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A NEW EXTENSION OF KOMLÓS' THEOREM IN INFINITE DIMENSIONS. APPLICATION: WEAK COMPACTNESS IN L^1_X

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Abstract: A new version of Komlós' theorem in infinite dimensions is presented. It gives a different approach to well-known theorems (C. Castaing [11], A. Ülger [30] and J. Diestel–W.M. Ruess–W. Schachermayer [17]).

1 – Introduction

The purpose of this paper is to prove Komlós' theorem for bounded sequences in L^1_X (X being a separable Banach space) that satisfy a tightness condition formulated in the manner of A. Ülger [30]. We obtain our result (Theorems 1 and 2) by combining truncation arguments and the diagonal process based on successively applying the scalar version of Komlós' theorem, via the biting lemma. Corollaries 3–6 are special cases of Theorems 1–2 and include results of E.J. Balder [5].

As applications, we recover directly from Corollary 3, by using the Lebesgue–Vitali theorem, a relative weak compactness criterion of C. Castaing [[11], Th. 4.1, p. 2.14]: (Theorem 7). More generally, we obtain, from Theorem 1, a general weak compactness criterion (Theorem 8): the relative weak compactness in L^1_X is characterized by the uniform integrability and the tightness condition used in Theorem 1. This result involves recent criterions obtained by A. Ülger [30] and J. Diestel–W.M. Ruess–W. Schachermayer [17].

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2 – Notations, definitions and background

Throughout this paper the triple $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Without loss of generality we assume $\mu(\Omega) = 1$. $(X, |\cdot|)$ is a separable Banach space. The norm topology on X will be referred to by the symbol s. The weak topology $\sigma(X, X^*)$ on X will be indicated similarly by the symbol w. Here X^* stands for the topological dual of X, the duality between X and X^* is denoted by $\langle\ ,\ \rangle$. The (prequotient) space of all X-valued Bochner-integrable functions will be denoted by L^1_X.

Recall (see [23]) that the dual of L^1_X is the (quotient) space $L^\infty_{X^*}[X]$ of scalarly measurable bounded functions from Ω into X^*. We denote by $L^0_{X^*}[X]$ the (prequotient) space of scalarly measurable functions from Ω into X^*.

Also recall that a subset H of L^1_X is uniformly integrable (briefly UI) if

$$\lim_{t\to\infty} \sup_{u\in H} \int_{[|u|>t]} |u|\, d\mu = 0 \ .$$

It is well known that H is UI if it is bounded (i.e. $\sup_{u\in H} \int_\Omega |u|\, d\mu < +\infty$) and

$$\lim_{\mu(A)\to 0} \sup_{u\in H} \int_A |u|\, d\mu = 0 \ .$$

Definition. A set H of measurable functions is w-*tight* (resp. s-*tight*) if $\forall \epsilon > 0$, there exists a measurable multifunction (for measurability of multifunctions see [13]) with w-compact (resp. s-compact) values K_ϵ such that

$$\forall\, u \in H, \ \ \mu\Big(\Big\{\omega \in \Omega \colon\ u(\omega) \notin K_\epsilon(\omega)\Big\}\Big) \le \epsilon \ .$$

The following equivalent formulation of tightness is given in [4].

Definition. A set H of measurable functions is w-*tight* (resp. s-*tight*) if there exists an $\mathcal{F}\otimes\mathcal{B}$-measurable integrand $h\colon\ \Omega\times X \to [0, +\infty]$ which is w-inf-compact (resp. s-inf-compact) and such that

$$(\ddagger) \qquad \sup_{u\in H} \int_\Omega h(\omega, u(\omega))\, d\mu < +\infty \ .$$

This notion of tightness goes back to 1979 (see C. Castaing [10] and K.T. Andrews [2]). For more on tightness see [1], [7], [24], [27]. Note that every bounded subset of L^1_X is w-tight whenever X is reflexive.

Remark. In the first formulation of tightness the multifunctions K_ϵ can be assumed to be convex valued but in the second formulation its not always possible to replace h by a <u>convex</u> inf-compact function: the following is due to M. Valadier. Let $d\omega$ be the Lebesgue measure on [0, 1].

Proposition. *Let X be a Banach space containing a bounded sequence $(e_n)_{n\in\mathbb{N}^*}$ which does not admit any w-convergent subsequence. Then there exists a bounded set H in $L^1_X([0,1],d\omega)$ which is s-tight and such that there exists no measurable integrand $h\colon [0,1]\times X\to[0,+\infty]$ which is convex and w-inf-compact in x and which satisfies (‡).*

Proof: We may suppose without loss of generality that $m\neq n \Rightarrow me_m \neq ne_n \neq 0$. Let us consider $H:=\{ne_n 1_A:\ n\in\mathbb{N}^*,\ \mu(A)=1/n\}\cup\{0\}$. Then H is bounded in L^1_X. Let h_0 be defined by $h_0(0)=0$, $h_0(ne_n)=n$ for $n\geq 1$ and $h_0(x)=+\infty$ elsewhere. Thus h_0 is s-inf-compact and satisfies

$$\sup_{u\in H}\int_{[0,1]}(h_0\circ u)(\omega)\,d\omega\leq 1\ ,$$

hence H is s-tight.

