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EXISTENCE OF MINIMIZERS FOR SOME NON CONVEX ONE-DIMENSIONAL INTEGRALS *

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Abstract: We consider integrals of the type $\int_a^b \{h(u') + g(u)\}\,dx$, where h is a non-convex function such that $h^{**}(0) = h(0)$. It is still not known whether this condition alone on h is sufficient to get existence of minimizers for general g. In this paper we prove it under very mild assumptions on g, e.g. it can be any combination of elementary functions.

It is well-known that the integral

$$\int_a^b \Big\{(u'^2-1)^2 + u^2\Big\}\,dx$$

has no minimum in the class of the absolutely continuous functions satisfying $u(a) = u(b) = 0$. Indeed one may easily prove that, in the same class, a minimizer of the integral

$$\int_a^b \Big\{(u'-\alpha)^2\,(u'-\beta)^2 + u^2\Big\}\,dx \qquad (\alpha<\beta)$$

exists if and only if $0 \notin (\alpha,\beta)$. In this example the condition which plays a role in order to get existence, for any boundary data, is

$$h^{**}(0) = h(0)\ , \tag{1}$$

where $h(\xi) = (\xi-\alpha)^2\,(\xi-\beta)^2$ and $h^{**}\colon \mathbb{R}\to\mathbb{R}$ is the convex envelope of h.

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More generally we prove (see Theorem 1 below) existence of minimizers for integrals of the type

$$\int_a^b \Big\{ h(u') + g(u) \Big\}\, dx \ , \tag{2}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is a coercive, not necessarily convex, function satisfying (1) and $g\colon \mathbb{R} \to \mathbb{R}$ is for example one of the functions

$$g_\theta(s) = (1+|s|)^\theta \sin \frac{1}{s} \quad \text{for} \quad s \neq 0, \quad g_\theta(0) \leq -1, \quad \theta \in \mathbb{R} \ . \tag{3}$$

The peculiarity of this example is that the functions in (3) have infinitely many strict local minima on bounded intervals, a situation that seems to be not included in the results available in the literature.

Nonconvex problems have been extensively studied in the literature, especially in the scalar, one-dimensional case. References can be found in [M2]. More specific to functionals of the type (2) are the results proved in [AT], [M1], [Ray], [CC], [AC], [CM], [MO].

Examples of functions g which are critical for our first result, Theorem 1, concerning the integral in (2) are those given by the family of functions $g_r : \mathbb{R} \to \mathbb{R}$, for $r \geq 3$,

$$g_r(s) = [\operatorname{dist}(s, C_r)]^2 \ ,$$

where $C_r = \cap_{i=1}^\infty C_r^i$ is a Cantor type set (it is the standard Cantor set in case $r = 3$). As usual, C_r^1 is the set obtained by removing from $[0,1]$ the open interval of length $1/r$ centered at $s = \frac{1}{2}$; C_r^2 is obtained from C_r^1 removing from each of the remaining intervals the open interval with the same midpoint and length $1/r^2$; and so on. The measure of C_r is easily seen to be

$$\operatorname{meas}(C_r) = 1 - \sum_{i=1}^\infty \frac{2^{i-1}}{r^i} = 1 - \frac{1}{r-2} \ .$$

The set C_r is a level set of the function g_r and coincides with its boundary; it is also the set of minimum points of g_r. A consequence of Theorem 1 below is that if $\operatorname{meas}(C_r) = 0$, i.e. $r = 3$, a minimizer exists for the integral in (2) with any boundary data. If $r > 3$, namely the (boundary of the) level set C_r has positive measure, we are able to prove existence of minimizers in some special cases, for example

$$\int_a^b \Big\{ u'^2 (u' - \beta)^2 + g_r(u) \Big\}\, dx \tag{4}$$

(see Theorem 6 below). In fact we are able to prove existence of minimizers of (2) for any lower semicontinuous function g provided we assume, for instance,

$$\Big\{\xi \in \mathbb{R}\colon\ h^{**}(\xi) < h(\xi)\Big\} = (0, \beta) \ ,$$

as it happens in (4).

Theorem 1. *Let $h, g\colon \mathbb{R} \to \mathbb{R}$ be lower semicontinuous functions such that:*

$$h^{**}(0) = h(0)\,, \qquad \lim_{|\xi|\to\infty} \frac{h(\xi)}{|\xi|} = +\infty \ ; \tag{5}$$

g is bounded below and the boundary of each level set,

$$\partial\{s\colon\ g(s) = \text{const.}\}, \quad \textit{has zero measure} \ . \tag{6}$$

Then for any A, B the integral

$$\int_a^b \Big\{h(u'(x)) + g(u(x))\Big\}\, dx \tag{7}$$

has a minimizer u in the class of the absolutely continuous functions satisfying $u(a) = A$, $u(b) = B$.

Proof: Let us denote by v a minimizer of the relaxed integral

$$\int_a^b \Big\{h^{**}(v'(x)) + g(v(x))\Big\}\, dx \ , \tag{8}$$

under the boundary conditions $v(a) = A$, $v(b) = B$. There exist at most countably many real numbers, which we may order in a sequence c_i, whose corresponding level sets

$$L_i = \Big\{s \in \mathbb{R}\colon\ g(s) = c_i\Big\}$$

have positive measure. We may decompose the interior of each such L_i into a sequence of mutually disjoint open intervals L_{ij}, $j = 1, 2, ...$; and by assumptions (6) we then have

$$L_i = \Big(\bigcup_j L_{ij}\Big) \cup N_i \ , \tag{9}$$

where $N_i \subset \partial L_i$, so that N_i is a null set. Since v is continuous, the set $v^{-1}(L_{ij})$ is open and so it may be represented as the union of at most countably many pairwise disjoint open intervals (a_{ijk}, b_{ijk}), $k = 1, 2, \ldots$.

