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WEIL NEARNESS SPACES *

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Abstract: In this paper the concept of nearness for spaces is introduced in terms of Weil's notion of entourage, motivated by the study of the corresponding frame-theoretical objects. It is shown that these spaces, although distinct from the classical nearness spaces of Herrlich, also form a nice topological category. Indeed, it is proved that this category contains, as nicely embedded full subcategories, various categories of topological nature such as, for example, the categories of symmetric topological spaces and continuous maps, proximal spaces and proximal maps and uniform spaces and uniformly continuous maps.

Introduction

Topological spaces are the result of the axiomatization of the concept of nearness between a point x and a set A (expressed by the relation $x \in cl(A)$). On the other hand, proximal spaces are obtained by an axiomatization of the concept of nearness between two sets A and B (usually denoted by $A\delta B$, i.e., "A is near B" [17]) and contigual spaces express the concept of nearness between the elements of a finite family $\mathcal{A}$ of sets (usually denoted by $\sigma(\mathcal{A})$ [14]).

The concept of nearness space was introduced by Herrlich [10] as an axiomatization of the concept of "nearness of an arbitrary collection $\mathcal{A}$ of sets" (usually denoted by $\mathcal{A} \in \xi$, i.e., "$\mathcal{A}$ is near" [10]), with the goal of unifying several types of topological structures; as the author says in [11]:

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> "*The aim of this approach is to find a basic topological concept — if possible intuitively accessible — by means of which any topological concept or idea can be expressed*".

This goal was achieved by proving that the category of nearness spaces contains the categories of all symmetric topological spaces [23] and continuous maps, of all proximal spaces and proximal maps (δ-maps) [7], of all uniform spaces and uniformly continuous maps ([24], [25]), of all contigual spaces and contigual maps [14] as nicely embedded (either bireflective or bicoreflective) full subcategories ([10], [11]).

Nearness spaces arise by dropping the star-refinement condition on the uniform covers of a uniform space. As it is well-known, uniform spaces were first axiomatized by Weil [25] using entourages instead of covers. Unfortunately, the nice well-known bijective correspondence between entourages and uniform covers, which holds for uniform spaces, is not extendable to nearnesses (cf. [5] and Bentley's review to it in Mathematical Reviews [4]). In the final step of our investigation (in [19]) of the frame-theoretic version of the correspondence

$$\text{covers} \longleftrightarrow \text{Weil entourages}\ ,$$

we arrived to the consideration of the category of Weil nearness frames. This category naturally asks for its spatial companion. This motivates the study of the category of Weil nearness spaces, whose objects arise as the natural notion of spatial nearness via entourages. Although distinct from the classical nearness spaces of Herrlich, these spaces form a topological category (Proposition 3.1) which also fulfils the goal sought for by Herrlich in [10]: it is a nice supercategory of the categories of symmetric topological spaces (Propositions 3.4, 3.5, 3.6 and 3.7), proximal spaces (Propositions 3.12 and 3.13 and Corollary 3.14) and uniform spaces (Remark 2.2 (c) and Proposition 3.8).

This way, it is shown that the notion of Weil entourage is also a basic topological concept by means of which several topological ideas can be expressed.

1 – Motivation

Our main references are: for nearness spaces — [5], [10], [11]; for frames — [15]; for (covering) structured frames — [2], [3], [9], [12]; for (entourage) structured frames — [18], [19], [20]; for topological categories — [1].

Pointfree topology focuses on the open sets rather than the points of a space, regarding the latter as subsidiary to the former, and deals with abstractly defined "lattices of open sets" called *frames* (or *locales*, whatever terminology one may adopt) and their homomorphisms. For the record, a frame L is a complete lattice in which the infinite distributivity law

$$x \wedge \bigvee S = \bigvee \Big\{ x \wedge y \mid y \in S \Big\}$$

holds for all $x \in L$ and $S \subseteq L$. The motivating examples of frames are topologies: for every topological space $(X, \mathcal{T})$ the lattice $(\mathcal{T}, \subseteq)$ of open sets is a frame. A *frame homomorphism* is a map between frames which preserves finitary meets and arbitrary joins. There is a natural contravariant functor Ω defined by

$$\Omega(X, \mathcal{T}) = (\mathcal{T}, \subseteq) \quad \text{and} \quad \Omega(f) = f^{-1}$$

that assigns frames to topological spaces. This functor restricts to a full embedding of the subcategory of sober spaces into the dual of the category of frames (the category of locales); hence locales can be viewed as a generalization of sober spaces — see Johnstone [15].

The idea of endowing a frame with a uniform-type structure first appeared in Isbell [13]. In [9], Frith introduced the notion of quasi-uniform frame and the first definition of nearness frame appeared in Banaschewski and Pultr [3]. All these notions are presented by means of covers (a *cover* of a frame L is a subset C of L satisfying $\bigvee C = 1$). They are guided by adjunctions between the respective categories and the corresponding categories of spaces (cf. [9] and [12]) which lift the well-known adjunction between topological spaces and frames [15].