Now suppose that $h\colon [0,1]\times X\to[0,+\infty]$ is any convex weakly inf-compact integrand satisfying (‡). Then $h(\omega,e_n)\to+\infty$ because otherwise there would exist $M<+\infty$ and a subsequence $(e_{n_k})_k$ such that $\sup_k h(\omega,e_{n_k})\leq M$ and the sequence $(e_{n_k})_k$ would admit a w-convergent further subsequence. If $u_n = ne_n 1_{A_n}\in H$,

$$\int_{[0,1]}h(\omega,u_n(\omega))\,d\omega\geq\int_{A_n}h(\omega,ne_n)\,d\omega\ .$$

It is possible to choose A_n such that $\int_{[0,1]}h(\omega,u_n(\omega))\,d\omega\to+\infty$. Indeed, since $0\in H$, one has $h(\omega,0)<+\infty$ a.e. By convexity $h(\omega,ne_n)\geq n[h(\omega,e_n)-(1-\frac{1}{n})\,h(\omega,0)]\geq n[h(\omega,e_n)-h(\omega,0)]$. Then by Egorov's theorem, there exists a Borel subset B of $[0,1]$ such that $\mu(B)\geq 1/2$ and such that the sequence $(h(.,e_n)-h(.,0))_n$ converges uniformly to $+\infty$ on B. It remains to choose, for $n\geq 2$, A_n contained in B. ∎

Let us introduce the following notion of convergence [7].

Definition. Let $(u_n)_{n\in\mathbb{N}^*\cup\{\infty\}}$ be a sequence of measurable functions from Ω into X and Y be a subset of X^*.

- $(u_n)_n$ is said to *K-converge* (resp. $\sigma(X,Y)$*-K-converge*) almost everywhere on Ω to u_∞, if for every subsequence $(u_{n_i})_i$ of $(u_n)_n$ there exists a null set

$N \in \mathcal{F}$ such that for every $\omega \in \Omega \backslash N$,

$$\frac{1}{n}\sum_{i=1}^{n} u_{n_i}(\omega) \to u_\infty(\omega)$$

$$\left(\text{resp. } \forall x^* \in Y, \ \left\langle x^*, \frac{1}{n}\sum_{i=1}^{n} u_{n_i}(\omega)\right\rangle \to \langle x^*, u_\infty(\omega)\rangle\right) ;$$

– $(u_n)_n$ is said to *K-converge in measure* to u_∞, if for every subsequence $(u_{n_i})_i$ of $(u_n)_n$

$$\frac{1}{n}\sum_{i=1}^{n} u_{n_i} \to u_\infty \quad \text{in measure} .$$

Remark. Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{R}$ and $a \in \mathbb{R}$. Then, $(a_n)_n$ K-converges to a (consider each a_n as a constant function from Ω into $\mathbb{R}$ in the definition above) is equivalent to $a_n \to a$.

A celebrated discovery of Komlós [25] is as follows:

Theorem. *Let $(u_n)_n$ be a bounded sequence of integrable functions from Ω into $\mathbb{R}$. Then $(u_n)_n$ has a subsequence which K-converges a.e. to an integrable function.*

This result has received a new proof by S.D. Chatterji [14], (see also [29] for a short proof).

3 – Generalized Komlós' theorem

The main result of this paper is the following theorem under a w-tightness assumption.

Theorem 1. *Let $(u_n)_n$ be a bounded sequence in L^1_X such that the following tightness condition holds*

T_1: *Given any subsequence $(u_{k_i})_i$ of $(u_n)_n$, there exists a w-tight sequence $(v_n)_n$ with $v_n \in co\{u_{k_i} \colon i \geq n\}$.*

Let D be a countable $\sigma(X^*, X)$-dense subset of the unit ball B^* of X^*. Then there exist a function u_∞ in L^1_X and a subsequence $(u'_n)_n$ of $(u_n)_n$ such that

(a) *$(u'_n)_n$ $\sigma(X, D)$-K-converges a.e. to u_∞;*

($\mathbf{b}_1$) *every w-tight sequence* $(v_n)_n$, *with* $v_n \in co\{u'_i\colon\ i \geq n\}$ *has a subsequence* $(v'_n)_n$ *such that*

$$(v'_n)_n \quad \sigma(X,D)\text{-}K\text{-converges a.e. to } \ u_\infty\ ,$$
$$\forall\, h \in L^0_{X^*}[X]\,, \qquad (\langle h, v'_n\rangle)_n \quad K\text{-converges in measure to } \ \langle h, u_\infty\rangle\ .$$

Proof: The proof will be given in three steps.