Fix i, j, k and consider the minimization problem

$$\min\left\{ \int_{a_{ijk}}^{b_{ijk}} h^{**}(u'(x))\,dx\colon\ u(a_{ijk}) = v(a_{ijk}),\ u(b_{ijk}) = v(b_{ijk}) \right\} .$$

This problem has a minimizer which in general is not unique. We wish to choose now one such minimizer u_{ijk} as follows. Define the slope

$$\xi = \frac{v(b_{ijk}) - v(a_{ijk})}{b_{ijk} - a_{ijk}} .$$

If $h^{**}(\xi) = h(\xi)$ we choose

$$u_{ijk}(x) = \xi(x - a_{ijk}) + v(a_{ijk}) .$$

Otherwise, by assumption (5), $\xi \neq 0$, say $\xi > 0$. Moreover there exists a unique interval (α, β) containing ξ, with $0 \leq \alpha < \beta$, such that

$$h^{**} \ \text{ is affine and } \ < h \ \text{ in } (\alpha, \beta) , \qquad h^{**} = h \ \text{ at } \alpha, \beta .$$

In this case we take $u_{ijk}(x)$ to be any continuous piecewise affine function with slopes α and β which satisfies the given boundary conditions. In both cases the chosen minimizer u_{ijk} has range contained in the interval with endpoints $v(a_{ijk})$, $v(b_{ijk})$.

Letting now i, j, k run over all the positive integers, since $u_{ijk}((a_{ijk}, b_{ijk})) \subset L_{ij}$ and g is constant there, by defining

$$u(x) = \begin{cases} u_{ijk}(x) & \text{for } x \in (a_{ijk}, b_{ijk}), \\ v(x) & \text{elsewhere} , \end{cases}$$

we obtain another absolutely continuous minimizer of the relaxed functional (8), with the property that

$$h^{**}(u'(x)) = h(u'(x)) \ \text{ for a.e. } x \quad \text{ such that } \ u(x) \in \bigcup_i \operatorname{int} L_i . \tag{10}$$

We want to show that u is a minimizer of the integral (7). By Theorem 4.1 in [AAB], u satisfies the *DuBois-Reymond differential inclusion*, i.e. there exists a constant c and a measurable function $p(x)$ such that for a.e. $x \in (a, b)$,

$$\begin{cases} p(x) \in \partial h^{**}(u'(x)), \\ c = p(x)\, u'(x) - h^{**}(u'(x)) - g(u(x)) . \end{cases} \tag{11}$$

Let us define the open set

$$K = \left\{ \xi \in \mathbb{R} \colon \; h^{**}(\xi) < h(\xi) \right\} .$$

Then $K = \bigcup_r (\alpha_r, \beta_r)$, where the intervals (α_r, β_r) are pairwise disjoint. Since h^{**} is affine on each interval (α_r, β_r), it may be represented in the form $h^{**}(\xi) = m_r\,\xi + q_r$ for $\xi \in (\alpha_r, \beta_r)$. If we set

$$E_r = \left\{ x \in [a,b] \colon \; u'(x) \in (\alpha_r, \beta_r) \right\} \tag{12}$$

then from (11) we get that

$$g(u(x)) = -c - q_r \qquad \text{for a.e. } \; x \; \text{ in } E_r \; . \tag{13}$$

Consider now the level set

$$\left\{ s \in \mathbb{R} \colon \; g(s) = -c - q_r \right\} . \tag{14}$$

If this set has zero measure then, by (13), using Lemma 2 below we deduce that $u'(x) = 0$ a.e. in E_r; hence by the assumption $h^{**}(0) = h(0)$ and by the definition of E_r in (12), we have $\text{meas}(E_r) = 0$. If the level set (14) has positive measure, it coincides with one of the sets L_i defined above. By the representation (9) and by (10), $u(E_r) \subset N_i$, and since $\text{meas}(N_i) = 0$ we have again $u'(x) = 0$ a.e. in E_r, hence, as before, $\text{meas}(E_r) = 0$.

In conclusion, the set

$$\left\{ x \in [a,b] \colon \; h^{**}(u'(x)) < h(u'(x)) \right\}$$

has zero measure and so u is a minimizer of the integral (7) too. ∎

Lemma 2. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function. If $E \subset [a,b]$ is a measurable set such that* $\text{meas}(u(E)) = 0$, *then $u'(x) = 0$ a.e. on E.*

This lemma can be easily obtained as a consequence of the general area formula, which holds also for absolutely continuous functions (see [F, Theorem 3.2.6]). Here we give a self-contained proof, specific for the one dimensional case.

Proof of Lemma 2

Step 1. We first assume that $u \in C^1([a,b])$ and set $A_0 = \{x \in (a,b) : u'(x) = 0\}$, $A = (a,b) \backslash A_0$. Since A is open, it can be decomposed into a sequence of mutually disjoint open intervals (a_j, b_j); on each interval u' has constant sign, therefore u is a diffeomorphism in (a_j, b_j) and so from the change of variable formula and from the assumption we get, for any $j = 1, 2, \ldots$,

$$\int_{a_j}^{b_j} \chi_E(x)\, |u'(x)|\, dx = \operatorname{meas}\Big(u(a_j, b_j) \cap E\Big) = 0 \ .$$

From this we obtain

$$\int_E |u'|\, dx = \sum_{j=1}^{+\infty} \int_{a_j}^{b_j} \chi_E\, |u'|\, dx + \int_{A_0 \cap E} |u'|\, dx = 0$$

and then the result follows.