In [18], the author introduced the notion of Weil uniformity for frames, expressed in terms of Weil entourages, which constitutes the frame-theoretic version of Weil's approach to uniform spaces [25]. This notion is proved to be equivalent to the covering one of Isbell. The category of Weil quasi-uniform frames is presented in [20] and it is also isomorphic to the corresponding category of covering structured frames (the quasi-uniform frames of Frith). In [19], the concept of Weil nearness frames is introduced to generalize Weil uniform frames, in the same spirit as nearness frames generalize uniform frames. Let us briefly recall it:

We follow Kříž and Pultr [16] in defining the coproduct

$$L \xrightarrow{u_1^L} L \oplus L \xleftarrow{u_2^L} L$$

of a frame L by itself: a subset A of a partially ordered set $(X, \leq)$ is said to be a down-set if $A = \downarrow A$ where $\downarrow A$ denotes the set $\{x \in X \mid x \leq a \text{ for some } a \in A\}$. Take the cartesian product $L \times L$ with the usual order. A down-set A of $L \times L$

is a *C-ideal* if

$$\{x\} \times S \subseteq A \implies \left(x, \bigvee S\right) \in A$$

and

$$S \times \{y\} \subseteq A \implies \left(\bigvee S, y\right) \in A\,.$$

Put $L \oplus L$ as the frame of all C-ideals of $L \times L$. Observe that the case $S = \emptyset$ implies that every C-ideal contains the set $\mathbb{O} := \downarrow\{(1,0)\} \cup \downarrow\{(0,1)\}$. Obviously, each $\downarrow\{(x,y)\} \cup \mathbb{O}$ is a C-ideal. It is denoted by $x \oplus y$. Finally put $u_1^L(x) = x \oplus 1$ and $u_2^L(y) = 1 \oplus y$.

If $f\colon L \to M$ is a frame homomorphism, we write

$$f \oplus f\colon L \oplus L \longrightarrow M \oplus M$$

for the unique frame homomorphism given by $(f \oplus f) \cdot u_i^L = u_i^M \cdot f$, for $i \in \{1,2\}$.

In any frame L, a *Weil entourage* of L is an element E of $L \oplus L$ such that

$$\bigvee_{(x,x) \in E} x = 1$$

or, equivalently, for which there exists a cover U of L satisfying $\bigvee_{x \in U}(x \oplus x) \subseteq E$. The collection $\mathrm{WEnt}(L)$ of all Weil entourages of L may be partially ordered by inclusion. This is a partially ordered set with finitary meets.

For $E, F \in \mathrm{WEnt}(L)$, we define the composition $E \circ F$ as the Weil entourage

$$\bigvee\Big\{x \oplus y \mid \exists\, z \in L\backslash\{0\}\colon\ (x,z) \in E,\ (z,y) \in F\Big\}$$

and the inverse E^{-1} as $\{(y,x) \mid (x,y) \in E\}$.

Definition 1.1. A *Weil nearness frame* is a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a non-empty filter of $(\mathrm{WEnt}(L), \subseteq)$ such that:

(**1**) $E^{-1} \in \mathcal{E}$ for every $E \in \mathcal{E}$;

(**2**) for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$, where $y \overset{\mathcal{E}}{\lhd} x$ means that $E \circ (y \oplus y) \subseteq x \oplus x$ for some $E \in \mathcal{E}$.

Let $(L, \mathcal{E})$ and $(L', \mathcal{E}')$ be Weil nearness frames. A *Weil frame homomorphism* $f : (L, \mathcal{E}) \to (L', \mathcal{E}')$ is a frame map $f : L \to L'$ such that $(f \oplus f)(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$.

This definition motivates the following problems: Which is the right spatial concept in analogy with the chosen notion of Weil nearness frame? May this concept be expressed in terms of Weil's entourages for sets? In the next section we present the answers to these questions.

2 – Framed Weil nearness spaces

Let us denote by $\mathrm{WEnt}(X)$ the collection of all *entourages* (i.e., reflexive relations) of the set X. For any $E \in \mathrm{WEnt}(X)$, $x \in X$ and $A \subseteq X$ let

$$E[x] := \Big\{ y \in X \mid (x,y) \in E \Big\}$$

and

$$E[A] := \bigcup_{a \in A} E[a] \ .$$

If X is endowed with a topology $\mathcal{T}$, the *open entourages* of $(X, \mathcal{T})$ are the entourages of X which are open in the topological product of X by itself. In other words, E is open if and only if it coincides with

$$\mathrm{int}(E) := \Big\{ (x,y) \in X \times X \mid \exists\, U, V \in \mathcal{T}\colon (x,y) \in U \times V \subseteq E \Big\} \ .$$

Evidently, when E is open, $E[x]$ and $E^{-1}[x]$, for every $x \in X$, are open.

