\alpha}^n_{n+1}$ can be written in the form

$$q^n_1\, \widetilde{\alpha}^n_{n+1} = Q^n_1 . P^n_1 \ ,$$

where $Q^n_1\colon C_{[0,1]} \to C_{[0,1]}$ and $P^n_1\colon F'_n \to C_{[0,1]}$ are compact linear maps.

Again applying the result of Bellenot to P^n_1 we find compact linear maps $Q^n_2\colon C_{[0,1]} \to C_{[0,1]}$ and $P^n_2\colon F'_n \to C_{[0,1]}$ such that

$$P^n_1 = Q^n_2 . P^n_2 \ .$$

Continuing this process we get sequences of compact linear maps $\{Q_i^n\}$ and $\{P_i^n\}$ such that

$$q_1^n\,\widetilde{\alpha}_{n+1}^n = Q_1^n\,P_1^n, \quad \dots, \quad P_j^n = Q_{j+1}^n\,P_{j+1}^n, \quad \dots$$

where q_1^n are embeddings of $\operatorname{Im}\widetilde{\alpha}_{n+1}^n$ into $C_{[0,1]}$. For every $u \in F_n'$ put

$$\begin{aligned} p_n(u) = {} & \|P_1^{n+1\prime}\,\omega_{n+1}^n(u)\|/(n+1)^2 \prod_{j=2}^{n} \|Q_j^{n-1}\|\,\|P_n^{n+1}\| \\ & + \|P_2^{n+2\prime}\,\omega_{n+2}^n(u)\|/(n+1)^2 \prod_{j=3}^{n} \|Q_j^{n+2}\|\,\|P_{n+1}^{n+2}\| \\ & + \dots\,. \end{aligned}$$

Obviously for each n, ${}'\omega_{n+1}^n$ induces naturally a continuous linear map $\widetilde{\omega}_{n+1}^n\colon F_n'/p_n \to F_{n+1}'/p_{n+1}$ and $\pi_n\,j_n'\colon E_n' \to F_n'/p_n$ is an embedding, where $\pi_n\colon F_n' \to F_n'/p_n$ is the canonical map. Define the maps

$$\begin{aligned} \gamma_n\colon F_n'/p_n &\to \ell_1 \mathbin{\widehat{\otimes}_\pi} C_{[0,1]}\,, \\ \alpha_{n+1}\colon F_{n+1}'/p_{n+1} &\to \ell_1 \mathbin{\widehat{\otimes}_\pi} C_{[0,1]}\,, \end{aligned}$$

by the formulas

$$\gamma_n(u) = \left\{Q_2^{n+2}\,P_1^{n+2\prime}\,\omega_{n+2}^n(u)/(n+2)^2 \prod_{j=2}^{n+1} \|Q_j^{n+2}\|\,\|P_{n+1}^{n+2}\|,\ \dots\right\},$$

$$\alpha_{n+1}(u) = \left\{P_1^{n+2\prime}\,\omega_{n+2}^{n+1}(u)/(n+2)^2 \prod_{j=2}^{n+1} \|Q_j^{n+2}\|\,\|P_{n+1}^{n+2}\|,\ \dots\right\}.$$

Obviously $\gamma_n = \alpha_{n+1}\,\widetilde{\omega}_{n+1}^n$. Since the definition of α_{n+1} and Q_j^n are compact it follows γ_n is compact and hence $\widetilde{\omega}_{n+1}^n$ is compact.

Putting $\widetilde{E} = \varprojlim [F_n'/p_n]'$ and $h:\ \widetilde{E} \to F'' \cong F$, the map induced by the canonical maps $F_n' \to F_n'/p_n$, we obtain a Fréchet-Schwartz space $\widetilde{E}$ and a continuous linear continuous linear map $h\colon \widetilde{E} \to F$. It remains to check $jh\colon \widetilde{E} \to E$ is surjective, equivalently $(jh)'\colon E' \to \widetilde{E}'$ is an embedding. Since E' is a (DFS)-space and $\widetilde{E}$ is a Fréchet space, by [2] it suffices to show that $(jh)'^{-1}(Q)$ is bounded in E' for every bounded set Q in $\widetilde{E}'$. Given such a set $Q \subset \widetilde{E}'$, choose $n \geq 1$ such that Q is contained and bounded in $[F_n'/p_n]''$. From the relations $E_n' \hookrightarrow F_n'/p_n \hookrightarrow [F_n'/p_n]''$, we infer that $(jh)'^{-1}(Q)$ is contained and bounded in E_n'. Thus the case where $E \in \mathcal{S}$ is proved.