Step 2. *For any $\varepsilon > 0$ there exist $v_\varepsilon \in C^1([a,b])$ and a compact set $K_\varepsilon \subset [a,b]$ such that* $\operatorname{meas}([a,b] \backslash K_\varepsilon) < \varepsilon$ *and $v_\varepsilon(x) = u(x)$, $v'_\varepsilon(x) = u'(x)$ for any $x \in K_\varepsilon$.*

We follow [S], sect. 5.3, 5.4. Fix $\varepsilon > 0$. Applying *Lusin's Theorem* to u' we find a compact subset K_0 of $[a,b]$ such that u is differentiable on K_0, u' is continuous on K_0 and $\operatorname{meas}([a,b] \backslash K_0) < \frac{\varepsilon}{2}$. For any $x, y \in K_0$ with $x \neq y$ we set

$$R(x,y) = \frac{u(y) - u(x)}{y - x} - u'(x) \ .$$

If we define for any $j = 1, 2, \ldots$ and any $x \in K_0$

$$\varrho_j(x) = \sup\left\{ |R(x,y)| \colon\ y \in K_0,\ 0 < |x - y| < \frac{1}{j} \right\} \ ,$$

then $\varrho_j(x) \to 0$ as $j \to +\infty$ for any $x \in K_0$. Therefore, by *Egoroff's Theorem* there exists a compact set $K_\varepsilon \subset K_0$, with $\operatorname{meas}(K_0 \backslash K_\varepsilon) < \frac{\varepsilon}{2}$ such that $\varrho_j(x) \to 0$ uniformly on K_ε. Since u' is continuous on K_ε we may conclude that there exists an increasing function $\omega \colon (0, +\infty) \to (0, +\infty)$, with $\lim_{t \to 0^+} \omega(t) = 0$, such that for any $x, y \in K_\varepsilon$

$$|R(x,y)| + |u'(y) - u'(x)| \le \omega(|x - y|) \ . \tag{15}$$

To construct the function v_ε we notice that $(a,b) \backslash K_\varepsilon$ can be decomposed into a sequence of pairwise disjoint intervals (a_j, b_j). For any $j = 1, 2, \ldots$ we define u_j as the third order polynomial such that

$$u_j(a_j) = u(a_j), \quad u_j(b_j) = u(b_j), \quad u'_j(a_j) = u'(a_j), \quad u'_j(b_j) = u'(b_j) \ .$$

Therefore

$$u_j(x) = u(a_j) + u'(a_j)\,(x-a_j) + \Big[3R(a_j,b_j) + u'(a_j) - u'(b_j)\Big]\,\frac{(x-a_j)^2}{b_j-a_j}$$
$$+ \Big[u'(b_j) - u'(a_j) - 2R(a_j,b_j)\Big]\,\frac{(x-a_j)^3}{(b_j-a_j)^2}\ .$$

Using (15) we have

$$\begin{aligned}\max_{a_j\le x<y\le b_j} |u_j'(x)-u_j'(y)| &\le c\,|R(a_j,b_j)| + |u_j'(b_j)-u_j'(a_j)| \\ &\le c\,\omega(|b_j-a_j|)\ .\end{aligned} \tag{16}$$

Setting now for any $x\in[a,b]$

$$v_\varepsilon(x) = \begin{cases} u(x) & \text{if } x\in K_\varepsilon, \\ u_j(x) & \text{if } x\in[a_j,b_j] \ \text{ for some } j\ , \end{cases}$$

and using (16) one easily proves that $v_\varepsilon'(x)$ exists for any $x\in[a,b]$, v_ε' is continuous and $v_\varepsilon'(x) = u'(x)$ on K_ε.

Step 3. From Step 2 we deduce now that for any $j=1,2,...$ there exist a function $v_j\in C^1([a,b])$ and a compact set $K_j\subset[a,b]$ such that $v_j(x)=u(x)$, $v_j'(x)=u'(x)$ on K_j and $\operatorname{meas}([a,b]\backslash K_j) < \frac{1}{j}$. Using Step 1 we then get:

$$\int_{K_j\cap E} |u'|\,dx = \int_{K_j\cap E} |v_j'|\,dx = 0\ ,$$

since $\operatorname{meas}(v_j(K_j\cap E)) = \operatorname{meas}(u(K_j\cap E)) = 0$. Therefore $u'=0$ a.e. on $K_j\cap E$ for any j and the result follows. ∎

Also for integrands of product type it is possible to exhibit examples of non-existence, like the integral

$$\int_a^b \Big\{(1+u^2)\,\Big[(u'^2-1)^2+1\Big]\Big\}\,dx$$

which has no minimum under the boundary conditions $u(a)=u(b)=0$. As in the above case of the sum, also for integrands of the type $g(s)\,h(\xi)$, a crucial role is played by the assumption (1).

Theorem 3. *Let $h,g\colon \mathbb{R}\to\mathbb{R}$ be lower semicontinuous functions such that:*

$$h(\xi)\ge h(0)\ge 0 \quad \textit{for every } \xi\,, \qquad g(s)\ge 1 \quad \textit{for every } s\ . \tag{17}$$

Let h, g satisfy the assumptions (5) and (6). Then for any A, B the integral

$$\int_a^b g(u(x))\, h(u'(x))\, dx \tag{18}$$

has a minimizer u in the class of the absolutely continuous functions satisfying $u(a) = A$, $u(b) = B$.