We say that an entourage E of $(X, \mathcal{T})$ is an *interior entourage* provided that $\mathrm{int}(E)$ remains an entourage, or, equivalently, if E contains an open entourage.

Any Weil nearness $\mathcal{E}$ on the frame $\mathcal{T}$ of open sets of a topological space $(X, \mathcal{T})$ gives rise to a set

$$\mathcal{E}' := \Big\{ \bigcup_{(A,B) \in E} A \times B \mid E \in \mathcal{E} \Big\}$$

of open entourages of $(X, \mathcal{T})$ which satisfies

(**Fr0**) $E^{-1} \in \mathcal{E}'$ for every $E \in \mathcal{E}'$,

(**Fr1**) $\mathcal{E}'$ is a filter base (with respect to $\subseteq$),

(**Fr2**) for every $U \in \mathcal{T}$ and for every $x \in U$ there is $V \in \mathcal{T}$ and $E \in \mathcal{E}'$ such that $x \in V$ and $E \circ (V \times V) \subseteq U \times U$.

Indeed, conditions (Fr0) and (Fr1) are obvious since

$$\Big(\bigcup_{(A,B) \in E} A \times B \Big)^{-1} = \bigcup_{(A,B) \in E^{-1}} A \times B$$

and

$$\bigcup_{(A,B) \in E \cap F} A \times B \subseteq \Big(\bigcup_{(A,B) \in E} A \times B \Big) \cap \Big(\bigcup_{(A,B) \in F} A \times B \Big) \ ,$$

and condition (Fr2) is just the pointwise formulation of the admissibility condition (2) for Weil nearness frames:

If $E \circ (V \oplus V) \subseteq U \oplus U$ then, for any $(x,y) \in A \times B$, with $(A,B) \in E$, and for any $(y,z) \in V \times V$, we have $B \cap V \neq \emptyset$, $(A, B \cap V) \in E$ and $(B \cap V, V) \in V \oplus V$ hence $(A,V) \in U \oplus U$, which implies $(x,z) \in A \times V \subseteq U \times U$.

We designate the topological spaces $(X,\mathcal{T})$ endowed with a filter of open entourages of $(X,\mathcal{T})$ satisfying conditions (Fr0) and (Fr2) as *framed Weil nearness spaces*. The motivation for this designation comes from the corresponding (covering) framed nearness spaces of Hong and Kim [12].

Note that $E \circ (V \times V) \subseteq U \times U$ is equivalent to $E^{-1}[V] \subseteq U$. Thus, in presence of (Fr0), (Fr2) means that, for any $U \in \mathcal{T}$, $U = \bigcup\{V \in \mathcal{T} \mid E[V] \subseteq U$ for some $E \in \mathcal{E}'\}$.

The morphisms of the category $\mathsf{FrWNear}$ of framed Weil nearness spaces are the maps

$$f\colon (X,\mathcal{T},\mathcal{E}) \to (X',\mathcal{T}',\mathcal{E}')$$

for which $(f \times f)^{-1}(E) \in \mathcal{E}$ for any $E \in \mathcal{E}'$.

On the reverse direction, any framed Weil nearness $\mathcal{E}$ on $(X,\mathcal{T})$ gives rise to a Weil nearness on the frame $\mathcal{T}$ so that the spatial and frame notions coincide in this context. In order to conclude this, just take the family of all

$$\bigvee_{x \in X} \Big(E[x] \oplus E[x]\Big) \qquad (E \in \mathcal{E})\ ,$$

as a base. Since

$$\bigvee_{x \in X} \Big((E \cap F)[x] \oplus (E \cap F)[x]\Big) \subseteq \Big(\bigvee_{x \in X} \Big(E[x] \oplus E[x]\Big)\Big) \cap \Big(\bigvee_{x \in X} \Big(F[x] \oplus F[x]\Big)\Big)$$

and

$$\Big(\bigvee_{x \in X} \Big(E[x] \oplus E[x]\Big)\Big)^{-1} = \bigvee_{x \in X} \Big(E[x] \oplus E[x]\Big)$$

this is, in fact, a filter base.

The proof of the admissibility condition runs as follows:

For any $U \in \mathcal{T}$ and $x \in U$ consider $W \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in W$ and $E \circ (W \times W) \subseteq U \times U$, and pick $V \in \mathcal{T}$ and $F \in \mathcal{E}$ such that $x \in V$ and $F \circ (V \times V) \subseteq W \times W$. For $G = (F \cap E) \cap (F \cap E)^{-1} \in \mathcal{E}$,

$$\Big(\bigvee_{x \in X} \Big(G[x] \oplus G[x]\Big)\Big) \circ (V \oplus V) \subseteq U \oplus U\ ,$$

as can be easily proved.

These correspondences are functorial and establish an equivalence between the categories of framed Weil nearness spaces and spatial Weil nearness frames.