Proof of Theorem 2.2 in the case $E \in \mathcal{N}_\varepsilon$

In this case we need to use the following lemmas

2.4 Lemma. *Let* $Q_j\colon B \to B_j$ *be* s*-nuclear maps between Banach spaces and let* $\{a_j\} \in \ell_1$, $\nu_j = 1/j$. *Then the map*

$$\widetilde{Q}\colon B \to \ell_1(\{B_j\}) := \Big\{(x_j \in B_j)\colon \sum_{j\geq 1} \|x_j\| < \infty\Big\}$$

given by the formula $\widetilde{Q}(x) = \{a_j\, Q_j(x) \,/\, \sigma^j_{\nu_j}\, \rho_{\nu_j}(Q_j)\}$ *is* s*-nuclear.*

Proof: For each n put

$$\widetilde{Q}_n(x) = \Big\{a_1 Q_1(x)/\sigma_{\nu_1}\rho_{\nu_1}(Q_1),\ \ldots,\ a_n Q_n(x)/\sigma^n_{\nu_n}\rho_{\nu_n}(Q_n),\ 0,\ \ldots\Big\}\ .$$

By (1.3) it remains to check that $\rho_\nu(\widetilde{Q}_{n+p} - \widetilde{Q}_n) \to 0$ as $n,p \to \infty$ for every $\nu > 0$. Fix $\nu > 0$. Take j_0 such that $\nu_{j_0} < \nu$. By (1.1) and (1.2) we have

$$\begin{aligned}\rho_\nu(\widetilde{Q}_{n+p} - \widetilde{Q}_n) &\leq \rho_{\nu_{j_0}}(\widetilde{Q}_{n+p} - \widetilde{Q}_n)\\ &\leq \sigma_{\gamma_{j_0}} \frac{|a_{n+1}|\,\|Q_{n+1}\|}{\sigma^{n+1}_{\gamma_{n+1}}\,\rho_{\gamma_{n+1}}(Q_{n+1})} + \cdots + \sigma^p_{\gamma_{j_0}} \frac{|a_{n+p}|\,\|Q_{n+p}\|}{\sigma^{n+p}_{\gamma_{n+p}}\,\rho_{\gamma_{n+p}}(Q_{n+p})}\\ &< |a_{n+p}| + \cdots + |a_{n+1}| \to 0, \qquad \text{when } n,p \to \infty\ . \blacksquare\end{aligned}$$

2.5 Lemma [11]. *Let* $\alpha\colon A \to B$ *and* $\beta\colon B \to C$ *be continuous linear maps between Banach spaces. Let* $e\alpha$ *be* s*-nuclear for some embedding* e *and* β *be quasi-nuclear. Then* $\beta\alpha$ *is* s*-nuclear.*

Now we prove Theorem 2.2 in the case $E \in \mathcal{N}_\varepsilon$. As in the case $E \in \mathcal{S}$, for each n consider the commutative diagram (2.1). Then ${}'\alpha^n_{n+1}$ is s-nuclear for every $n \geq 1$. Let $\widetilde{\alpha}^n_{n+1}$ be an s-nuclear linear extension of ${}'\alpha^n_{n+1}$ from F'_n into E'_{n+1}. Such an extension exists by the Hahn–Banach theorem and by (1.6). From (1.6) we find s-nuclear maps P^n_j and Q^n_j such that

$$\widetilde{\alpha}^n_{n+1} = Q^n_1 P^n_1, \quad \ldots, \quad P^n_j = Q^n_{j+1} P^n_{j+1}, \quad \ldots\ .$$

