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ON DETERMINANT PRESERVERS OVER SKEW FIELDS

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Abstract: We characterize all the semilinear transformations on matrices over skew fields that preserve the Dieudonné determinant.

1 – Introduction

Throughout this work, D is a skew field (division ring) which is considered as finite-dimensional vector space over a field K, which may be identified with a subfield of the center of D.

In [3], Draxl gave some equivalent conditions to the invertibility of a matrix.

Theorem 1.1. *Let $A \in M_n(D)$. Then the following conditions are equivalent:*

i) $A \in GL_n(D)$, *i.e., A is invertible;*

ii) $AB = I$, *for some $B \in M_n(D)$, i.e., A has a right inverse;*

iii) $BA = I$, *for some $B \in M_n(D)$, i.e., A has a left inverse;*

iv) *the rows of A are left linearly independent over D;*

v) *the columns of A are right linearly independent over D.*

The theorem below says that we can decompose any invertible matrix in the *Bruhat normal form* ([3]).

Theorem 1.2. *Let $A \in M_n(D)$ be an invertible matrix. There exists a decomposition of A such that $A = TUP(\pi)V$, where T is a lower triangular matrix, V is an upper triangular matrix, both with 1's on the main diagonal, and*

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$u_{ij} = u_i\,\delta_{ij}$ with $u_i \neq 0$, for each i, with π and U uniquely determined by A, i.e.,

$$A = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} \begin{bmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{bmatrix} P(\pi) \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} . \tag{1.1}$$

Of course we can extend the concept of the Bruhat normal form of a matrix to singular matrices in analogous way of the one done by Draxl to prove Theorem 1.2, i.e., any matrix can be expressed in the form (1.1), where some of the u_i's may be zeroes.

Definition 1.3. Let $A \in M_n(D)$ be a matrix which admits a decomposition of the type (1.1), such that only $m\,(\leq n)$ of the u_i's are nonzero. Then we say that A has rank m.

Therefore, one can say that A has rank 1 if and only if it has the form xy where x is a $n \times 1$ matrix and y is a $1 \times n$ matrix, both non zero. Notice, also, that if X is invertible and Y is a matrix with rank m, then XY has rank m.

We will need the next lemma in a very important proof in the last section.

Lemma 1.4. *Let $A \in M_n(D)$ a matrix with rank r. Then there exists a matrix $B \in M_n(D)$ with rank $n - r$ such that $A + B$ is nonsingular.*

Proof: Let $A = T\,U P(\pi)\,V$ and, without loss of generality, one supposes that in (1.1), the $u_1, ..., u_r$ are nonzero and the $u_{r+1} = ... = u_n = 0$. Thus, let $U' \in M_n(D)$ be a diagonal matrix such that $u'_1 = ... = u'_r = 0$ and $u'_{r+1} = ... = u'_n = 1$. Now, one only has to consider the matrix $B = T\,U' P(\pi)\,V$. ∎

2 – The Dieudonné determinant

Let D^* be the non abelian multiplicative group $D - \{0\}$. We denote by $[D^*, D^*]$ the normal subgroup of the commutators of D^*. The *Dieudonné determinant* is the application

$$\det\colon\, M_n\,(D) \longrightarrow D^*/[D^*, D^*] \cup \{0\} ,$$

such that, in the conditions of Theorem 1.2,

$$\det A = \begin{cases} 0 & \text{if } A \notin GL_n(D), \\ \Big[\operatorname{sgn}(\pi) \prod_{t=1}^{n} u_t\Big] & \text{if } A \in GL_n(D) , \end{cases}$$

where $[x]$ represents the class of x. This determinant has the important property

$$\det AB = \det A \, \det B \ .$$

Note that $D^*/[D^*, D^*] \cup \{0\}$ is a multiplicative semigroup with a zero, such that all nonzero elements are invertible.

3 – Nonsingular rank 1 preservers

Let $\{t_1, ..., t_m\}$ be a basis of D over K. We define $e^{i,l}$ as the $n \times 1$ matrix with t_l in position i and 0 elsewhere, and $e_{j,l}$ as the $1 \times n$ matrix with t_l in position j and 0 elsewhere defined. We suppose that $t_1 = 1$ and define $e^i = e^{i,1}$ and $e_j = e_{j,1}$.