The proof of this result follows basically the same lines of that of Theorem 1. Obviously the *DuBois–Reymond inclusion* becomes, instead of (11),

$$\begin{cases} p(x) \in g(u(x))\, \partial h^{**}(u'(x)), \\ c = p(x)\, u'(x) - g(u(x))\, h^{**}(u'(x)) \ , \end{cases}$$

condition (13) being thus replaced by

$$q_r\, g(u(x)) = -c \ . \tag{19}$$

However, in case some q_r, say q_1, is zero and the corresponding set E_1, defined as in (12), has positive measure then the constant c must be zero and the above differential inclusion becomes

$$h^{**}(u'(x)) \ \in \ u'(x)\, \partial h^{**}(u'(x)) \ .$$

The existence of minimum can be proved in this case using the same method as in the proof of Theorem 7 below.

Now we extend Theorem 1 to the case in which g is any lower semicontinuous function, provided the function h satisfies an additional assumption.

Lemma 4. *Given any continuous function $\varphi\colon \mathbb{R} \to \mathbb{R}$, there exists a sequence g_n of C^1 functions converging to φ uniformly on compact sets and such that for any interval $[a,b]$ and for any n*

$$\Big\{ x \in [a,b] \colon\ g_n'(x) = 0 \Big\} \quad \textit{is finite} \ .$$

Moreover if φ is bounded below then also the sequence g_n is equibounded below.

Proof: For each n let us define g_n in the interval $[\frac{i}{n}, \frac{i+1}{n}]$, i any integer. In case $\varphi(\frac{i}{n}) = \varphi(\frac{i+1}{n})$ we take

$$g_n(x) = \varphi\Big(\frac{i}{n}\Big) + \Big(x - \frac{i}{n}\Big)^2 \Big(x - \frac{i+1}{n}\Big)^2 \ ;$$

otherwise we take

$$g_n(x) = \varphi\left(\frac{i}{n}\right) + 6\,n^3\left[\varphi\left(\frac{i+1}{n}\right) - \varphi\left(\frac{i}{n}\right)\right]\int_{i/n}^{x}\left(\tau - \frac{i}{n}\right)\left(\frac{i+1}{n} - \tau\right)d\tau\ .$$

With this choice of g_n the result immediately follows. ∎

Lemma 5. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a lower semicontinuous function. Then there exists a sequence g_n of $C^1(\mathbb{R})$ functions such that*

(**i**) *$g_n(x) \to g(x)$ for every x in $\mathbb{R}$;*

(**ii**) *For each n and each interval $[a,b]$*

$$\Big\{x \in [a,b]\colon\ g_n'(x) = 0\Big\} \quad \textit{is finite}\ ;$$

(**iii**) *For each interval $[a,b]$ there exists n_0 such that*

$$g_n(x) \le g_{n+1}(x) \quad \textit{for all}\ \ x \in [a,b]\ \ \textit{and all}\ n > n_0\ .$$

Moreover if g is bounded below then also the sequence g_n is equibounded below.

Proof: For each n and each integer i, set

$$m_{i,n} = \inf\left\{g(x)\colon\ \frac{i}{n} \le x < \frac{i+1}{n}\right\}$$

and define

$$\psi_n(x) = m_{i,n} \quad \text{for}\ \ x \in \left[\frac{i}{n} + \frac{1}{3^n}, \frac{i+1}{n} - \frac{1}{3^n}\right]\ .$$

On the intervals of the type $[\frac{i}{n} - \frac{1}{3^n}, \frac{i}{n} + \frac{1}{3^n}]$ define

$$\psi_n(x) = m_{i,n}$$

in case $m_{i,n} = m_{i-1,n}$;

$$\psi_n(x) = \begin{cases} m_{i-1,n} & \text{for } x \in [\frac{i}{n} - \frac{1}{3^n}, \frac{i}{n}],\\ 3^n(m_{i,n} - m_{i-1,n})\,(x - \frac{i}{n}) + m_{i-1,n} & \text{for } x \in [\frac{i}{n}, \frac{i}{n} + \frac{1}{3^n}]\ , \end{cases}$$

in case $m_{i-1,n} < m_{i,n}$; and

$$\psi_n(x) = \begin{cases} 3^n(m_{i,n} - m_{i-1,n})\,(x - \frac{i}{n}) + m_{i,n} & \text{for } x \in [\frac{i}{n} - \frac{1}{3^n}, \frac{i}{n}],\\ m_{i,n} & \text{for } x \in [\frac{i}{n}, \frac{i}{n} + \frac{1}{3^n}]\ , \end{cases}$$

in case $m_{i-1,n} > m_{i,n}$. Then $\psi_n(x)$ is continuous and

$$\text{(20)} \qquad \begin{cases} \psi_n(x) \le g(x) & \text{for all } x \in \mathbb{R}, \\ \psi_n(x) \ge \min\{m_{i-1,n}; m_{i,n}; m_{i+1,n}\} & \text{for all } x \in [\frac{i}{n}, \frac{i+1}{n}[\ . \end{cases}$$

We show that

$$\psi_n(x) \to g(x) \quad \text{ for any } \ x\ .$$

Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ and let $\delta > 0$ be such that if $x \in (x_0-\delta, x_0+\delta)$ then $g(x) > g(x_0) - \varepsilon$. If $n > \frac{2}{\delta}$ and $x_0 \in [\frac{i}{n}, \frac{i+1}{n}[$ for some integer i, then $[\frac{i-1}{n}, \frac{i+2}{n}] \subset (x_0-\delta, x_0+\delta)$ and from (20) we get

$$g(x_0) - \varepsilon \le \psi_n(x_0) \le g(x_0)\ .$$

Let us now define for each n and each x,

$$\varphi_n(x) = \max\Big\{\psi_1(x), ..., \psi_n(x)\Big\}\ ,$$

thus obtaining an increasing sequence of continuous functions converging to $g(x)$ for any x.