Step 1 – *Existence of* $(u'_n)_n$: by the biting lemma (see [18], a proof of the biting lemma is reproduced in [26]), there exist an increasing sequence of measurable sets A_p with $\lim_{p\to\infty} \mu(A_p) = 1$ and a subsequence $(u'_n)_n$ of $(u_n)_n$ such that

$$\forall\, p, \quad \text{the sequence } (1_{A_p} u'_n)_n \text{ is UI}\ . \tag{1}$$

On the other hand, using Komlós' theorem and the diagonal sequence process, we obtain the existence of integrable functions α_{x^*}, $(x^* \in D)$, from Ω into $\mathbb{R}$, and of a subsequence of $(u'_n)_n$, which we may still denote by $(u'_n)_n$, such that for every further subsequence $(u'_{n_i})_i$ of $(u'_n)_n$

$$\forall\, x^* \in D\,, \qquad \frac{1}{n}\sum_{i=1}^{n} \langle x^*, u'_{n_i}(\omega)\rangle \to \alpha_{x^*}(\omega) \quad \text{a.e.} \tag{2}$$

Let $(v_n)_n$ be an arbitrary fixed and w-tight sequence, with $v_n \in co\{u'_i\colon\ i \geq n\}$. Then, for each $q \in \mathbb{N}^*$, there exists a measurable multifunction with w-compact values K_q such that

$$\forall\, n\,, \qquad \mu(\Omega \backslash B_{n,q}) \leq \frac{1}{q}\ , \tag{3}$$

where

$$B_{n,q} := \Big\{\omega \in \Omega\colon\ v_n(\omega) \in K_q(\omega)\Big\}\ .$$

The same diagonal process used above gives for the sequences $(\langle x^*, v_n\rangle)_n$ and $(\langle x^*, 1_{B_{n,q}} v_n\rangle)_n$, the existence of integrable functions β_{x^*} and $\beta_{x^*,q}$ ($x^* \in D$, $q \in \mathbb{N}^*$), and of a subsequence $(v'_n)_n$ of $(v_n)_n$ such that for every further subsequence $(v'_{n_i})_i$ of $(v'_n)_n$

$$\forall\, x^* \in D\,, \qquad \frac{1}{n}\sum_{i=1}^{n} \langle x^*, v'_{n_i}(\omega)\rangle \to \beta_{x^*}(\omega) \quad \text{a.e.}\ , \tag{4}$$

$$\forall\, x^* \in D, \ \ \forall\, q \in \mathbb{N}^*\ , \qquad \frac{1}{n}\sum_{i=1}^{n} \langle x^*,\, 1_{B'_{n_i,q}} v'_{n_i}(\omega)\rangle \to \beta_{x^*,q}(\omega) \quad \text{a.e.}\ , \tag{5}$$

where for each $q \in \mathbb{N}^*$, $(B'_{n,q})_n$ is the subsequence of $(B_{n,q})_n$ corresponding to $(v'_n)_n$.

By (1), (2), (4) and Vitali's theorem we have, for every $x^* \in D$, every p, every $A \in A_p \cap \mathcal{F}$, every subsequence $(u'_{n_i})_i$ of $(u'_n)_n$ and every subsequence $(v'_{n_i})_i$ of $(v'_n)_n$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \int_A \langle x^*, u'_{n_i}\rangle \, d\mu = \int_A \alpha_{x^*} \, d\mu$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \int_A \langle x^*, v'_{n_i}\rangle \, d\mu = \int_A \beta_{x^*} \, d\mu \ .$$

Hence, by using the remark given in section 2, we get, for every $x^* \in D$, every p and every $A \in A_p \cap \mathcal{F}$,

$$\lim_{n\to\infty} \int_A \langle x^*, u'_n\rangle \, d\mu = \int_A \alpha_{x^*} \, d\mu \ , \tag{6}$$

$$\lim_{n\to\infty} \int_A \langle x^*, v'_n\rangle \, d\mu = \int_A \beta_{x^*} \, d\mu \ . \tag{7}$$

Each v'_n is of the form

$$v'_n = \sum_{i\le J_n} \lambda_i \, u'_{i+n}, \qquad \text{with } \ \lambda_i \ge 0 \ \text{ and } \ \sum_{i\le J_n} \lambda_i = 1 \ .$$

Using (6), (7) and this expression of v'_n, one can easily see that for every $x^* \in D$, every p and every $A \in A_p \cap \mathcal{F}$,

$$\int_A \alpha_{x^*} \, d\mu = \int_A \beta_{x^*} \, d\mu \ .$$

Therefore, since D is countable and $\lim_{p\to\infty} \mu(A_p) = 1$, we deduce that for every $x^* \in D$

$$\alpha_{x^*}(\omega) = \beta_{x^*}(\omega) \quad \text{a.e.} \tag{8}$$

Step 2 – *Existence of* u_∞: we define for all $n \in \mathbb{N}^*$ and $q \in \mathbb{N}^*$ the following partial sums

$$S'_n := \frac{1}{n} \sum_{i=1}^{n} v'_i \quad \text{ and } \quad S'_{n,q} := \frac{1}{n} \sum_{i=1}^{n} 1_{B'_{i,q}} \, v'_i \ .$$