Let us consider $\mathcal{C}$, the right vector space over D generated by e^i's, and $\mathcal{R}$, the left vector space over D generated by e_j's.

The set $\{e^{i,l} \mid i = 1, ..., n \text{ and } l = 1, ..., m\}$ is a basis of $\mathcal{C}$ considered as vector space over K and $\{e_{j,l} \mid j = 1, ..., n \text{ and } l = 1, ..., m\}$ is a basis of $\mathcal{R}$ consider as vector space over K.

The dimension of $\mathcal{C}$ over D, $\dim_D \mathcal{C}$, is n, and over K is $[D : K] \dim_D \mathcal{C}$. Similarly, $\dim_D \mathcal{R} = n$ and $\dim_K \mathcal{R} = [D : K] \dim_D \mathcal{R}$.

Throughout this work $n > 1$. The case $n = 1$ is trivial.

Definition 3.1. Let f be a transformation of $M_n(D)$ which is additive and $f(\alpha X) = \alpha f(X)$, for all $\alpha \in K$ and $X \in M_n(D)$. Then we say that f is *semilinear*.

The next theorem is a generalization of results by Mink in [7], Wong in [8] and Marcus and Moyls in [6]. Though the first part of the proof is similar to the one done by Mink in [7], we use different arguments and for a sake of completeness we present the entire proof. Also, Jacob in [5], proved in a very intricated way some results similars to ours. He used some results of affine geometry and a concept of *coherence invariant mapping*. Our proofs only appeal to matrix concepts.

Theorem 3.2. *Let f be a nonsingular semilinear transformation on $M_n(D)$. If the set of rank 1 matrices is invariant under f, then there exist nonsingular matrices A, B and a bijective map σ of D, with $\sigma_{|K} = \mathrm{Id}_K$, such that either σ is a homorphism and*

$$f(X) = A\,\overline{X}\,B \ ,$$

for all $X \in M_n(D)$, or σ is an anti-homorphism and

$$f(X) = A\,\overline{X'}\,B\ ,$$

for all $X \in M_n(D)$, where $\overline{X} = (\overline{x}_{ij}) = (\sigma(x_{ij}))$.

Proof: Let u and v be any vectors of $\mathcal{C}$ and $\mathcal{R}$, respectively. Then uv is a matrix of $M_n(D)$ which rank is 1. Therefore the rank of $f(uv)$ is also 1, i.e.,

$$f(u\,v) = x^{uv}\,y_{uv}\ , \tag{3.1}$$

where $x^{uv} \in \mathcal{C}$ and $y_{uv} \in \mathcal{R}$, not equal to zero.

First, let us prove that, for any v and v', at least one of the conditions

1) $x^{uv} \,\|\, x^{uv'}$ (i.e., x^{uv} and $x^{uv'}$ are linearly dependent);

2) $y_{uv} \,\|\, y_{uv'}$,

holds. The matrix $u\,v + u\,v'$ has rank 1, and so $f(u\,v + u\,v') = x^{uv}\,y_{uv} + x^{uv'}y_{uv'}$ also has rank 1. But

$$\begin{aligned} f(u\,v + u\,v') &= x^{uv}\,y_{uv} + x^{uv'}y_{uv'} \\ &= \begin{bmatrix} x^{uv} & x^{uv'} \end{bmatrix} \begin{bmatrix} y_{uv} \\ y_{uv'} \end{bmatrix}\ . \end{aligned}$$

If $x^{uv} \not\parallel x^{uv'}$, then there exists an invertible matrix $P \in M_n(D)$ such that $P[\ x^{uv} \quad x^{uv'}\ ] = [e^1\ e^2]$. Hence

$$\begin{aligned} f(u\,v + u\,v') &= \begin{bmatrix} x^{uv} & x^{uv'} \end{bmatrix} \begin{bmatrix} y_{uv} \\ y_{uv'} \end{bmatrix} \\ &= P^{-1}\Big[e^1\ e^2\Big] \begin{bmatrix} y_{uv} \\ y_{uv'} \end{bmatrix} \\ &= P^{-1} \begin{bmatrix} y_{uv} \\ y_{uv'} \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ , \end{aligned}$$

and thus $y_{uv} \,\|\, y_{uv'}$.

