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FINITE MORPHISMS BETWEEN DIFFERENTIABLE ALGEBRAS

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Abstract: We characterize, given a morphism $A \to B$ between differentiable algebras, when B is a finite A-module by properties of the continuous map $Y \to X$ induced between the respective spectra and properties of the fibre rings $B/\mathfrak{m}_x B$.

1 – Introduction

In this paper we show that a finite morphism between diferentiable algebras (quotients of $\mathcal{C}^\infty(\mathbb{R}^n)$ by closed ideals) $A \to B$ induces a proper map between the topological spaces of its maximal ideals with residue field $\mathbb{R}$. But we show the more important fact that, with another property concerning rings of germs of functions, we can characterize when B is a finite A-module.

The property of finiteness of a homomorfism between rings of functions is classical in the study of aplications between spaces. For example, the Main Theorem of Zariski states that a proper morphism of finite fibres between algebraic varieties induces a finite morphism between their coordinate rings. A similar relation is found in Mulero [6]; for a proper map of finite fibres between topological spaces, the induced morphism between the algebras of continuous functions is finite and flat.

To reach the caracterization, we need some results about localization of topological algebras [7], the notion of fibre ring, and as the key step the Malgrange preparation theorem [4].

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2 – Previous results

Let A be a topological $\mathbb{R}$-algebra, i.e. a $\mathbb{R}$-algebra whith a topology such that A is a topological $\mathbb{R}$-vector space, the ring multiplication $A \times A \dot{\to} A$ is continuous, and the map $a \rightsquigarrow a^{-1}$ is continuous.

The *topological spectrum* $\mathrm{Spec}_t A$ is defined to be the set of all continuous morphisms of algebras $x\colon A \to \mathbb{R}$. Each element $a \in A$ defines a map

$$\begin{aligned} a\colon \mathrm{Spec}_t A &\to \mathbb{R} \\ x &\rightsquigarrow a(x) = x(a) \ . \end{aligned}$$

The *Gelfand topology* in $\mathrm{Spec}_t A$ is the initial topology of these maps.

A Fréchet $\mathbb{R}$-algebra is a $\mathbb{R}$-algebra with a Hausdorff topology defined by a numerable family of multiplicative seminorms. It can be shown [2] that a Fréchet $\mathbb{R}$-algebra is a topological algebra. As an usual example, let $A = \mathcal{C}^r(\mathbb{R}^n)$ denotes the algebra of all real-valued functions on $\mathbb{R}^n$ of class r, endowed with the topology of uniform convergence on compact sets of the function and their derivatives up to order r $(r = 1, 2, ..., \infty)$. Then A is a Fréchet $\mathbb{R}-$algebra and it can be shown that $\mathrm{Spec}_t A$ is homeomorphic to $\mathbb{R}^n$ ([7], [5]).

In a topological $\mathbb{R}$-algebra A, each point $x \in \mathrm{Spec}_t A$ defines a closed maximal ideal $\mathfrak{m}_x = \{a \in A\colon a(x) = 0\}$ of A. There is a bijection between $\mathrm{Spec}_t A$ and the set of closed maximal ideals of residue field $\mathbb{R}$. A topological $\mathbb{R}$-algebra A is said to be *rational* if any continuous $\mathbb{R}$-algebra morphism $A \to \mathbb{C}$ is $\mathbb{R}$-valued. Equivalently, A is rational if every closed maximal has real residue field.

The localization or ring of fractions of A with respect to the multiplicative closed system $S_x = \{a \in A\colon a(x) \neq 0\}$ will be denoted by $A_x = \{\frac{a}{s}\colon s \in S_x\}$, and a_x will be the fraction $\frac{a}{1}$ for any $a \in A$. It must be understood as the germ of a at x. In fact, for any open subset U in $\mathrm{Spec}_t A$, we denote by A_U the localization of A by $S_U = \{a \in A\colon a(x) \neq 0, \text{ for any } x \in U\}$, so we obtain a presheaf of algebras on $\mathrm{Spec}_t A$, and A_x is just the stalk at x of the associated sheaf $\widetilde{A}$. This is the *sheaf of localization*.