Therefore, the notion of framed Weil nearness space is the right spatial analog of the frame concept of Weil nearness.

Let us see how framed Weil nearness spaces can be equated within the framework of a generalization of Weil's uniform spaces.

Definitions 2.1. Let X be a set and let $\mathcal{E}$ be a non-empty set of entourages of X. Consider the following axioms:

(**WN0**) $E^{-1} \in \mathcal{E}$ for every $E \in \mathcal{E}$;

(**WN1**) If $E \subseteq F$ and $E \in \mathcal{E}$ then $F \in \mathcal{E}$;

(**WN2**) $E \cap F \in \mathcal{E}$ for every $E, F \in \mathcal{E}$;

(**WN3**) for each $E \in \mathcal{E}$,

$$\Big\{(x,y) \in X \times X \mid \exists\, U, V \subseteq X\colon U = \mathrm{int}_{\mathcal{E}}(U),\ V = \mathrm{int}_{\mathcal{E}}(V),\ (x,y) \in U \times V \subseteq E\Big\} \in \mathcal{E}\ ,$$

where, for any $A \subseteq X$,

$$\mathrm{int}_{\mathcal{E}}(A) = \Big\{x \in X \mid \exists\, E \in \mathcal{E}\colon E[x] \subseteq A\Big\}\ .$$

$\mathcal{E}$ is called a *Weil prenearness* on X if it satisfies (WN0) and (WN1); $\mathcal{E}$ is called a *Weil seminearness* on X if it satisfies (WN0), (WN1) and (WN2) and $\mathcal{E}$ is called a *Weil nearness* on X if it fulfils (WN0), (WN1), (WN2) and (WN3). The pair $(X, \mathcal{E})$ is called a *Weil prenearness space* (respectively, *Weil seminearness space*, *Weil nearness space*) if $\mathcal{E}$ is a Weil prenearness (respectively, Weil seminearness, Weil nearness) on X.

A *Weil nearness map* is just a map $f\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$ between Weil prenearness spaces for which $(f \times f)^{-1}(E) \in \mathcal{E}$ for every $E \in \mathcal{E}'$.

We denote by PWNear the category of Weil prenearness spaces and Weil nearness maps and by SWNear and WNear its full subcategories of, respectively, Weil seminearness spaces and Weil nearness spaces.

Remarks 2.2.

(**a**) If $(X, \mathcal{E})$ is a Weil prenearness space then $\mathrm{int}_{\mathcal{E}}$ is an operator on $\mathcal{P}(X)$ satisfying the following axioms:

(**T0**) $x \in \mathrm{int}_{\mathcal{E}}(X \backslash \{y\})$ if and only if $y \in \mathrm{int}_{\mathcal{E}}(X \backslash \{x\})$, for every pair x, y of elements of X;

(T1) $\mathrm{int}_{\mathcal{E}}(X) = X$;

(T2) $\mathrm{int}_{\mathcal{E}}(A) \subseteq A$ for every $A \subseteq X$;

(T3) If $A \subseteq B$ then $\mathrm{int}_{\mathcal{E}}(A) \subseteq \mathrm{int}_{\mathcal{E}}(B)$.

If $(X, \mathcal{E})$ is a Weil seminearness space then, in addition, $\mathrm{int}_{\mathcal{E}}$ satisfies the axiom

(T4) $\mathrm{int}_{\mathcal{E}}(A \cap B) = \mathrm{int}_{\mathcal{E}}(A) \cap \mathrm{int}_{\mathcal{E}}(B)$.

Finally, if $(X, \mathcal{E})$ is a Weil nearness space, then $\mathrm{int}_{\mathcal{E}}$ also satisfies the axiom

(T5) $\mathrm{int}_{\mathcal{E}}(\mathrm{int}_{\mathcal{E}}(A)) = \mathrm{int}_{\mathcal{E}}(A)$.

Thus any Weil nearness structure $\mathcal{E}$ on X induces on X a symmetric topology $\mathcal{T}_{\mathcal{E}}$ (i.e., a topology satisfying the axiom (T0) of Šanin [23]). Axiom (WN3) says that, with respect to the product topology $\mathcal{T}_{\mathcal{E}} \times \mathcal{T}_{\mathcal{E}}$, $\mathrm{int}(E) \in \mathcal{E}$ whenever $E \in \mathcal{E}$.

(**b**) The Weil seminearness spaces are the semi-uniform spaces of Čech [6]. In this case $\mathrm{int}_{\mathcal{E}}$ may not be an interior operator. It only defines a closure operator in the sense of Čech [6].
The non-symmetric (i.e., without (WN0)) Weil seminearnesses are studied by H.W. Pu and H.H. Pu in [21] and [22].