Fix n and set $\widetilde{\varphi}_n(x) = \varphi_n(x) - \frac{1}{2^n}$; by Lemma 4 there exists a C^1 function g_n satisfying (ii) and such that

$$|\widetilde{\varphi}_n(x) - g_n(x)| < \frac{1}{2^{n+2}} \quad \text{ for all } \ x \ \text{ in } [-n,n]\ .$$

The sequence g_n satisfies (i); moreover if $x \in [-n_0, n_0]$ and $n > n_0$,

$$\begin{aligned} g_{n+1}(x) &\ge \widetilde{\varphi}_{n+1}(x) - \frac{1}{2^{n+3}} \ge \widetilde{\varphi}_n(x) + \frac{1}{2^n} - \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}} \\ &\ge g_n(x) + \frac{1}{2^n} - \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} - \frac{1}{2^{n+3}} > g_n(x)\ . \end{aligned}$$

Theorem 6. *Let $h, g\colon \mathbb{R} \to \mathbb{R}$ be lower semicontinuous functions such that:*

$$\text{(21)} \qquad \Big\{\xi \in \mathbb{R}\colon\ h^{**}(\xi) < h(\xi)\Big\} = (0,\beta)\,, \qquad \lim_{|\xi|\to\infty} \frac{h(\xi)}{|\xi|} = +\infty\ ,$$

g is bounded below.

Then for any A, B the integral (7) has a minimizer u in the class of the absolutely continuous functions satisfying $u(a) = A$, $u(b) = B$.

Proof: By subtracting a linear function to $h^{**}(\xi)$ we may assume that

$$\text{(22)} \qquad h^{**}(0) = \min h^{**}(\xi)\ .$$

Throughout the proof we shall assume $A \leq B$ since the case $A > B$ can be treated with a similar argument. Let g_n be a sequence of C^1 functions satisfying (i), (ii), (iii) of Lemma 5, and equibounded below.

Fix n. Let v_n be a minimizer of the functional

$$F_n^{**}(v) = \int_a^b \{h^{**}(v') + g_n(v)\}\, dx \ ,$$

under the boundary conditions $v(a) = A$, $v(b) = B$. Define

$$[A_n, B_n] = v_n([a,b])\,, \qquad m_n = \min\Big\{g_n(s)\colon\ s \in [A_n, B_n]\Big\}\ .$$

We will consider two cases.

First case: $[A_n, B_n] = [A, B]$.

Step 1. *We may assume that for each n there exist $c_n \leq d_n$ in $[a,b]$ such that $v_n(x) = s_n$ if and only if $x \in [c_n, d_n]$, where s_n is the largest point of absolute minimum of g_n in $[A,B]$; and that if $s < s_n$ is any other point of absolute minimum of g_n in $[A,B]$ then there exists a unique $x \in [a,b]$ such that $v_n(x) = s$.*

We start by showing that if $a \leq x_1 \leq b$ and $v_n(x_1) = v_n(x_2) = s$ then v_n is constant on $[x_1, x_2]$. In fact if it were not so then, setting $\widetilde{v}_n(x) = s$ in $[x_1, x_2]$, $\widetilde{v}_n(x) = v_n(x)$ in $[a,b]\backslash[x_1,x_2]$, by (22) and the definition of s, we would get $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$ which is impossible.

From Lemma 4, g_n has only a finite number of absolute minimizers $s_1^n < \ldots < s_{N_n}^n$ in $[A,B]$. Hence there exist N_n disjoint intervals $[a_1,b_1]$, ..., $[a_{N_n}, b_{N_n}]$ (each of which may possibly reduce to a point), such that for any $i = 1, ..., N_n$, $v_n(x) = s_i^n$ if and only if $x \in [a_i, b_i]$. If one of them, say $[a_1, b_1]$, has nonempty interior, setting

$$\widetilde{v}_n(x) = \begin{cases} v_n(x) & \text{if } a \leq x \leq a_1, \\ v_n(x + b_1 - a_1) & \text{if } a_1 \leq x \leq a_{N_n} - (b_1 - a_1), \\ s_{N_n}^n & \text{if } a_{N_n} - (b_1 - a_1) \leq x \leq a_{N_n}, \\ v_n(x) & \text{if } a_{N_n} \leq x \leq b\ , \end{cases}$$

we have $F_n^{**}(\widetilde{v}_n) = F_n^{**}(v_n)$. By repeating, if necessary, such a modification of v_n at most $N_n - 1$ times, Step 1 is proved.

Step 2. *v_n is strictly increasing in $[a, c_n]$ and in $[d_n, b]$.*

In fact if $a < x_1 < x_2 < c_n$ and $v_n(x_1) = v_n(x_2)$, by Step 1 this value could not be an absolute minimum of g_n in $[A, B]$, therefore setting

$$\widetilde{v}_n(x) = \begin{cases} v_n(x) & \text{if } a \le x \le x_1, \\ v_n(x + x_2 - x_1) & \text{if } x_1 \le x \le c_n - (x_2 - x_1), \\ s_n & \text{if } c_n - (x_2 - x_1) \le x \le c_n, \\ v_n(x) & \text{if } c_n \le x \le b \ , \end{cases}$$

we would get $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$, absurd.