A is said to be *strictly regular* if for any closed Y in $\mathrm{Spec}_t A$ and any point $x \notin Y$, there exists $a \in A$ such that $a(x) \neq 0$ and $a_y = 0$ for any $y \in Y$. This is stronger than the classical hypothesis of regularity (that there are $a \in A$ with $a(x) \neq 0$ and $a(y) = 0$ for any $y \in Y$).

The algebra $\mathcal{C}^r(\mathbb{R}^n)$, and any quotient of it by a closed ideal, is a strictly regular rational Fréchet $\mathbb{R}$-algebra. In particular $\mathcal{C}^r(M)$ with M a differentiable manifold of class r (for we have M is a closed submanifold of $\mathbb{R}^n$ for some n). Besides $\mathrm{Spec}_t\, \mathcal{C}^r(M)$ is homeomorphic to M. In the example case, it can be shown that the sheaf of localization coincides with the sheaf of differentiable functions: $\widetilde{\mathcal{C}^r(M)}(U) = \mathcal{C}^r(U)$ with $U \subset M$ (see [7]).

3 – The fibre ring

Let $f: A \to B$ be a continuous homomorphism between topological $\mathbb{R}$-algebras. We can define the *corresponding continuous map* $\pi\colon \operatorname{Spec}_t B \to \operatorname{Spec}_t A$; for any $y \in \operatorname{Spec}_t B$, let $\pi(y)$ be such that $\mathfrak{m}_{\pi(y)} = f^{-1}(\mathfrak{m}_y) = \{a \in A\colon f(a) \in \mathfrak{m}_y\}$. It is the map defined by composition: $\pi(y) = y \circ f$ for any $y \in \operatorname{Spec}_t B$.

In our example, if $A = \mathcal{C}^r(M)$ and $B = \mathcal{C}^r(N)$ with M and N manifolds of class r, the corresponding continuous map between the spectra $\pi: N \to M$ is differentiable of the same class. Every differentiable aplication $\phi\colon N \to M$ is of this kind: ϕ is the corresponding continuous map of the homomorphism $A \to B$ defined by composition with ϕ.

If $x \in \operatorname{Spec}_t A$, we denote by B_x the localization of the A-module B by $S_x = \{a \in A\colon a(x) \neq 0\}$ and we say that $B/\mathfrak{m}_x B$ is the *fibre ring* at x. We can see that $B/\mathfrak{m}_x B = B_x/\mathfrak{m}_x B_x$ because $\mathfrak{m}_x B \supset \operatorname{Ker}(B \to B_x)$. If $x \in \operatorname{Spec}_t A$, it is $\pi^{-1}(x) \simeq \operatorname{Spec}_t(B_x/\mathfrak{m}_x B_x)$, because $y \in \pi^{-1}(x)$ when $\mathfrak{m}_y \supset \mathfrak{m}_x B$. There is a similar notion in commutative algebra ([1]).

Lemma. *Let $A \to B$ be a continuous morphism between rational strictly regular algebras. If the corresponding continuous map $\pi\colon \operatorname{Spec}_t B \to \operatorname{Spec}_t A$ is a closed map, and for any point $x \in \operatorname{Spec}_t A$ the fibre $\pi^{-1}(x) = \{y_1, \ldots, y_s\}$ is finite, then we have*

$$B_x = B_{y_1} \oplus \cdots \oplus B_{y_s} \ .$$

Proof: There is a morphism $B_x \to \varinjlim B_{\pi^{-1}(U)}$, where U runs over all open neighborhoods of x in $\operatorname{Spec}_t A$. In fact, for each element $s \in A$ with $s(x) \neq 0$, there is a neighborhood U with $s(x') \neq 0$ when $x' \in U$. Then $\frac{1}{s} \in B_x$ is sent to the same fraction in $B_{\pi^{-1}(U)}$, wich defines an element of $\varinjlim B_{\pi^{-1}(U)}$. First we prove that this natural morphism is an isomorphism.