(**c**) Every uniform space is a Weil nearness space since the refinement condition

$$\forall E \in \mathcal{E} \;\; \exists F \in \mathcal{E} \colon \quad F \circ F \subseteq E$$

implies condition (WN3). Indeed, if we pick a symmetric $F \in \mathcal{E}$ such that $F^3 \subseteq E$ then $F \subseteq \mathrm{int}(E)$ so $\mathrm{int}(E) \in \mathcal{E}$.

The category SWNear is bicoreflective in PWNear. If $(X, \mathcal{E})$ is a Weil prenearness space and $\mathcal{E}_S$ is the set of all entourages of X which contain the intersection of a finite number of elements of $\mathcal{E}$, then $1_X \colon (X, \mathcal{E}_S) \to (X, \mathcal{E})$ is the bicoreflection of $(X, \mathcal{E})$ with respect to SWNear.

The category WNear is bireflective in SWNear. For $(X, \mathcal{E}) \in \mathsf{SWNear}$ define, for every ordinal α, the operator int^{α} on $\mathcal{P}(X)$ by

- $\mathrm{int}^0(A) := A$,
- $\mathrm{int}^{\alpha}(A) := \mathrm{int}^{\beta}(A) \backslash \{x \in \mathrm{int}^{\beta}(A) \mid \forall E \in \mathcal{E} \;\; E[x] \cap (X \backslash A) \neq \emptyset\}$ if $\alpha = \beta + 1$,
- $\mathrm{int}^{\alpha}(A) := \bigcap_{\beta < \alpha} \mathrm{int}^{\beta}(A)$ if α is a limit ordinal.

Then

$$\operatorname{int}(A) := \bigcap_{\alpha \in Ord} \operatorname{int}^{\alpha}(A)$$

is the "largest" operator on $\mathcal{P}(X)$ satisfying axioms (T0), (T1), (T2), (T3), (T4) and (T5) and so it defines a symmetric topology $\mathcal{T}$ on X. Putting

$$\mathcal{E}_N := \Big\{ E \subseteq X \times X \mid \operatorname{int}(E) \in \mathcal{E} \Big\} ,$$

$1_X \colon (X, \mathcal{E}) \to (X, \mathcal{E}_N)$ is the bireflection of $(X, \mathcal{E})$ with respect to WNear.

Let us now consider on a Weil nearness space $(X, \mathcal{E})$ the spatial analog of the partial order $\overset{\mathcal{E}}{\lhd}$.

For subsets A and B of X, we write $A <_{\mathcal{E}} B$ whenever there is $E \in \mathcal{E}$ such that $E[A] \subseteq B$. In particular, $x <_{\mathcal{E}} A$ means that $x \in \operatorname{int}_{\mathcal{T}_{\mathcal{E}}}(A)$. Moreover, if $A <_{\mathcal{E}} B$ then $A \subseteq \operatorname{int}_{\mathcal{T}_{\mathcal{E}}}(B)$.

Proposition 2.3. *Let $(X, \mathcal{T}, \mathcal{E})$ be a framed Weil nearness space and let $\overline{\mathcal{E}}$ be the filter of* $(\mathrm{WEnt}(X), \subseteq)$ *generated by $\mathcal{E}$. Then:*

(**a**) $\mathcal{T}_{\overline{\mathcal{E}}} = \mathcal{T}_{\mathcal{E}} = \mathcal{T}$;

(**b**) $(X, \overline{\mathcal{E}})$ *is a Weil nearness space satisfying the condition*

$$x <_{\overline{\mathcal{E}}} A \implies \exists\, B \subseteq X \colon x <_{\overline{\mathcal{E}}} B <_{\overline{\mathcal{E}}} A .$$

Proof: (**a**) Let $A \in \mathcal{T}$ and $x \in A$. By assumption, there are $V \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in V$ and $E[V] \subseteq A$. Then $E[x] \subseteq A$, so $x \in \operatorname{int}_{\mathcal{T}_{\mathcal{E}}}(A)$ and $A \in \mathcal{T}_{\mathcal{E}}$. Conversely, if $A \in \mathcal{T}_{\mathcal{E}}$, there is, for each $x \in A$, $E_x \in \mathcal{E}$ with $E_x[x] \subseteq A$. Therefore

$$A = \bigcup \Big\{ E_x[x] \mid x \in A \Big\} \in \mathcal{T} .$$

The other equality is now obvious.

(**b**) The proof that $(X, \overline{\mathcal{E}})$ is a Weil nearness space is trivial.