Step 3. $v_n'(x) \ge \beta$ a.e. *in* $[a, c_n] \cup [d_n, b]$

Let us define the set

$$E = \Big\{ x \in [a, c_n] \colon \ v_n'(x) \ \text{ exists and belongs to } [0, \beta) \Big\} \ .$$

Then the *DuBois–Reymond differential inclusion* implies that there exists a constant c such that

$$g_n(v_n(x)) = c \qquad \text{for all } \ x \in E \ .$$

By the assumption (ii) on g_n, the equation $g_n(s) = c$ may have only a finite number of solutions in $[A, B]$, therefore, by Step 2, E has finitely many points.

Second case: $[A_n, B_n] \ne [A, B]$.

We suppose, in steps 4, 5, 6 below, that $A_n < A$; if we had $A_n = A$ and $B < B_n$ the reasoning would be similar.

Step 4. A_n *is an absolute minimizer of* g_n *in* $[A_n, B_n]$.

Notice first that $g_n(A_n) < g_n(s)$ for $d \in \,]A_n, A]$; in fact if there existed $s_0 \in \,]A_n, A]$ with $g_n(s_0) \le g_n(s)$ for every $s \in [A_n, A]$, setting

$$\widetilde{v}_n(x) = \max\{v_n(x), s_0\} \ ,$$

from (22) and (ii) we would have $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$.

Let us assume that A_n is not an absolute minimizer of g_n in $[A_n, B_n]$. From what we just noticed it is clear that if s were any point of absolute minimum of g_n in $[A_n, B_n]$ then s would belong to the interval $]A, B_n]$. Let (x_1, x_2) be any connected component of the open set $\{x \in (a, b) : \ v_n(x) < A\}$ and let x_3 be a point such that $v_n(x_3) = s$; clearly $x_3 \notin [x_1, x_2]$ and to fix ideas suppose $x_2 < x_3 \le b$. Setting

$$\widetilde{v}_n(x) = \begin{cases} v_n(x) & \text{if } a \le x \le x_1, \\ v_n(x + x_2 - x_1) & \text{if } x_1 \le x \le x_3 - (x_2 - x_1), \\ s & \text{if } x_3 - (x_2 - x_1) \le x \le x_3, \\ v_n(x) & \text{if } x_3 \le x \le b \ , \end{cases}$$

we would then have $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$, which is impossible.

Similarly one may show, in case $B < B_n$, that B_n is an absolute minimizer of g_n in $[A_n, B_n]$.

Step 5. *The open set* $\{x \in (a,b)\colon\ v_n(x) < A\}$ *is an interval* (a, x_n).

Let (x_1, x_2) be a connected component of this set containing a point $\overline{x}$ such that $v_n(\overline{x}) = A_n$; if (x_3, x_4) were another connected component with, say, $x_3 > x_2$, with the function

$$\widetilde{v}_n(x) = \begin{cases} v_n(x) & \text{if } a \le x \le \overline{x}, \\ A_n & \text{if } \overline{x} \le x \le \overline{x} + x_4 - x_3, \\ v_n(x - (x_4 - x_3)) & \text{if } \overline{x} + x_4 - x_3 \le x \le x_4, \\ v_n(x) & \text{if } x_4 \le x \le b\ , \end{cases}$$

we would get $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$, since, by Step 4, A_n is an absolute minimizer of g_n in $[A_n, B_n]$. Using this fact again, one can check, in the same way, that the interval $\{x \in (a,b)\colon\ v_n(x) < A\}$ has left extremity a.

If $B < B_n$ one can prove, using the same method, that $\{x \in (a,b)\colon\ v_n(x) > B\}$ is an interval (y_n, b).

Step 6. v_n *is decreasing in* $[a, a_n]$, $a_n \in (a, x_n)$ *being the largest point where* v_n *attains the value* A_n.

First notice that if $x \in (a, a_n)$ is any point with $v_n(x) = A_n$ then $v_n \equiv A_n$ in $[x, a_n]$, as in Step 4. So denote by a_n^- the smallest point where v_n attains the value A_n, so that $v_n \equiv A_n$ in $[a_n^-, a_n]$. If there were points $x_1 < x_2$ in $[a, a_n^-]$ such that $v_n(x_1) = v_n(x_2)$, this value would be a number $s \in (A_n, A]$ with $g_n(A_n) < g_n(s)$, as in Step 4; and we could construct a function $\widetilde{v}_n$ such that $F_n^{**}(\widetilde{v}_n) < F_n^{**}(v_n)$. This shows that v_n is strictly decreasing in $[a, a_n^-]$.

Similarly one can show that v_n is decreasing in $[b_n, b]$, if b_n is the smallest point in (y_n, b) where v_n attains the value $B_n > B$.

We summarize now what we have shown in the two cases above considered.

Step 7. *We may assume there exist points* $a_n \le c_n \le d_n \le b_n$ *in* $[a,b]$ *such that*

$$\begin{aligned} v_n'(x) &\le 0 \quad \text{a.e. in } [a, a_n]\ , \\ v_n'(x) &\ge \beta \quad \text{a.e. in } [a_n, c_n]\ , \\ v_n'(x) &= 0 \quad \text{a.e. in } [c_n, d_n]\ , \\ v_n'(x) &\ge \beta \quad \text{a.e. in } [d_n, b_n]\ , \\ v_n'(x) &\le 0 \quad \text{a.e. in } [b_n, b]\ . \end{aligned}$$

In fact, if $v_n([a,b]) = [A,B]$, we just take $a_n = a$, $b_n = b$ and apply Step 3. If instead $v_n([a,b]) \neq [A,B]$, we apply Step 6 to determine a_n and b_n; and then notice that v_n is a minimizer of

$$\int_{a_n}^{b_n} \left\{ h^{**}(v') + g_n(v) \right\} dx$$

under the boundary conditions $v_n(a_n) = A_n$, $v_n(b_n) = B_n$. Since $v_n([a_n, b_n]) = [A_n, B_n]$, c_n and d_n are determined applying Step 3 to v_n relatively to the interval $[a_n, b_n]$. Indeed one could prove even better, namely that in both cases either $a_n = a$ or $b_n = b$ or both equalities hold.