1) Let $q \in \mathbb{N}^*$. We may assume that $K_q(\omega)$ is convex and contains 0 for each $\omega \in \Omega$ by considering the corresponding multifunction $\overline{co}(K_q(.) \cup \{0\})$, thanks to Krein and Eberlein–Šmulian's theorem [[22], Th. 19 E]. The measurability of this new map follows from [[13], Th. III.40] and [[21], Remark (1), p. 163].

Since $S'_{n,q}(\omega)$ belongs to the w-compact set $K_q(\omega)$ for all $n \in \mathbb{N}^*$ and all $\omega \in \Omega$, there exists a subsequence $(S'_{n_i,q}(\omega))_i$ of $(S'_{n,q}(\omega))_n$ (possibly depending upon ω), which w-converges to some limit point $x_{q,\omega}$. But, because of (5) the entire sequence $(S'_{n,q}(\omega))_n$ w-converges to the same limit point whenever ω is outside some negligible set N. Thus, we obtain

$$\forall\, x^* \in D, \ \ \forall\, \omega \in \Omega \backslash N\,, \qquad \beta_{x^*,q}(\omega) = \langle x^*, x_{q,\omega} \rangle \quad \text{a.e.} \tag{9}$$

Define $u_q^\infty(\omega) := x_{q,\omega}$ for $\omega \in \Omega\backslash N$ and $u_q^\infty(\omega) := 0$ for $\omega \in N$. Then it is easily seen by the Pettis measurability theorem [[16], Th. 2, II] that u_q^∞ is measurable.

2) Let $h \in L^\infty_{X^*}[X]$. Since the sequence $(\langle h, 1_{A_p} S'_{n,q}\rangle)_n$ is UI and converges a.e. to $\langle h, 1_{A_p} u_q^\infty\rangle$ for all p and all $q \in \mathbb{N}^*$, the Lebesgue–Vitali theorem gives

$$\int_{A_p} \Big| \langle h, S'_{n,q}\rangle - \langle h, u_q^\infty \rangle \Big|\, d\mu \to 0\ , \tag{10}$$

hence

$$\forall\, p, \ \ \forall\, q \in \mathbb{N}^*\,, \qquad 1_{A_p}\, S'_{n,q} \to 1_{A_p}\, u_q^\infty \qquad \text{weakly in } \ L^1_X\ .$$

Furthermore, $\forall\, p$, $\forall\, q \in \mathbb{N}^*$,

$$\begin{aligned} \sup_n \int_{A_p} |S'_n - S'_{n,q}|\, d\mu &\le \sup_n \frac{1}{n} \sum_{i=1}^n \int_{A_p \cap (\Omega \backslash B'_{i,q})} |v'_i|\, d\mu \\ &\le \sup_n \int_{A_p \cap (\Omega\backslash B'_{n,q})} |v'_n|\, d\mu \\ &\le \sup_n \sup_m \int_{A_p \cap (\Omega\backslash B'_{n,q})} |u'_m|\, d\mu\ . \end{aligned}$$

As (1) and (3) ensure

$$\forall\, p\,, \qquad \lim_{q\to\infty} \sup_n \sup_m \int_{A_p \cap (\Omega\backslash B'_{n,q})} |u'_m|\, d\mu = 0\ ,$$

we obtain

$$\forall\, p\,, \qquad \lim_{q\to\infty} \sup_n \int_{A_p} |S'_n - S'_{n,q}|\, d\mu = 0\ . \tag{11}$$

Consequently, by Grothendieck's weak relative compactness lemma [[20], ch. 5, 4, n° 1], for each p, the sequence $(1_{A_p} S'_n)_n$ is relatively weakly compact in L^1_X.

3) For each p choose a weak limit point $u_{\infty,p}$ of the sequence $(1_{A_p} S'_n)_n$. Then, by (7), we obtain $\forall\, x^* \in D$, $\forall\, A \in A_p \cap \mathcal{F}$,

$$\begin{aligned}\int_A \beta_{x^*}\, d\mu &= \lim_{n\to\infty} \int_A \langle x^*, v'_n\rangle\, d\mu\\ &= \lim_{n\to\infty} \int_A \langle x^*, S'_n\rangle\, d\mu\\ &= \int_A \langle x^*, u_{\infty,p}\rangle\, d\mu\ ,\end{aligned}$$

hence

$$\forall\, x^* \in D\,, \qquad \beta_{x^*}(\omega) = \langle x^*, u_{\infty,p}(\omega)\rangle \quad \text{a.e. in} \ \ A_p\ .$$