Similarly, it can be proved that, for any u and u', $x^{uv} \,\|\, x^{u'v}$ or $y_{uv} \,\|\, y_{u'v}$.

Let us show now that, for all v', $x^{uv} \,\|\, x^{uv'}$ or, for all v', $y_{uv} \,\|\, y_{uv'}$, for a given u. For, if $x^{uv} \not\parallel x^{uw}$, for some $w \in \mathcal{R}$, then $y_{uv} \,\|\, y_{uw}$. In this case $y_{uv} \,\|\, y_{uv'}$, for all v', since, given $x^{uw'}$, $x^{uw'} \nparallel x^{uv}$ or $x^{uw'} \nparallel x^{uw}$, i.e., $y_{uw'} \parallel y_{uv}$ or $y_{uw'} \parallel y_{uw}$. Similarly, we can prove that, for any v, $x^{uv} \parallel x^{u'v}$, for all u', or $y_{uv} \parallel y_{u'v}$, for all u'.

Each non zero vector x^{uv}, or each y_{uv}, of (3.1), is determined only to within a scalar. For, supposing that $x^{uv} \parallel x^{uv'}$, for all v', i.e., $x^{uv'} = x^{uv}\alpha_{v'}$, for all v', it follows that $f(uv') = x^{uv'}\alpha_{v'}^{-1}\alpha_{v'}\, y_{uv'}$; thus, without loss of generality, for each u,

$$x^{uv} = x^{uv'} \ , \tag{3.2}$$

for all v', or

$$y_{uv} = y_{uv'} \ , \tag{3.3}$$

for all v', and, for each v,

$$x^{uv} \parallel x^{u'v} \tag{3.4}$$

for all u', or

$$y_{uv} \parallel y_{u'v} \ , \tag{3.5}$$

for all u'.

It is clear that, for each u, either equation (3.2) holds or (3.3) does. Otherwise, it would follow that $f(u\,v - u\,v') = 0$, for $v \neq v'$, which is impossible, since $u(v-v')$ is of rank 1.

On other hand, if $x^{uv} = x^{uv'}$, for all v', then there exists $w \in \mathcal{R}$ such that $y_{uv} \nparallel y_{uw}$. For, if $y_{uw} = \alpha_w\, y_{uv}$, for all w, then the map defined by $w \to x^{uv}\alpha_w\, y_{uv}$, is injective. But $\dim_K \mathcal{R} = [D : K] \dim_D \mathcal{R}$ and the dimension of the image over K is less or equal than $[D : K]$, a contradiction. (Note we have assumed that $\dim_D \mathcal{R} > 1$.)

Either the equation (3.2) holds, for all u, or (3.3) does, for all u. In fact, suppose that $x^{uv} = x^{uv'}$ and $y_{u'v} = y_{u'v'}$, for some u different from u' and for all v'. As we have seen in the last paragraph, there exists w such that $y_{uv} \nparallel y_{uw}$. Then $y_{uv} \nparallel y_{u'v}$ or $y_{uw} \nparallel y_{u'v}$, and, therefore, $x^{u'v} = x^{uv}\beta$, for some β, or $x^{u'v} = x^{uw}\beta$, for some β. Suppose that the first case happens. Choose $v' \neq v$ such that $x^{u'v} \nparallel x^{u'v'}$. Then

$$\begin{aligned} f\Big((u+u')\,(c\,v+v')\Big) &= c\,x^{uv}y_{uv} + x^{uv'}y_{uv'} + c\,x^{u'v}y_{u'v} + x^{u'v'}y_{u'v'} \\ &= x^{uv}\,(c\,y_{uv} + y_{uv'} + c\,\beta\, y_{u'v}) + x^{u'v'}y_{u'v} \ , \end{aligned}$$

for an arbitrary scalar $c \in K$. Since $x^{uv} \parallel x^{u'v} \nparallel x^{u'v'}$, as we have proven in the beginning of this proof, $(c\,y_{uv} + y_{uv'} + c\,\beta\, y_{u'v}) \parallel y_{u'v}$. But $c\,\beta\, y_{u'v} \parallel y_{u'v}$ and thus

$(c\,y_{uv} + y_{uv'}) \parallel y_{u'v}$, for all scalar $c \in K$, which is impossible, since $y_{uv} \nparallel y_{u'v}$. Similarly, we can also prove that the second case can not happen.