Step 8. *Conclusion of the proof.*

Now we use the fact that h grows at infinity more than linearly and that g_n is a sequence uniformly bounded from below. Letting $n \to \infty$ we may assume, passing possibly to a subsequence, that there exist $u(x)$ and points $a' \leq c' \leq d' \leq b'$ in $[a,b]$ such that $v_n \rightharpoonup u$, $w - W^{1,1}$, $a_n \to a'$, $b_n \to b'$, $c_n \to c'$, $d_n \to d'$.

From Step 7 we have also that

$$\begin{cases} u'(x) \leq 0 & \text{a.e. in } [a,a'] \cup [b',b], \\ u'(x) \geq \beta & \text{a.e. in } [a',c'] \cup [d',b'], \\ u'(x) = 0 & \text{a.e. in } [c',d'] \ . \end{cases} \tag{23}$$

Take A_0, B_0 such that $[A_0, B_0] \supset [A_n, B_n]$ for every n. Using (iii) we get that there exists n_0 such that

$$g_{n+1}(s) \geq g_n(s) \quad \text{for any} \quad n \geq n_0 \quad \text{and} \quad s \in [A_0, B_0] \ .$$

Therefore if $k \geq n_0$, since h^{**} is convex and $v_n \rightharpoonup u$ $w - W^{1,1}$, we have

$$\begin{aligned} \liminf_n F_n^{**}(v_n) &\geq \liminf_n \int_a^b h^{**}(v_n')\, dx + \liminf_n \int_a^b g_n(v_n)\, dx \geq \\ &\geq \int_a^b h^{**}(u')\, dx + \lim_n \int_a^b g_k(v_n)\, dx = \int_a^b \left\{ h^{**}(u') + g_k(u) \right\} dx \ , \end{aligned}$$

and so, letting $k \to \infty$,

$$\liminf_n F_n^{**}(v_n) \geq \int_a^b \left\{ h^{**}(u') + g(u) \right\} dx \ .$$

Then if v is any absolutely continuous function satisfying the boundary conditions we have

$$\int_a^b \left\{ h^{**}(v') + g(v) \right\} dx = \lim_n F_n^{**}(v) \geq \liminf_n F_n^{**}(v_n) \geq \int_a^b \left\{ h^{**}(u') + g(u) \right\} dx \ .$$

Therefore u is a minimizer of the functional

$$\int_a^b \Big\{h^{**}(v') + g(v)\Big\}\,dx\ ,$$

hence, by (23) and (21), also a minimizer of the functional (7).

Remark. It is clear that Theorem 6 still holds if we replace in (21) the interval $(0,\beta)$ by $(\alpha,0)$. Moreover notice that in Theorems 1 and 6 the assumption that g is bounded below can be replaced by any of the usual assumptions ensuring the coercivity of the integral.

It is possible to obtain also a result of existence of minimizers for integrals of "affine" type. Consider the set

$$T_q = \Big\{\xi \in \mathbb{R}\colon\ h^{**}(\xi) \in q + \xi\,\partial h^{**}(\xi)\Big\} \tag{24}$$

of points over which the tangent to the graph of h^{**} meets the vertical axis at the point $(0,q)$. We suppose in Theorem 7 below that there exists a unique number q such that the set $\{\xi \in \mathbb{R}\colon\ h^{**}(\xi) < h(\xi)\}$ is contained in T_q.

In case $q = 0$ and $\varphi(s) \equiv 0$ one obtains the special case of integrals of product type, considered in Theorem 3, in which there exists exactly one number q_r, as in (19), and is equal to zero.

Theorem 7. *Let $h,\varphi,\rho\colon\ \mathbb{R}\to\mathbb{R}$ be lower semicontinuous functions satisfying $\rho(s)\geq 1$ for every s, and (5). Suppose there exists a unique number q such that*

$$\Big\{\xi \in \mathbb{R}\colon\ h^{**}(\xi) < h(\xi)\Big\} \subset T_q\ ,$$

$s \mapsto q\,\rho(s)$ is lower semicontinuous and (6) holds true with $g(s) = \varphi(s) + q\,\rho(s)$, and $h(\xi) \geq h(0)\ \ \forall\,\xi \in \mathbb{R}$.

Then for any A, B the integral

$$\int_a^b \Big\{\varphi(u(x)) + \rho(u(x))\,h(u'(x))\Big\}\,dx \tag{25}$$

has a minimizer u in the class of the absolutely continuous functions satisfying $u(a) = A$, $u(b) = B$.