Since $\lim_{p\to\infty} \mu(A_p) = 1$, there exists a measurable function u_∞ defined on all Ω such that

$$\forall\, x^* \in D\,, \qquad \beta_{x^*}(\omega) = \langle x^*, u_\infty(\omega)\rangle \quad \text{a.e.} \tag{12}$$

Moreover, by (4) and (12), we have

$$\forall\, x^* \in D\,, \qquad \langle x^*, S'_n(\omega)\rangle \to \langle x^*, u_\infty(\omega)\rangle \quad \text{a.e.}$$

which implies

$$|u_\infty(\omega)| \le \liminf_{n\to\infty} |S'_n(\omega)| \quad \text{a.e.}$$

and hence

$$\int_\Omega |u_\infty|\, d\mu \le \int_\Omega \liminf_{n\to\infty} |S'_n|\, d\mu\ .$$

By Fatou's lemma we get

$$\int_\Omega |u_\infty|\, d\mu \le \sup_n \int_\Omega |S'_n|\, d\mu \le \sup_n \int_\Omega |u_n|\, d\mu < +\infty$$

and hence $u_\infty \in L^1_X$.

Step 3 – $(u'_n)_n$ and u_∞ have the required properties.

1) By combining (2), (4), (8) and (12), we obtain

$$(u'_n)_n \ \text{ and } \ (v'_n)_n \quad \sigma(X,D)\text{-K-converge a.e. to } \ u_\infty\ . \tag{13}$$

Furthermore, from (5) and (9) we deduce that

$$\forall\, q \in \mathbb{N}^*\,, \qquad (1_{B'_{n,q}} v'_n)_n \ \ \sigma(X,X^*)\text{-K-converges a.e. to } u_q^\infty\ . \tag{14}$$

Fix h in $L^0_{X^*}[X]$. We shall show that $(\langle h, v'_n\rangle)_n$ K-converges in measure to $\langle h, u_\infty\rangle$. By considering the function $h/(1+|h|)$, we may suppose that $|h|_\infty < 1$. Let $(v'_{n_i})_i$ be a subsequence of $(v'_n)_n$. By (13) and (14), we have

$$\forall\, x^* \in D\,, \qquad \langle x^*, S''_n(\omega)\rangle \to \langle x^*, u_\infty(\omega)\rangle \quad \text{a.e.}$$

and

$$\forall\, x^* \in D, \quad \forall\, q \in \mathbb{N}^*\,, \qquad \langle x^*, S''_{n,q}(\omega)\rangle \to \langle x^*, u^\infty_q(\omega)\rangle \quad \text{a.e.}$$

where

$$S''_n := \frac{1}{n}\sum_{i=1}^n v'_{n_i} \quad \text{and} \quad S''_{n,q} := \frac{1}{n}\sum_{i=1}^n 1_{B'_{n_i,q}}\, v'_{n_i} \qquad (n \in \mathbb{N}^*)\ .$$

Then

$$\forall\, x^* \in D, \quad \forall\, q \in \mathbb{N}^*\,, \qquad \Big|\Big\langle x^*,\, S''_n(\omega) - S''_{n,q}(\omega)\Big\rangle\Big| \to \Big|\Big\langle x^*,\, u_\infty(\omega) - u^\infty_q(\omega)\Big\rangle\Big| \quad \text{a.e.}$$

Hence for all $q \in \mathbb{N}^*$ and for a.e. ω we have

$$\begin{aligned}\Big|u_\infty(\omega) - u^\infty_q(\omega)\Big| &= \sup_{x^*\in D}\Big|\Big\langle x^*,\, u_\infty(\omega) - u^\infty_q(\omega)\Big\rangle\Big| \\ &\le \liminf_n \Big|S''_n(\omega) - S''_{n,q}(\omega)\Big|\ .\end{aligned}$$

This inequality implies that $\forall\, n \in \mathbb{N}^*$, $\forall\, p$ and $\forall\, q \in \mathbb{N}^*$

$$\begin{aligned}\int_{A_p}\Big|\langle h, S''_n\rangle - \langle h, u_\infty\rangle\Big|\, d\mu &\le \int_{A_p}\Big|\langle h, S''_n\rangle - \langle h, S''_{n,q}\rangle\Big|\, d\mu + \int_{A_p}\Big|\langle h, S''_{n,q}\rangle - \langle h, u^\infty_q\rangle\Big|\, d\mu \\ &\quad + \int_{A_p}\Big|\langle h, u^\infty_q\rangle - \langle h, u_\infty\rangle\Big|\, d\mu \\ &\le \int_{A_p}|S''_n - S''_{n,q}|\, d\mu + \int_{A_p}\Big|\langle h, S''_{n,q}\rangle - \langle h, u^\infty_q\rangle\Big|\, d\mu \\ &\quad + \int_{A_p}\liminf_n |S''_n - S''_{n,q}|\, d\mu\ .\end{aligned}$$