Let $\nu_j = 1/j$. For every $u \in F'_n$ put

$$\begin{aligned}p_n(u) = &\ \|P^{n+1\prime}_1\, \omega^n_{n+1}(u)\|/(n+1)\,\sigma_{\nu_1} \prod_{j=2}^{n} \rho_{\nu_1}(Q^{n+1}_j)\,\rho_{\nu_1}(P^{n+1}_n)\\ &+ \|P^{n+2\prime}_2\, \omega^n_{n+2}(u)\|/(n+1)^2\,\sigma^2_{\nu_2} \prod_{j=3}^{n+1} \rho_{\nu_2}(Q^{n+2}_j)\,\rho_{\nu_2}(P^{n+2}_{n+1})\\ &+ \ldots\ .\end{aligned}$$

Let us show that the map $\widetilde{\omega}^n_{n+2}\colon F'_n/p_n \to F'_{n+2}/p_{n+2}$ induced by ${}'\omega^n_{n+2}$ is s-nuclear. Consider the maps

$$\gamma_n\colon F'_n/p_n \to \ell_1\Big(\overline{\{\operatorname{Im} P^{n+2}_j\}}\Big) ,$$

$$\alpha_{n+1}\colon F'_{n+1} \to \ell_1\Big(\overline{\{\operatorname{Im} P^{n+2}\}}\Big) ,$$

given by the formulas

$$\gamma_n(u) = \Big\{ Q^{n+2}_2 P^{n+2\prime}_2\, \omega^n_{n+2}(u)/(n+2)^2\, \sigma^2_{\nu_2} \prod_{j=2}^{n+1} \rho_{\nu_2}(Q^{n+2}_j)\, \rho_{\nu_2}(P^{n+2}_{n+1}),\ \dots \Big\} ,$$

$$\alpha_{n+1}(u) = \Big\{ P^{n+2\prime}_1\, \omega^{n+1}_{n+2}(u)/(n+2)^2\, \sigma^2_{\nu_2} \prod_{j=2}^{n+1} \rho_{\nu_2}(Q^{n+2}_j)\, \rho_{\nu_2}(P^{n+2}_{n+1}),\ \dots \Big\} .$$

Then α_{n+1} is an embedding. Since Q^n_j is s-nuclear for every j, $n \geq 1$, by Lemma 2.4, $\gamma_n = \alpha_{n+1}\, \widetilde{\omega}^n_{n+1}$ is s-nuclear. Thus $\widetilde{\omega}^{n+1}_{n+2}$ is quasi-nuclear. Hence the proof is complete by Lemma 2.5 and by an argument as in the case where $E \in \mathcal{S}$. ∎

Question. How to solve the problem on lifting nuclear structures?

3 – Some applications

In this section we give some examples on applications of Theorem 2.1 and 2.2 to problems on extending and lifting continuous linear maps and holomorphic maps.

3.1. Extending and lifting continuous linear maps

We first give the following two definitions for a class of locally convex spaces.

3.1.1 Definition. A locally convex space P is said to have the extension property with respect to the class $\mathcal{D}$ if every continuous linear map from subspace $E \in \mathcal{D}$ of a space $F \in \mathcal{D}$ with values in P can be extended to a continuous linear map on F.

3.1.2 Definition. We say that the space P has the lift property with respect to the class $\mathcal{D}$ if for every continuous linear open map j from a space $F \in \mathcal{D}$ onto a space $E \in \mathcal{D}$, the map $\widehat{j}\colon L_b(P,F) \to L_b(P,E)$ induced by j, is surjective.

The following lemma has been proved by Palamodov

3.1.3 Lemma [10]. *Let*

$$0 \longrightarrow \{G_n, \alpha_m^n\} \xrightarrow{\{f_n\}} \{F_n, \omega_m^n\} \xrightarrow{\{g_n\}} \{E_n, \beta_m^n\} \longrightarrow 0$$

be a complexe of projective systems of Fréchet spaces and let $k \geq 0$. *Assume that*

$$\operatorname{Ker} g_n = \operatorname{Im} f_n \quad \text{and} \quad \operatorname{Ker} f_n = 0 \qquad \text{for every} \quad n > 1 \ .$$

Then the map $\varprojlim g_n \colon \varprojlim F_n \to \varprojlim E_n$ *is surjective if the maps* g_n, α_m^n *and* β_m^n *satisfy the following conditions for every* $n \geq 1$

(ML_1) $\operatorname{Im} \beta_{n+k}^n \subset \operatorname{Im} g_n$;

(ML_2) $\operatorname{Im} \alpha_{n+k+1}^n$ *is dense in* $\operatorname{Im} \alpha_{n+k}^n$.