Assume $x <_{\overline{\mathcal{E}}} A$, i.e., $x \in \operatorname{int}_{\mathcal{T}_{\overline{\mathcal{E}}}}(A)$. Then $x \in \operatorname{int}_{\mathcal{T}}(A)$. By hypothesis, there are $B \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in B$ and $E[B] \subseteq \operatorname{int}_{\mathcal{T}}(A)$. Since B is open, $x <_{\overline{\mathcal{E}}} B$. On the other hand, $B <_{\overline{\mathcal{E}}} A$ because $E \in \mathcal{E} \subseteq \overline{\mathcal{E}}$. ∎

Proposition 2.4. *Let $(X, \mathcal{E})$ be a Weil nearness space satisfying axiom*

(**WN4**) $x <_{\mathcal{E}} A \Longrightarrow \exists\, B \subseteq X \colon x <_{\mathcal{E}} B <_{\mathcal{E}} A$,

and let $\mathring{\mathcal{E}} = \{\operatorname{int}(E) \mid E \in \mathcal{E}\}$ be the set of open entourages in $\mathcal{E}$. Then $(X, \mathcal{T}_{\mathcal{E}}, \mathring{\mathcal{E}})$ is a framed Weil nearness space.

Proof: Since $\mathrm{int}(E_1) \cap \mathrm{int}(E_2) = \mathrm{int}(E_1 \cap E_2)$, it is evident that $\overset{\circ}{\mathcal{E}}$ is a filter of open entourages of $(X, \mathcal{T}_{\mathcal{E}})$.

Axiom (Fr0) is a consequence of the fact that $(\mathrm{int}(E))^{-1} = \mathrm{int}(E^{-1})$.

Let us check axiom (Fr2): consider $U \in \mathcal{T}_{\mathcal{E}}$ and $x \in U$. Then $x <_{\mathcal{E}} U$, so, by assumption, there is some $B \subseteq X$ such that $x <_{\mathcal{E}} B <_{\mathcal{E}} U$. This means that $x \in \mathrm{int}_{\mathcal{T}_{\mathcal{E}}}(B)$ and $E[\mathrm{int}_{\mathcal{T}_{\mathcal{E}}}(B)] \subseteq E[B] \subseteq U$, for some $E \in \mathcal{E}$, so it suffices to take $V = \mathrm{int}_{\mathcal{T}_{\mathcal{E}}}(B)$ and $\mathrm{int}(E) \in \overset{\circ}{\mathcal{E}}$. ∎

Condition (WN4) is an analog, for Weil nearnesses, of condition ([12], Definition 1.1).

Corollary 2.5. *The category* $\mathsf{FrWNear}$ *is isomorphic to the full subcategory* $\mathsf{WNear}_{(\mathsf{WN4})}$ *of* WNear *of all Weil nearness spaces satisfying (WN4).*

Proof: Let us begin by showing that, for any morphism $f\colon (X, \mathcal{E}) \to (X', \mathcal{E}')$ in $\mathsf{WNear}_{(\mathsf{WN4})}$,

$$f\colon (X, \mathcal{T}_{\mathcal{E}}, \overset{\circ}{\mathcal{E}}) \to (X', \mathcal{T}_{\mathcal{E}'}, \overset{\circ}{\mathcal{E}'})$$

belongs to $\mathsf{FrWNear}$. So, consider $E \in \overset{\circ}{\mathcal{E}'}$, that is, $E \in \mathcal{E}'$ with $\mathrm{int}(E) = E$. By hypothesis, $(f \times f)^{-1}(E) \in \mathcal{E}$. But

$$(f \times f)^{-1}(\mathrm{int}(E)) \subseteq \mathrm{int}((f \times f)^{-1}(E)) \; .$$

In fact, if $(f(x), f(y)) \in \mathrm{int}(E)$, there are $U, V \in \mathcal{T}_{\mathcal{E}'}$ such that $(f(x), f(y)) \in U \times V \subseteq E$ which implies $(x, y) \in f^{-1}(U) \times f^{-1}(V) \subseteq (f \times f)^{-1}(E)$ with $f^{-1}(U), f^{-1}(V) \in \mathcal{T}_{\mathcal{E}}$.

Therefore $(f \times f)^{-1}(E)$ is an open entourage of $\mathcal{E}$ and, thus, it belongs to $\overset{\circ}{\mathcal{E}}$.

On the reverse direction, the fact that $f : (X, \overline{\mathcal{E}}) \to (X', \overline{\mathcal{E}'})$ belongs to $\mathsf{WNear}_{(\mathsf{WN4})}$ whenever $f : (X, \mathcal{T}, \mathcal{E}) \to (X', \mathcal{T}', \mathcal{E}')$ belongs to $\mathsf{FrWNear}$ is obvious.

Now the existence of the isomorphism is an immediate corollary of Propositions 2.3 and 2.4 and the following two obvious facts:

- $\overset{\circ}{\overline{\mathcal{E}}} = \mathcal{E}$ for any framed Weil nearness $\mathcal{E}$;
- $\overline{\overset{\circ}{\mathcal{E}}} = \mathcal{E}$ for any Weil nearness $\mathcal{E}$ satisfying (WN4). ∎

3 – The category WNear as a unified theory of (symmetric) topology, proximity and uniformity

The classical correspondence between uniform covers and uniform entourages still works for nearnesses; in fact, for any nearness space (X, μ), the collection of all

$$\bigcup_{U \in \mathcal{U}} U \times U \qquad (\mathcal{U} \in \mu)$$

forms a base for a Weil nearness on X and, conversely, for any Weil nearness space $(X, \mathcal{E})$, the covers

$$\Big\{E[x] \colon x \in X\Big\} \qquad (E \in \mathcal{E})$$

form a base for a nearness on X. These correspondences are functorial and define a Galois correspondence which is an isomorphism precisely when restricted to uniformities. Furthermore, as Bentley pointed out in [4], there is no concrete isomorphism between the categories of nearness spaces and Weil nearness spaces (considered as concrete categories over the category Set of sets and functions).