Then by Fatou's lemma, $\forall\, n \in \mathbb{N}^*$, $\forall\, p$ and $\forall\, q \in \mathbb{N}^*$,

$$\int_{A_p}\Big|\langle h, S''_n\rangle - \langle h, u_\infty\rangle\Big|\, d\mu \le 2\sup_n \int_{A_p}|S''_n - S''_{n,q}|\, d\mu + \int_{A_p}\Big|\langle h, S''_{n,q}\rangle - \langle h, u^\infty_q\rangle\Big|\, d\mu\ .$$

Using similar arguments to those used to prove (10) and (11), we obtain

$$\forall\, p, \quad \forall\, q \in \mathbb{N}^*\,, \qquad \lim_{n\to\infty}\int_{A_p}\Big|\langle h, S''_{n,q}\rangle - \langle h, u^\infty_q\rangle\Big|\, d\mu = 0$$

and

$$\forall\, p\,, \qquad \lim_{q\to\infty} \sup_n \int_{A_p} |S''_n - S''_{n,q}|\, d\mu = 0$$

hence

$$\forall\, p\,, \qquad \lim_{n\to\infty} \int_{A_p} \Big|\langle h, S''_n\rangle - \langle h, u_\infty\rangle\Big|\, d\mu = 0\ .$$

Let $\epsilon > 0$. For p and n sufficiently large, we have

$$\mu(\Omega\backslash A_p) < \frac{\epsilon}{2} \qquad \text{and} \qquad \int_{A_p} \Big|\langle h, S''_n\rangle - \langle h, u_\infty\rangle\Big|\, d\mu \le \frac{\epsilon^2}{2}\ .$$

Therefore, by Markov's inequality, we get

$$\mu\bigg(\Big\{\omega \in A_p\colon\ \Big|\langle h(\omega), S''_n(\omega)\rangle - \langle h(\omega), u_\infty(\omega)\rangle\Big| > \epsilon\Big\}\bigg) \le \frac{\epsilon}{2}$$

then

$$\mu\bigg(\Big\{\omega \in \Omega\colon\ \Big|\langle h(\omega), S''_n(\omega)\rangle - \langle h(\omega), u_\infty(\omega)\rangle\Big| > \epsilon\Big\}\bigg) \le \mu(\Omega\backslash A_p) + \frac{\epsilon}{2} < \epsilon\ .$$

Thus $\langle h, S''_n\rangle \to \langle h, u_\infty\rangle$ in measure.

2) Finally, if $(w_n)_n$ is any other w-tight sequence, with $w_n \in co\{u'_i :\ i \ge n\}$, then the same method applied to $(v_n)_n$ gives for $(w_n)_n$ the existence of a subsequence $(w'_n)_n$ of $(w_n)_n$ and of a function $v_\infty \in L^1_X$ such that

$$(u'_n)_n \ \text{ and } \ (w'_n)_n \quad \sigma(X,D)\text{-K-converge a.e. to } \ v_\infty$$

$$\forall\, h \in L^0_{X^*}[X]\,, \qquad (\langle h, w'_n\rangle)_n \quad \text{K-converges in measure to } \ \langle h, v_\infty\rangle\ .$$

Since $(u'_n)_n$ $\sigma(X,D)$-K-converges a.e. to u_∞, we get $v_\infty = u_\infty$ a.e., so the proof is complete. ∎

Now we give the version of Theorem 1 relative to the s-tightness assumption.

Theorem 2. *Let $(u_n)_n$ be a bounded sequence in L^1_X such that the following tightness condition holds*

T_2**:** *Given any subsequence $(u_{k_i})_i$ of $(u_n)_n$, there exists a s-tight sequence $(v_n)_n$ with $v_n \in co\{u_{k_i}\colon\ i \ge n\}$.*

Let D be a countable $\sigma(X^*,X)$-dense subset of the unit ball, B^*, of X^*. Then there exist a function u_∞ in L^1_X and a subsequence $(u'_n)_n$ of $(u_n)_n$ such that

(**a**) *$(u'_n)_n$ $\sigma(X,D)$-K-converges a.e. to u_∞;*

($\mathbf{b}_2$) *Every s-tight sequence $(v_n)_n$, with $v_n \in co\{u'_i : i \geq n\}$ has a subsequence $(v'_n)_n$ such that*

$$\begin{array}{ll} (v'_n)_n & \sigma(X,D)\text{-}K\text{-converges a.e. to } u_\infty \ , \\ (v'_n)_n & K\text{-converges in measure to } u_\infty \ . \end{array}$$

Proof: The proof is almost the same as the one given above. We will only consider the points which need to be modified. Since each multifunction K_q has s-compact values, the sequence $(S'_{n,q})_n$ s-converges a.e. to u_q^∞. Therefore, by Lebesgue–Vitali's theorem,

$$\forall\, p, \ \ \forall\, q\,, \qquad \int_{A_p} |S'_{n,q} - u_q^\infty|\, d\mu \to 0 \ .$$