Examples

1) By the Sobczyk's theorem [13], the Banach space C_0 has the extension property for the class separable locally convex spaces.

2) From the Hahn-Banach theorem and from Lemma 3.1.3 it follows that every nuclear Fréchet space has the extension property for the class $D\mathcal{F}$ of DF-spaces.

3) Similarly from Lemma 3.1.3 it is to see that each nuclear DF-space has the lift property for the class $\mathcal{F}$ of Fréchet spaces.

3.1.4 Lemma [8]. *Let* J *be a Montel space and let*

$$\theta \colon J \to Q = \pi\Big\{J(U) \colon U \in \mathcal{U}(J)\Big\}$$

be the canonical embedding of J *into* Q. *Then* J *is isomorphic to* $\boldsymbol{C}^{\wedge}$ *for some set* $\wedge$ *if and only if there exists a continuous linear projection of* Q *onto* $\operatorname{Im} J$.

Two following theorems are examples of applications of Theorem 2.1 and 2.2.

3.1.5 Theorem. *Let* $\mathcal{D} \in \{\mathcal{S}, \mathcal{N}_\varepsilon\}$ *and* $P \in \mathcal{D}$. *Then the following conditions are equivalent*

(i) $P = \boldsymbol{C}^{\wedge}$ *for some set* $\wedge$;

(ii) P *has the extension property for the class* $\mathcal{L}$ *of all locally convex spaces;*

(iii) P *has the extension property for the class* $\mathcal{D}$.

Proof: From Theorem 2.1 and Lemma 3.1.4. ∎

3.1.6 Theorem. *Let $\mathcal{D} \in \{\mathcal{S}, \mathcal{N}_\varepsilon\}$ and $P \in \mathcal{D} \cap \mathcal{F}$. Then the following conditions are equivalent*

(**i**) $P = \boldsymbol{C}^m$ *for some* $m \geq 0$;

(**ii**) P *has the lift property for the class* $\mathcal{F}$ *of Fréchet spaces;*

(**iii**) P *has the lift property for the class* $\mathcal{F} \cap \mathcal{S}$.

Proof: (i)⇒(ii)⇒(iii) are trivial. By Theorem 2.2 and by a result of Geijler [3] it follows that (iii)⇒(ii)⇒(i). ∎

3.2. Extending holomorphic maps

In [6] Meise and Vogt have proved that for an exact sequence of Fréchet spaces

$$0 \longrightarrow G \longrightarrow F \to E \to 0 \tag{3.1}$$

the restriction map $R_P: H(F', P) \to H(E', P)$ is surjective for every Fréchet space P if F' has a fundamental system of continuous Hilbert semi-norms and E is a Fréchet–Montel space. In a particular case this result has been proved in [9].

In this section using Theorem 2.2 we shall prove some results on the extension of holomorphic functions on E' with values in Fréchet spaces when either E is locally projective or G is C-local Schwartz and F is Montel.

We first give some definitions.

Assume now that D is an open subset of a locally convex space P and L is a locally convex space. A holomorphic map $f: D \to L$ is called a holomorphic map of finite type if $D^n f(z): P \times ... \times P \to L$ can be approximated by elements belonging to $L_b(P, L) \mathbin{\widehat{\otimes}_\pi} ... \mathbin{\widehat{\otimes}_\pi} L_b(P, L)$ for every $z \in D$ and every $n \geq 1$. By $H_f(D, L)$ we denote the subspace of $H(D, L)$ consisting of holomorphic maps of finite type.