If $\frac{b}{s} \in B_x$ vanishes in some $B_{\pi^{-1}(U)}$, then $b_y = 0$ for all $y \in \pi^{-1}(U)$. There exists $a \in A$ such that $a(x) \neq 0$ and $a_z = 0$ for all $z \in X - U$, by the strict regularity of A. Then $(a \cdot b)_y = 0$ for all $y \in \operatorname{Spec}_t B$ and, by 3.4(a) of [7] we obtain that $a \cdot b = 0$ in B. Since $a \in S_x$, we conclude that $\frac{b}{s} = 0$ in B_x, so that the morphism is injective. On the other hand, if $\frac{b}{s} \in B_{\pi^{-1}(U)}$, where $s \in S_{\pi^{-1}(U)} = \{c \in B\colon c(y) \neq 0 \text{ for any } y \in \pi^{-1}(U)\}$, by 4.4 of [7] there exists an open neighborhood U' of x such that $\frac{1}{s} = \frac{b'}{1}$ in $B_{\pi^{-1}(U')}$ for some $b' \in B$; hence $\frac{b}{s} = \frac{b \cdot b'}{1}$ in $\varinjlim B_{\pi^{-1}(U)}$; and $\frac{b}{s}$ comes from an element of B_x.

Since the map is closed and of finite fibres, it is easy to see that there are a neighbourhood $U \ni x$ and disjoint neighborhoods $V_i \ni y_i$, $i = 1, \ldots, s$, of each point $y_i \in \pi^{-1}(x)$, with $\pi^{-1}(U) = V_1 \bigcup \cdots \bigcup V_s$. Besides U can be chosen inside any given neighbourhood of x and $V_1 \cup \cdots \cup V_s$ inside any given open with $\pi^{-1}(x)$. The sheaf $\widetilde{B}$ coincides with the presheaf of localization in a basis $\mathcal{B}$ of neighborhoods ([7], 4.1). If $V, V' \in \mathcal{B}$, we have $B_{V \cup V'} = B_V \oplus B_{V'}$. We take the direct limit indexed by this basis of neighborhoods, and so $\varinjlim_{U \ni x} B_{\pi^{-1}(U)} = \varinjlim_{V_i \ni y_i} B_{V_1 \cup \cdots \cup V_s} = \varinjlim (B_{V_1} \oplus \cdots \oplus B_{V_s}) = \bigoplus_{i=1}^{s} B_{y_i}$. ∎

Lemma. *In the hypothesis of the former lemma, and if besides B_y is an $A_{\pi(y)}$-module of finite type for any $y \in \operatorname{Spec}_t B$, then a given finite family $b_1, \ldots, b_r$ generates the A-module B if and only if it generates the fibre ring $B/\mathfrak{m}_x B$ as a real vector space for any $x \in \operatorname{Spec}_t A$.*

Proof: If any B_{y_i} is assumed to be an A_x-module of finite type, then B_x is an A_x-module of finite type by the former lemma and, by Nakayama's lemma [1], we have that $(b_1)_x, \ldots, (b_r)_x$ generate the A_x-module B_x if and only if they generate the vector space $B_x/\mathfrak{m}_x B_x = B/\mathfrak{m}_x B$.

Finally, a standard argument with partitions of unity (which existence in our case is proved in [7]) shows that a given finite family $m_1, \ldots, m_r \in M$ of a Fréchet A-module M generates M if and only if $(m_1)_x, \ldots, (m_r)_x$ generate the A_x-module M_x for any $x \in \operatorname{Spec}_t A$. ∎

4 – Preparation Theorem

A real Fréchet algebra is said to be a *differentiable algebra* if it is isomorphic to $\mathcal{C}^\infty(\mathbb{R}^n)/I$, where I is a closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$. So, if M is a differentiable manifold, $\mathcal{C}^\infty(M)$ is a differentiable algebra. This concept is more general than that of algebra of differentiable functions on a manifold. For example $\mathcal{C}^\infty(\mathbb{R})/(x^2)$ is not an algebra of differentiable functions.