Consequently, by virtue of (11), for each p the sequence $(1_{A_p} S'_n)_n$ is relatively strongly compact in L^1_X. Hence, $(S'_n)_n$ is precompact in measure. If u_∞ is a limit point of $(S'_n)_n$ with respect to the topology of convergence in measure, then by (4),

$$\forall\, x^* \in D\,, \qquad \beta_{x^*}(\omega) = \langle x^*, u_\infty(\omega)\rangle \quad \text{a.e.}$$

Therefore $(S'_n)_n$ converges in measure to u_∞. The remainder of the proof follows easily from the proof of Theorem 1. ∎

Every w-tight (resp. s-tight) sequence satisfies T_1 (resp. T_2). So we have the following

Corollary 3. *Let $(u_n)_n$ be a bounded sequence in L^1_X and D a countable $\sigma(X^*,X)$-dense subset of B^*. If $(u_n)_n$ is w-tight (resp. s-tight), then there exist a function u_∞ in L^1_X and a subsequence $(u'_n)_n$ of $(u_n)_n$ for which (a) and (b'_1) (resp. (a) and (b'_2)) hold, where*

($\mathbf{b}'_1$) $\forall\, h \in L^0_{X^*}[X]$, $(\langle h, u'_n\rangle)_n$ *K-converge in measure to* $\langle h, u_\infty\rangle$;

($\mathbf{b}'_2$) $(u'_n)_n$ *K-converges in measure to* u_∞.

The following result is due to E.J. Balder [[5], Th. B].

Corollary 4. *Assume that X is not necessarily separable. Let $(u_n)_n$ be a bounded sequence in L^1_X such that $\{u_n(\omega) : n \in \mathbb{N}\}$ is a.e. relatively w-compact. Then there exist a subsequence of $(u_n)_n$ that w-K-converges a.e. to a Bochner integrable function.*

Proof: By the Pettis measurability theorem, we may assume that X is separable. So by Corollary 3, there exist a subsequence of $(u_n)_n$ and a function in L^1_X for which (a) holds. Since $\{u_n(\omega)\colon\, n \in \mathbb{N}\}$ is a.e. relatively w-compact, we conclude, by Krein and Eberlein–Šmulian's theorem, that the convergence in (a) extends to all $x^* \in X^*$. ∎

Corollary 5. *Assume that X^* is strongly separable. Let $(u_n)_n$ be a bounded sequence in L^1_X such that the condition T_1 holds. Then there exist a subsequence of $(u_n)_n$ that w-K-converges a.e. to a Bochner integrable function.*

Proof: Let D be a countable strongly dense subset of B^*. Let $(u'_n)_n$ and u_∞ be as obtained in Theorem 1. By Komlós' theorem and by extracting a subsequence if necessary, we may suppose that the sequence $(|u'_n|)_n$ K-converges a.e. to an integrable function from Ω into $\mathbb{R}^+$. Hence, for every further subsequence $(u'_{n_i})_i$ of $(u'_n)_n$, the sequence $(\frac{1}{n}\sum_{i=1}^n u'_{n_i}(\omega))_n$ is a.e. bounded. Therefore, since $(u'_n)_n$ $\sigma(X,D)$-K-converges a.e. to u_∞ and D is dense, the sequence $(u'_n)_n$ w-K-converges a.e. to u_∞. ∎

From Corollary 5 we deduce the following result, which is due to E.J. Balder [[5], Th. A].

Corollary 6. *Assume that X is a reflexive Banach space (not necessarily separable). Let $(u_n)_n$ be a bounded sequence in L^1_X. Then there exist a subsequence of $(u_n)_n$ that w-K-converges a.e. to a Bochner integrable function.*

Proof: By the Pettis measurability theorem, we may assume that X is separable. Furthermore, since X is reflexive, X^* is strongly separable. And, by the boundedness assumption and reflexivity of X, the sequence $(u_n)_n$ is w-tight. Thus we return to the situation of Corollary 5. ∎

4 – Applications

As an immediate application of our results we have the following theorem, which is due to C. Castaing [[11], Th. 4.1, p. 2. 14], see also [[1], Th. 6].

Theorem 7. *Let H be a w-tight uniformly integrable subset of L^1_X. Then H is sequentially relatively weakly compact (and hence relatively weakly compact) in L^1_X.*

Proof: Let $(u_n)_n$ be a sequence in H. Let $(u'_n)_n$ and u_∞ be as obtained in Corollary 3. By UI and (b'_1), it follows from Lebesgue–Vitali's theorem that

$$\forall\, h \in L^\infty_{X^*}[X]\,, \qquad \Big(\int_\Omega \langle h, u'_n\rangle\, d\mu\Big)_n \quad \text{K-converges to} \quad \int_\Omega \langle h, u_\infty\rangle\, d\mu\ .$$

Hence, by using the remark given in section 2, we get

$$u'_n \to u_\infty \quad \text{weakly in } \ L^1_X\ .$$

Thus H is sequentially relatively weakly compact. Then, by Eberlein–Šmulian's theorem [[22], Th. 18, A], H is relatively weakly compact in L^1_X. ∎

Now we give a very general criterion for relative weak compactness in L^1_X.