In [15] Vogt has said that a locally convex space E is locally projective if for every $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ such that $V \subset U$ and the map $\omega(V, U): E(V) \to E(U)$ is factorized through the space ℓ^1. Similarly we say that E is a C-local space if for every $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ such that $V \subset U$ and the map $\omega(V, U): E(V) \to E(U)$ is factorized through a space $C(S)$, where $C(S)$ is the Banach space of continuous functions on a Hausdorff compact space S. We have

3.2.1 Theorem. *Let (3.1) be an exact sequence of Fréchet spaces. Then the restriction map $R_P\colon H(F',P) \to H(E',P)$ (resp. $R_P\colon H_f(F',P) \to H_f(E',P)$) is surjective for every Fréchet space P if E is locally projective Schwartz (resp. G is C-local Schwartz) and F is Montel.*

Proof: (**i**) In notations as in the proof of Theorem 2.2. By Theorem 2.2 we can assume that F is a Schwartz space. Consider the complex of Fréchet spaces

$$(3.2) \qquad 0 \longrightarrow \{\operatorname{Ker}\widehat{j'_n}\} \longrightarrow \{H^b_f(F'_n,P)\} \xrightarrow{\widehat{j'_n}} \{H^b_f(E'_n,P)\} \longrightarrow 0\ ,$$

where for each locally convex space Q by $H^b_f(Q,P)$ we denote the space of holomorphic maps finite type which are bounded on every bounded set in Q with values in P. By hypothesis we can assume that $\alpha^n_{n+1}\colon E_{n+1} \to E_n$ are factorized through the space ℓ^1. Then for each $n \geq 1$ there exists a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E'_n & \xrightarrow{j'_n} & F'_n & \xrightarrow{e'_n} & G'_n & \longrightarrow & 0 \\
 & & \Big\downarrow {'\alpha^n_{n+1}} & \swarrow \gamma^n_{n+1} & \Big\downarrow {'\omega^n_{n+1}} & \swarrow \theta^n_{n+1} & \Big\downarrow {'\beta^n_{n+1}} & & \\
0 & \longrightarrow & E'_{n+1} & \xrightarrow{j'_{n+1}} & F'_{n+1} & \xrightarrow{e'_{n+1}} & G'_{n+1} & \longrightarrow & 0
\end{array}$$

such that

$$\gamma^n_{n+1}\, j'_n = {'\alpha^n_{n+1}}\,, \qquad e'_{n+1}\,\theta^n_{n+1} = {'\beta^n_{n+1}} \qquad \text{and} \qquad {'\omega^n_{n+1}} = \gamma^n_{n+1} + \theta^n_{n+1}\, e'_n\ .$$

Thus (3.2) satisfies (ML_1) with $k = 1$.

Now for each n define a continuous linear map

$$\theta_n\colon E'_n \times G'_n \to F'_{n+1}$$

by

$$\theta_n(u,v) = {'\alpha^n_{n+1}}(u) + \theta^n_{n+1}(v)\ .$$

Assume that $f \in \operatorname{Ker}\widehat{j'_n}$. Then $f\theta_{n-1} \in H^b_f(E'_{n-1} \times G'_{n-1},\ P)$ and $(f\theta_{n-1})(0,v) = (f\gamma^{n-1}_n(v)) = 0$. Thus $f\theta_{n-1}$ induces a holomorphic map $\widetilde{f\theta_{n-1}}\ :\ G'_{n-1} \to H^b_{f,0}(E'_{n-1},P)$, where $H^b_{f,0}(E'_{n-1},P) = \{g \in H^b_f(E'_{n-1},P)\ :\ g(0) = 0\}$. Since the maps ${'\alpha^n_{n+1}}$ and ${'\beta^n_{n+1}}$ are compact it follows that $\widetilde{f\theta}_{n-1}$ can be approximated by elements belonging to $H^b_f(G'_{n+1}, H_f, o(E'_{n+1},P))$ and hence $f|_{F'_{n-2}}$ can be approximated by elements belonging to $\operatorname{Ker}\widehat{j'_{n+1}}$. Thus (3.2) satisfies ($\mathrm{ML}_2$) with $k = 2$. Finally, since $H(E',P) = H_f(E',P) = \varinjlim H^b_f(E'_n,P)$ it follows that R_P is surjective.