If $f\colon A \to B$ is a continuous morphism between differentiable algebras, and π is the corresponding continuous map, Malgrange's Preparation theorem ([4]) states that B_y is a finite A_x-module, $\pi(y) = x$, whenever $B_y/\mathfrak{m}_x B_y$ is a finite-dimensional vector space.

We have $B_y/\mathfrak{m}_x B_y = (B/\mathfrak{m}_x B)_y$, so that $B_y/\mathfrak{m}_x B_y$ is finite-dimensional whenever $B/\mathfrak{m}_x B$ so is.

Theorem. *Let $f\colon A \to B$ be a continuous morphism between differentiable algebras. B is A-module of finite type if and only if the corresponding continuous map $\pi : \operatorname{Spec}_t B \to \operatorname{Spec}_t A$ is proper and the fibre rings $B/\mathfrak{m}_x B$, $x \in A$, are finite-dimensional real vector spaces of bounded dimension.*

Proof: First, to prove the converse, we fix an isomorphism $B \simeq \mathcal{C}^\infty(\mathbb{R}^n)/I$, and we consider the coordinate functions $x_1, \dots, x_n \in \mathcal{C}^\infty(\mathbb{R}^n)$. Since $B/\mathfrak{m}_x B$ is a finite dimensional quotient algebra of $\mathcal{C}^\infty(\mathbb{R}^n)$, then any fibre $\pi^{-1}(x) = \operatorname{Spec}_t B/\mathfrak{m}_x B$ is finite and monomials in $x_1, \cdots, x_n$ generate the vector space $B/\mathfrak{m}_x B$ for any $x \in \operatorname{Spec}_t A$. Now, if $\dim(B/\mathfrak{m}_x B) \le d$, then in this fibre ring x_i^d is linearly dependent of $1, x_i, \dots, x_i^{d-1}$; hence any monomial is linearly dependent of monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_i < d$, and we have a finite family $b_1, \dots, b_s \in B$ with their images generating $B/\mathfrak{m}_x B$ for any $x \in \operatorname{Spec}_t A$. Besides, by the Malgrange theorem, we have that B_y is a finite A_x-module. By the above lemma we conclude that such monomials generate the A-module B.

If $f : A \to B$ is any morphism between conmutative rings and B is an A-module of finite type, it is well known [1] that the corresponding continuous map $\hat{\pi}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ between their prime spectrum (with the Zariski topology) is a closed map with finite fibres. Now in our particular case, this map $\hat{\pi}$ extend the map π between the topological spectra, and the condition of A and B being regular just states that $\operatorname{Spec}_t A$ and $\operatorname{Spec}_t B$ are topological sub-spaces of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ respectively, and $\hat{\pi}^{-1}(\operatorname{Spec}_t A) = \operatorname{Spec}_t B$ because B is rational. In fact, if $\mathfrak{p}$ is a prime ideal of B such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}_x$ for some $x \in \operatorname{Spec}_t A$, then $B/\mathfrak{p}$ is a quotient algebra of the algebra $B/\mathfrak{m}_x B$, so that $B/\mathfrak{p}$ is a finite extension of $\mathbb{R}$. B is rational, then we have $B/\mathfrak{p} \simeq \mathbb{R}$ and $\mathfrak{p} = \mathfrak{m}_y$ for some $y \in \operatorname{Spec}_t B$. We conclude that $\pi\colon \operatorname{Spec}_t B \to \operatorname{Spec}_t A$ is a closed map with finite fibres, hence it is proper [3]. ∎

As a corollary, if M, N are manifolds and $\phi : M \to N$ is a differentiable aplication proper and of bounded dimension of the fibre ring, the homomorphism $\mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ of composition with ϕ is finite.

If a differentiable aplication ϕ is proper and regular with $\dim M \le \dim N$, it can be shown that it verifies the above hypotesis. Then the homomorphism induced by this aplication is finite.

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