Proof: Clearly we may write $T_q = [\alpha_1,\beta_1] \cup [\alpha_2,\beta_2]$ with $\alpha_1 \leq \beta_1 \leq 0 \leq \alpha_2 \leq \beta_2$ and

$$h^{**}(\xi) = q + m_1\,\xi \quad \text{ for }\ \xi\ \text{ in } [\alpha_1,\beta_1]\ ,$$

$$h^{**}(\xi) = q + m_2\,\xi \quad \text{ for }\ \xi\ \text{ in } [\alpha_2,\beta_2]\ .$$

Define the function

$$h_1(\xi) = h(\xi) - q$$

obtaining

$$h_1^{**}(\xi) = m_1\,\xi \quad \text{for } \xi \text{ in } [\alpha_1, \beta_1]\ ,$$

$$h_1^{**}(\xi) = m_2\,\xi \quad \text{for } \xi \text{ in } [\alpha_2, \beta_2]\ .$$

To find a minimizer of (25) is equivalent to obtaining a minimizer of

$$\int_a^b \Big\{ g(u(x)) + \rho(u(x))\, h_1(u'(x)) \Big\}\, dx \tag{26}$$

under the same boundary conditions $u(a) = A$, $u(b) = B$. Let us denote by v a minimizer of the relaxed integral corresponding to (26). As in the proof of Theorem 1 we may consider the minimization problem

$$\min\bigg\{ \int_{a_{ijk}}^{b_{ijk}} \rho(u(x))\, h_1(u'(x))\, dx\colon\ u(a_{ijk}) = v(a_{ijk}),\ u(b_{ijk}) = v(b_{ijk}) \bigg\}, \tag{27}$$

where $v((a_{ijk}, b_{ijk}))$ is an interval along which g is constant. Suppose that v itself does not solve (27); then at least one of the sets E_1, E_2, defined as in (12) with v in place of u, has positive measure. It follows that the *DuBois–Reymond inclusion* for the relaxed integral corresponding to (27) becomes, instead of (11), because the constant c is zero,

$$v'(x) \in \Big\{ \xi \in \mathbb{R}\colon\ h_1^{**}(\xi) \in \xi\, \partial h_1^{**}(\xi) \Big\}$$

for a.e. x in $[a_{ijk}, b_{ijk}]$.

Let d_1 be the smallest point of minimum of $\rho(v(x))$ in $[a_{ijk}, b_{ijk}]$ and set $D = v(d_1)$, $e_1 = \max v^{-1}(D)$. If, say, $D \le \min\{v(a_{ijk}), v(b_{ijk})\}$ then, since $v'(x) \in [\alpha_1, \beta_2]$ for a.e. x in $[a_{ijk}, b_{ijk}]$, it is possible to find points $d \le d_1 \le e_1 \le e$ in $[a_{ijk}, b_{ijk}]$ such that the function

$$u_{ijk}(x) = \begin{cases} D - \alpha_1(d - x) & \text{for } x \in [a_{ijk}, d], \\ D & \text{for } x \in [d, e], \\ D + \beta_2(x - e) & \text{for } x \in [e, b_{ijk}]\ , \end{cases}$$

satisfies $u_{ijk}(a_{ijk}) = v(a_{ijk})$, $u_{ijk}(b_{ijk}) = v(b_{ijk})$ and

$$u_{ijk}((a_{ijk}, b_{ijk})) \subset v((a_{ijk}, b_{ijk}))\ .$$

We show now that u_{ijk} minimizes the integral in (27):

$$\begin{aligned}\int_{a_{ijk}}^{b_{ijk}} \rho(v(x))\, h_1^{**}(v'(x))\, dx &= \int_{a_{ijk}}^{d_1} \rho(v(x))\, h_1^{**}(v'(x))\, dx + \\ &+ \int_{d_1}^{e_1} \rho(v(x))\, h_1^{**}(v'(x))\, dx + \int_{e_1}^{b_{ijk}} \rho(v(x))\, h_1^{**}(v'(x))\, dx \geq \\ \geq \int_{a_{ijk}}^{d_1} \rho(v(x))\, m_1\, v'(x)\, dx &+ \int_{d_1}^{e_1} \rho(D)\, h_1^{**}(0)\, dx + \\ &+ \int_{e_1}^{b_{ijk}} \rho(v(x))\, m_2\, v'(x)\, dx = \int_{a_{ijk}}^{b_{ijk}} \rho(u_{ijk}(x))\, h_1(u'_{ijk}(x))\, dx \ .\end{aligned}$$

In case $D \geq \max\{v(a_{ijk}), v(b_{ijk})\}$ or $v(b_{ijk}) < D < v(a_{ijk})$ or $v(a_{ijk}) < D < v(b_{ijk})$ one may construct similarly a minimizer.

Letting now i, j, k run over all the positive integers, since $u_{ijk}((a_{ijk}, b_{ijk})) \subset v((a_{ijk}, b_{ijk}))$ and g is constant along this interval, by defining

$$u(x) = \begin{cases} u_{ijk}(x) & \text{for } x \in (a_{ijk}, b_{ijk}), \\ v(x) & \text{elsewhere} \ , \end{cases}$$

we obtain another minimizer of the relaxed integral corresponding to (26) which satisfies the property (10).

We wish to show that u is a minimizer of the integral (26). If this were not true then one of the sets E_1, E_2, defined as in (12), would have positive measure and the *DuBois–Reymond inclusion* would assert the existence of a constant c such that, instead of (11),

$$g(u(x)) = -c \quad \text{for a.e. } x \text{ in } E_1 \cup E_2 \ .$$

It is enough to follow now the arguments of the final part of the proof of Theorem 1 to reach a contradiction. ∎

Remark. We may say that the condition, imposed in Theorem 7, that the level sets of $\varphi(s) + q\,\rho(s)$ have boundary with zero measure, is satisfied quite generally; in fact, its denial means there exists some vertical translate of the graph of $\varphi(s)$ whose points of intersection with the graph of $-q\,\rho(s)$ have vertical projection with boundary of positive measure. It surely takes some effort to exhibit explicit examples of functions φ, ρ which do not satisfy (6): obviously one may have to search them among special Cantor type functions like the ones considered in (4), with $r > 3$.

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