Either the equation (3.4) holds, for all v, or (3.5) does, for all v. Suppose for some v_1 and v_2, we have $x^{u_1 v_1} \parallel x^{u v_1}$, for all u, and $y_{u_1 v_2} \parallel y_{u v_2}$, for all u. Let us make

$$x\,\mathcal{R} = \Big\{ x\,y \mid y \in \mathcal{R} \Big\}$$

and

$$\mathcal{C}\,y = \Big\{ x\,y \mid x \in \mathcal{C} \Big\} \ .$$

We have

$$f(u\,v_1) = x^{uv_1}\,y_{uv_1} = x^{u_1 v_1}\,\alpha_u y_{uv_1} \ \in \ x^{u_1 v_1}\,\mathcal{R}$$

and

$$f(u\,v_2) = x^{uv_2}\,y_{uv_2} = x^{uv_2}\,\beta_u\,y_{u_1 v_2} \ \in \ \mathcal{C}\,y_{u_1 v_2} \ .$$

But $f(u(v_1 + v_2)) = x^{u_1 v_1}\alpha_u\,y_{uv_1} + x^{uv_2}\beta_u\,y_{u_1 v_2}$ has rank 1. Therefore either $x^{u_1 v_1} \parallel x^{uv_2}$ or $y_{uv_1} \parallel y_{u_1 v_2}$, i.e., either $f(u\,v_1)$ or $f(u\,v_2)$ is in $\mathcal{C}y_{u_1 v_2} \cap x^{u_1 v_1}\mathcal{R}$. Let us consider the semilinear applications φ_1 of $\mathcal{C}$ into $x^{u_1 v_1}\mathcal{R}$ defined by $\varphi_1(u) = f(u\,v_1)$ and φ_2 of $\mathcal{C}$ into $\mathcal{C}y_{u_1 v_2}$ defined by $\varphi_2(u) = f(u\,v_2)$. Then $\mathcal{C}$ is the union of the proper subspaces $\varphi_1^{-1}(\mathcal{C}y_{u_1 v_2} \cap x^{u_1 v_1}\mathcal{R})$ with $\varphi_2^{-1}(\mathcal{C}y_{u_1 v_2} \cap x^{u_1 v_1}\mathcal{R})$, which is impossible.

Suppose now the equations (3.2) and (3.4) both hold. Then $x^u = x^{uv}$, for all u and v, and $x^u = x^{u'}\alpha_u$ for all u. It is easy to see in these conditions, that, for example, all the $m\,n$ vectors $\alpha_{e^1}\,y_{e^1 e_{j,l}}$ are linearly independent. Then we may write $\alpha_{e^2}\,y_{e^2 e_1}$ as a linear combination of these vectors, for instance,

$$\alpha_{e^2}\,y_{e^2 e_1} = \sum_{j,l} c_{j,l}\,\alpha_{e^1}\,y_{e^1 e_{j,l}} \ .$$

But then

$$f\left(e^1 \Big(\sum_{j,l} c_{j,l}\,e_{j,l} \Big) - e^2\,e_1 \right) = 0 \ ,$$

which is absurd, since f is nonsingular.

Similarly we can prove that (3.3) and (3.5) could not both hold.

We are able to say now that either

$$x^{uv} = x^u \ ,$$

for all u and v, and

$$y_{uv} \parallel y_v \ ,$$

for all u and v, or

$$y_{uv} = y_u \ ,$$

for all u and v, and

$$x^{uv} \parallel x^v$$

for all u and v.