Theorem 8. *Let H be a bounded subset of L^1_X. Then the following statements are equivalent:*

(**i**) *H is relatively weakly compact in L^1_X.*

(**ii**) *H is UI and every sequence in H satisfies T_1.*

(**iii**) *H is UI and every sequence in H satisfies T_2.*

Proof: Suppose (i) holds. Then H is UI (see [16], [9]). Let $(u_n)_n$ be a sequence in H. There exists a subsequence $(u'_n)_n$ of $(u_n)_n$ that converges weakly to some $u \in L^1_X$. Then, by Mazur's theorem, there exists a sequence $(v_n)_n$, with $v_n \in co\{u'_i\colon\, i \geq n\}$ such that $|v_n - u|_{L^1_X} \to 0$. Hence, there exists a subsequence $(v'_n)_n$ of $(v_n)_n$ that converges a.e. to u. Therefore $(v'_n)_n$ is s-tight. This proves (iii). The implication (iii)⇒(ii) is obvious. Now let us prove the main implication of the theorem (ii)⇒(i). Suppose (ii) holds. By Eberlein–Šmulian's theorem, it is enough to show that H is sequentially relatively weakly compact. Let $(u_n)_n$ be a sequence in H. Since $(u_n)_n$ is bounded and satisfies T_1, by Theorem 1, there exist a subsequence $(u'_n)_n$ of $(u_n)_n$ and a function $u_\infty \in L^1_X$ for which (b_1) holds. For a fixed h in $L^\infty_{X^*}[X]$, choose a subsequence $(u'_{k_i})_i$ of $(u'_n)_n$ such that

$$\lim_{i\to+\infty} \int_\Omega \langle h, u'_{k_i}\rangle\, d\mu = \limsup_n \int_\Omega \langle h, u'_n\rangle\, d\mu\ .$$

By T_1, there exists a w-tight sequence $(v_n)_n$, with $v_n \in co\{u'_{k_i} : i \geq n\} \subset co\{u'_i\colon i \geq n\}$. So, by (b_1), every subsequence of $(v_n)_n$ has a further subsequence $(v'_n)_n$ such that

$$(\langle h, v'_n\rangle)_n \quad \text{K-converges in measure to} \ \ \langle h, u_\infty\rangle$$

then

$$\Big(\int_\Omega \langle h, v'_n\rangle \, d\mu\Big)_n \quad \text{K-converges to} \quad \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

This is equivalent to

$$\int_\Omega \langle h, v'_n\rangle \, d\mu \ \rightarrow \ \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

Consequently,

$$\int_\Omega \langle h, v_n\rangle \, d\mu \ \rightarrow \ \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

Since each v_n is of the form

$$v_n = \sum_{i\le k_n} \lambda_i \, u'_{k_{i+n}}, \qquad \text{with} \ \ \lambda_i \ge 0 \ \ \text{and} \ \ \sum_{i\le k_n} \lambda_i = 1 \ ,$$

we have

$$\begin{aligned} \int_\Omega \langle h, u_\infty\rangle \, d\mu &= \lim_{n\to+\infty} \sum_{i\le k_n} \lambda_i \int_\Omega \langle h, u'_{k_{i+n}}\rangle \, d\mu \\ &= \lim_{i\to+\infty} \int_\Omega \langle h, u'_{k_i}\rangle \, d\mu \ . \end{aligned}$$

Then

$$\limsup_n \int_\Omega \langle h, u'_n\rangle \, d\mu = \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

Similarly, we obtain

$$\liminf_n \int_\Omega \langle h, u'_n\rangle \, d\mu = \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

Hence

$$\lim_{n\to+\infty} \int_\Omega \langle h, u'_n\rangle \, d\mu = \int_\Omega \langle h, u_\infty\rangle \, d\mu \ .$$

This proves (i). ∎

Remark 1. Theorem 8 allows to recover a recent criterion for relative weak compactness, which is due to A. Ülger [30] and J. Diestel–W.M. Ruess–W. Schachermayer [17] (by the Pettis measurability theorem, the hypothesis "X is separable" can be relaxed). In [17] the proof is based on the characterization of the weak compactness by the iterated limit condition (see [[22], Lemma 1, 19 A]).

Remark 2. Considering constant functions in X, we recover also A. Ülger's lemma [[30], Lemma 2.1]. Recall that this author proved his result by using James's characterization of weak compactness via norm-attaining functionals.

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