In spite of this, our category of Weil nearness spaces still have the nice categorical properties that Herrlich was looking for when searching for a good axiomatization of nearness ([10], [11]).

For example:

Proposition 3.1. *The category* WNear *is a well-fibred topological category over the category* Set.

Proof: The well-fibreness is obvious: for any set X, the class of all Weil nearness spaces $(X, \mathcal{E})$ with underlying set X is a set and there exists exactly one Weil nearness space with underlying set X whenever X is of cardinality at most one.

It remains to show that the forgetful functor $\mathsf{WNear} \xrightarrow{|\cdot|} \mathsf{Set}$ is topological, i.e., that every $|\cdot|$-structured source $\Big(X \xrightarrow{f_i} |X_i, \mathcal{E}_i|\Big)_{i \in I}$ has a unique $|\cdot|$-initial lift

$$\Big((X, \mathcal{E}) \xrightarrow{f_i} (X_i, \mathcal{E}_i)\Big)_{i \in I} .$$

Consider

$$\mathcal{B} = \left\{\bigcap_{j=1}^{n} (f_{i_j} \times f_{i_j})^{-1}(E_j) \mid n \in \mathbb{N},\ i_j \in I,\ E_j \in \mathcal{E}_{i_j}\right\} \cup \{X \times X\} .$$

A straightforward checking ensures that this is a base for a Weil seminearness $\mathcal{E}$ on X. It is clear that this is the coarsest seminearness $\mathcal{E}$ on X for which every $(X, \mathcal{E}) \xrightarrow{f_i} (X_i, \mathcal{E}_i)$ is a Weil nearness map, so this is the initial lift in SWNear. Since WNear is bireflective in SWNear, as shown in Section 2, WNear is closed under the formation of initial sources in SWNear and, consequently, $(X, \mathcal{E})$ belongs to WNear. ∎

As we shall see in the sequel, WNear also unifies several types of topological structures such as the symmetric topological spaces, the proximal spaces and the uniform spaces.

Symmetric topological spaces

If we want to investigate whether topological spaces are embedded in WNear, our first step should be to try to axiomatize the concept of topological space in terms of entourages. We shall do this using interior entourages.

We have to make a restriction on the type of topological spaces we consider: they should be symmetric.

From now on, for subsets A and B of a set X, we denote the set

$$(X\backslash A \times X\backslash A) \cup (B \times B) \ ,$$

which is an entourage of X if and only if $A \subseteq B$, by $E^X_{A,B}$ (or, briefly, by $E_{A,B}$ whenever there is no danger of confusion). The set $E^X_{\{x\},B}$ will be denoted by $E^X_{x,B}$ (or $E_{x,B}$).

These sets characterize the order relation $<_{\mathcal{E}}$; we have $A <_{\mathcal{E}} B$ if and only if $E_{A,B} \in \mathcal{E}$.

Lemma 3.2. *In a symmetric topological space $(X, \mathcal{T})$ the following assertions are equivalent:*

(**i**) $x \in \mathrm{int}_{\mathcal{T}}(A)$;

(**ii**) $E_{x,A}$ *is an interior entourage of* $(X, \mathcal{T})$.

Proof: (**i**)$\Rightarrow$(**ii**): We need to show that, for any $y \in X$, there exists $U \in \mathcal{T}$ such that $(y, y) \in U \times U \subseteq E_{x,A}$.

Of course, if $y \in \mathrm{int}_{\mathcal{T}}(A)$, it suffices to pick $U = \mathrm{int}_{\mathcal{T}}(A)$.