Suppose the first case happens. Let $u' \in \mathcal{C}$ and $v' \in \mathcal{R}$ be vectors such that $x^u \nparallel x^{u'}$ and $y_v \nparallel y_{v'}$. Since $f(u\,v) = x^u\,\alpha(u,v)\,y_v$ we have

$$\begin{aligned} f\Big[(u{+}u')\,(v{+}v')\Big] &= x^u\,\alpha(u,v)\,y_v + x^u\,\alpha(u,v')\,y_{v'} + x^{u'}\alpha(u',v)\,y_v + x^{u'}\alpha(u',v')\,y_{v'} \\ &= x^u\Big(\alpha(u,v)\,y_v + \alpha(u,v')\,y_{v'}\Big) + x^{u'}\Big(\alpha(u',v)\,y_v + \alpha(u',v')\,y_{v'}\Big), \end{aligned}$$

which implies that

$$\Big(\alpha(u,v)\,y_v + \alpha(u,v')\,y_{v'}\Big) \parallel \Big(\alpha(u',v)\,y_v + \alpha(u',v')\,y_{v'}\Big)$$

and therefore $\alpha(u,v) = \alpha(u,v')\,\alpha(u',v')^{-1}\,\alpha(u',v)$. If we redefine x^u as $x^u\,\alpha(u,v')$ when $u \neq 0$, and $x^0 = 0$, and y_v as $\alpha(u',v')^{-1}\,\alpha(u',v)\,y_v$ when $v \neq 0$, and $y_0 = 0$, then

$$f(u\,v) = x^u\,y_v \ .$$

From the fact

$$x^{u\alpha}\,y_v = f\Big((u\alpha)\,v\Big) = f\Big(u(\alpha\,v)\Big) = x^u\,y_{\alpha v} \ ,$$

since $x^{u\alpha} \parallel x^u$, we deduce

$$x^{u\alpha} = x^u\,\overline{\alpha} \ ,$$

and

$$y_{\alpha v} = \overline{\alpha}\,y_v$$

where $\overline{\alpha} \in D$. These two equations allow us to conclude that $\overline{\alpha}$ depends neither on v nor on u, respectively.

The application σ which maps each α to $\overline{\alpha}$ is an automorphism of D. First, we have, for any $\alpha, \beta \in D$,

$$x^u\,\overline{\alpha\,\beta} = x^{u\alpha\beta} = x^{u\alpha}\,\overline{\beta} = x^u\,\overline{\alpha}\,\overline{\beta} \ ,$$

which means $\sigma(\alpha\beta) = \sigma(\alpha)\,\sigma(\beta)$. It is easy to see $\sigma(\alpha+\beta) = \sigma(\alpha) + \sigma(\beta)$. The endomorphism σ is also injective, since it extends the Id_K, hence it is an automorphism of D, since D is finite-dimensional over K.

For each $\alpha \in D$, we have $f(e^i \alpha e_j) = x^{e^i} \sigma(\alpha) y_{e_j}$, and therefore $f(X) = A\overline{X}B$, for all $X \in M_n(D)$, where A is the matrix whose columns are $x^{e^1}, ..., x^{e^n}$ and B is the matrix whose rows are $y_{e_1}, ..., y_{e_n}$, and

$$\overline{X} = (\overline{x}_{ij}) = \Big(\sigma(x_{ij})\Big) \ .$$

Let us note that $x^{e^1}, ..., x^{e^n}$ and $y_{e_1}, ..., y_{e_n}$ are linearly independent. For, if $\sum_i x^{e^i} \alpha_i = 0$, then

$$0 = \sum_i x^{e^i} \alpha_i \, y_v = f\Big(\sum_i e^i \, \sigma^{-1}(\alpha_i) \, v\Big)$$

which implies $\sum_i e^i \, \sigma^{-1}(\alpha_i) = 0$, i.e., $\alpha_1 = ... = \alpha_n = 0$. Similarly we prove $y_{e_1}, ..., y_{e_n}$ are left linearly independent. This fact leads us to the conclusion that A and B are invertible ([3]).

Using similar arguments, we conclude for the other case that $f(X) = A\,\overline{X'}B$, for all $X \in M_n(D)$, where A is the invertible matrix whose columns are $x^{e_1}, ..., x^{e_n}$ and B is the invertible matrix whose rows are $y_{e^1}, ..., y_{e^n}$. ∎

4 – Dieudonné determinant preservers

Next, we still work on generalizations for skew fields of results obtained before ([7]). The last theorem will have a crucial role and the concept of Dieudonné determinant will be needed.