(**ii**) With notations in (i). By the hypothesis we assume that the $\beta^n_{n+1} : G_{n+1} \to G_n$ are compact and factorized through the space $C(S)$. This implies that there exists a diagram as in (i). Considering the complex of locally convex spaces

$$0 \longrightarrow \{\mathrm{Ker}\, \widehat{j'_n}\} \xrightarrow{\{\widehat{j'_n}\}} \{H_f(F'_n, P)\} \xrightarrow{\{e'_n\}} \{H_f(E'_n, P)\} \longrightarrow 0 \tag{3.3}$$

as in (i) we give the surjectivity of R_P. The Theorem is proved. ∎

Remark. There exists a locally projective Schwartz (resp. a C-local Schwartz) space which is not nuclear.

In fact, let $\{a_i\}$ be a sequence of positive numbers decreasing to zero such that $\sum_{i\geq 1} a_i = \infty$. For each n define a continuous norm q_n on ℓ^1 (resp. on C_0) by the formula

$$q_n(\{x_i\}) = \sum_{i\geq 1} |a_i^{2-n}\, x_i| \qquad (\text{resp. } q_n(\{x_i\}) = \sup |a_i^{2-n}\, x_i|)\ .$$

Let F denote the ℓ^1 (resp. C_0) equipped with the norms q_n. Since $a_i \searrow 0$ it is easy to see that F is locally projective Schwartz (resp. C-local Schwartz) which is not nuclear because of the relation $\sum_{i\geq 1} a_i = \infty$.

ACKNOWLEDGEMENT – The authors would like to thank Prof. Nguyen Van Khue for helpful advises and suggestions in writing this paper. At the same time, we would like to express our thanks to referees for their helpful remarks.

REFERENCES

[**1**] Bellenot, F. – Factorization of compact operators and finite representability of Banach spaces with applications to Schwartz spaces, *Studia Math.,* 62 (1978), 273–286.

[**2**] Baerstein, A. – Representation of holomorphic functions by boundary integrals, *Trans. Amer. Math. Soc.,* 60 (1971), 27–37.

[**3**] Gejler, W. – Extending and lifting continuous linear mappings in topological vector spaces, *Studia Math.,* 62 (1975), 295–303.

[**4**] Kelley, J.L. – Banach spaces with the extension property, *Trans. Amer. Math. Soc.,* 72 (1952), 323–326.

[**5**] Lindenstrauss, J. – *The extension of Compact Operators I.*, Technical Note, 28, Jerusalem, 1962.

[**6**] Meise, R. and Vogt, D. – Extension of entire functions on nuclear locally convex spaces, *Proc. Amer. Math. Soc.,* 92 (1984), 495–500.

[**7**] NGUYEN VAN KHUE – Lifting uniformly continuous maps with values in nuclear Fréchet spaces, *Ann. Polon. Math.,* 50 (1990), 291–297.

[**8**] NGUYEN VAN KHUE – On the extension of continuous linear maps in function space and the splitness of Dolbeault complexes of holomorphic Banach bundles, *Studia Math.,* 75 (1982), 69–80.

[**9**] NGUYEN VAN KHUE – On the extension of holomorphic maps with values in Fréchet spaces, *Ann. Polon. Math.,* 44 (1984), 163–175.

[**10**] PALAMODOV, P.V. – Homology methods in theory of locally convex spaces, *Uspekhi Math. Nauk,* 1 (1971), 3–75.

[**11**] PIESTCH, A. – *Nuclear Locally Convex Spaces*, Akademie Verlag Berlin, 1972.

[**12**] SCHAEFER, H. – *Topological Vector Spaces*, New York, 1966.

[**13**] SEMADENI, Z. – *Banach Spaces of Continuous Functions I*, Warszawa, 1971.

[**14**] VOGT, D. – On two classes of (F)-spaces, *Arch. Math.,* 54 (1985), 255–266.

[**15**] VOGT, D. – On the functors $\mathrm{Ext}^1(E,F)$ for Fréchet spaces, *Studia Math.,* 85 (1987), 165–197.

Le Mau Hai and Pham Hien Bang,
Department of Mathematics, Pedagogical Institute 1 Hanoi,
Tu Liem, Hanoi – VIETNAM