Lemma 4.1. *Let f be a semilinear map on $M_n(D)$. If f preserves Dieudonné determinant, then f is nonsingular.*

Proof: Suppose that $f(X) = 0$. Let Y be a matrix of rank $n - \operatorname{rank} X$ such that $X + Y$ is nonsingular (Lemma 1.4). Then

$$\begin{aligned}
\det\Big(f(Y)\Big) &= \det\Big(f(X) + f(Y)\Big) \\
&= \det\Big(f(X+Y)\Big) \\
&= \det(X+Y) \\
&\neq 0 \ .
\end{aligned}$$

Hence $\det(Y) \neq 0$, i.e., Y is nonsingular and therefore $X = 0$. We conclude that f is nonsingular. ∎

We only have to prove now that if f preserves Dieudonné determinant, then f preserves rank 1.

Theorem 4.2. *A semilinear map f on $M_n(D)$ preserves the Dieudonné determinant if and only if there exist nonsingular matrices A, B and a bijective map σ of D, with $\sigma_{|K} = \mathrm{Id}_K$, such that either σ is a homorphism and*

$$f(X) = A\,\overline{X}B\ ,$$

for all $X \in M_n(D)$, or σ is an anti-homorphism and

$$f(X) = A\,\overline{X'}B\ ,$$

for all $X \in M_n(D)$, with $\overline{X} = (\overline{x}_{ij}) = (\sigma(x_{ij}))$.

Proof: Let X be a matrix of rank 1 and suppose that $f(X)$ has rank k. One assumes that $X = T_1\,U_1\,P(\pi_1)\,V_1$ and $f(X) = T_2\,U_2\,P(\pi_2)\,V_2$ with T_i, V_i, π_i as in Theorem 1.2, and, without loss of generality, $U_1 = \alpha\,E_{11}$ and $U_2 = \alpha_1\,E_{11} + \cdots + \alpha_k\,E_{kk}$. Let ξ be an indeterminate over K and make $Y = T_2\,U_Y\,P(\pi_2)\,V_2$, with $U_Y = E_{k+1k+1} + \cdots + E_{nn}$. Then

$$\begin{aligned}\det\Big(\xi\,f(X) + Y\Big) &= \det\Big(\xi\,T_2(U_2 + U_Y)\,P(\pi_2)\,V_2\Big)\\ &= \xi^k\,\alpha_1\cdots\alpha_k\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}\det\Big(\xi\,f(X) + Y\Big) &= \det\Big(f(\xi\,X + f^{-1}(Y)\Big)\\ &= \det\Big(\xi\,X + f^{-1}(Y)\Big)\ .\end{aligned}$$

Since $\det(f^{-1}(Y)) = \det(Y) = 0$, the rank of $f^{-1}(Y)$ is less than n. In other hand, it can't be less than $n-2$. This would imply that $\det(\xi\,f(X) + Y) = 0$. Then the rank of $f^{-1}(Y)$ is $n-1$ and determinant (4.1) is a polynomial in ξ. Thus $k = 1$, and T preserves rank 1 matrices.

We note now that if $X = T\,U P(\pi)\,V$,

$$X = \begin{bmatrix} 1 & & 0\\ & \ddots & \\ * & & 1\end{bmatrix} \begin{bmatrix} u_1 & & 0\\ & \ddots & \\ 0 & & u_n\end{bmatrix} P(\pi) \begin{bmatrix} 1 & & *\\ & \ddots & \\ 0 & & 1\end{bmatrix}$$

then $\overline{X} = \overline{T}\,\overline{U}\,P(\pi)\,\overline{V}$,

$$\overline{X} = \begin{bmatrix} 1 & & 0\\ & \ddots & \\ \overline{*} & & 1\end{bmatrix} \begin{bmatrix} \sigma(u_1) & & 0\\ & \ddots & \\ 0 & & \sigma(u_n)\end{bmatrix} P(\pi) \begin{bmatrix} 1 & & \overline{*}\\ & \ddots & \\ 0 & & 1\end{bmatrix}\ .$$

Therefore $\det(\overline{X}) = \sigma(\det(X))$. Since either $f(X) = A\,\overline{X}B$, for all X, or $f(X) = A\,\overline{X'}B$, for all X, and $\det(f(X)) = \det(X)$, the result follows. Note that $\sigma(x\,y) = \sigma(x)\,\sigma(y)$, implies that $\det(A\,B) = 1$.

The converse is obvious. ∎

Notice that if D is a (commutative) field, then the last theorem reduces to the one obtained by Frobenius and by Mink.

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