On the other hand, if $y \in X\backslash\mathrm{int}_{\mathcal{T}}(A)$ take $U = \mathrm{int}_{\mathcal{T}}(X\backslash\{x\})$. By the symmetry of $(X, \mathcal{T})$, y is indeed in U:

$$\begin{aligned} y \in X\backslash\mathrm{int}_{\mathcal{T}}(A) &\implies \mathrm{int}_{\mathcal{T}}(A) \subseteq X\backslash\{y\} \\ &\implies x \in \mathrm{int}_{\mathcal{T}}(X\backslash\{y\}) \\ &\implies y \in \mathrm{int}_{\mathcal{T}}(X\backslash\{x\}) \ . \end{aligned}$$

(**ii**)$\Rightarrow$(**i**): It is obvious. ∎

The implication (ii)⇒(i) can be generalized in the following way:

Lemma 3.3. *Given two subsets A and B of a symmetric topological space $(X,\mathcal{T})$, if $E_{B,A}$ is an interior entourage of $(X,\mathcal{T})$ then $B \subseteq \mathrm{int}_{\mathcal{T}}(A)$.*

Proof: For every $b \in B$, $(b,b) \in U \times U \subseteq E_{B,A}$ for some $U \in \mathcal{T}$. Hence $b \in U \subseteq A$. ∎

Proposition 3.4. *The set $\mathcal{E}$ of all interior entourages of a symmetric topological space $(X,\mathcal{T})$ is a Weil nearness on X satisfying the axiom*

(**WN5**) *$E \in \mathcal{E}$ whenever $\mathrm{int}(E)$ is an entourage of X,*

and the topology induced by $\mathcal{E}$ coincides with $\mathcal{T}$.

Proof: The fact that $\mathcal{E}$ is a Weil nearness on X satisfying (WN5) is obvious. Let us prove that $\mathcal{T}$ coincides with the topology induced by $\mathcal{E}$, i.e., that for any subset A of X,

$$\mathrm{int}_{\mathcal{T}}(A) = \Big\{ x \in X \mid \exists\, E \in \mathcal{E} \colon\ E[x] \subseteq A \Big\}\,.$$

For $x \in \mathrm{int}_{\mathcal{T}}(A)$, consider the entourage $E_{x,A}$, which, by Lemma 3.2, belongs to $\mathcal{E}$. Of course, $E_{x,A}[x] = A$. Conversely, if there is some $E \in \mathcal{E}$ with $E[x] \subseteq A$, then, since $\mathrm{int}(E) \in \mathrm{WEnt}(X)$, there exist $U, V \in \mathcal{T}$ satisfying $(x,x) \in U \times V \subseteq E$. It follows that $x \in \mathrm{int}_{\mathcal{T}}(E[x]) \subseteq \mathrm{int}_{\mathcal{T}}(A)$. ∎

Proposition 3.5. *If $\mathcal{E}$ is a Weil nearness on a set X satisfying (WN5), there exists precisely one symmetric topology $\mathcal{T}$ on X such that $\mathcal{E}$ is the set of all interior entourages of $(X,\mathcal{T})$.*

Proof: Take for $\mathcal{T}$ the topology $\mathcal{T}_{\mathcal{E}}$ induced by $\mathcal{E}$. We already observed that $\mathcal{T}_{\mathcal{E}}$ is a symmetric topology on X. By (WN5), $\mathcal{E}$ contains all interior entourages of $(X,\mathcal{T}_{\mathcal{E}})$.

The reverse inclusion follows from (WN3): take $E \in \mathcal{E}$; then $\mathrm{int}(E)$ belongs to $\mathcal{E}$ and, in particular, it is an entourage. Hence E is an interior entourage.

The uniqueness of $\mathcal{T}$ is a corollary of the previous proposition. ∎

The preceding propositions show that symmetric topological spaces can be always identified as Weil nearness spaces satisfying axiom (WN5).

Proposition 3.6. *Suppose $f\colon (X,\mathcal{T}) \to (X',\mathcal{T}')$ is a map between symmetric topological spaces and let $\mathcal{E}_{(X,\mathcal{T})}$ (respectively, $\mathcal{E}_{(X',\mathcal{T}')}$) denote the set of all* *interior entourages of $(X,\mathcal{T})$ (respectively, $(X',\mathcal{T}')$). The following conditions are equivalent:*

(i) *f is continuous;*

(ii) *$E \in \mathcal{E}_{(X',\mathcal{T}')}$ implies $(f\times f)^{-1}(E) \in \mathcal{E}_{(X,\mathcal{T})}$.*

Proof: Since the interior entourages are the ones that are refined by some open entourage, the implication (i)⇒(ii) is an immediate consequence of the fact, already proved in Corollary 2.5, that $(f\times f)^{-1}(\mathrm{int}E) \subseteq \mathrm{int}((f\times f)^{-1}(E))$.

Conversely, suppose $V \in \mathcal{T}'$ and let $v \in V$. Then, by Lemma 3.2, $E^{X'}_{v,V} \in \mathcal{E}_{(X',\mathcal{T}')}$. Thus $(f\times f)^{-1}(E^{X'}_{v,V}) \in \mathcal{E}_{(X,\mathcal{T})}$, that is, $E^{X}_{f^{-1}(v),f^{-1}(V)} \in \mathcal{E}_{(X,\mathcal{T})}$. By Lemma 3.3, $f^{-1}(v) \subseteq \mathrm{int}_{\mathcal{T}}(f^{-1}(V))$ for every $v \in V$. Consequently,

$$f^{-1}(V) \subseteq \mathrm{int}_{\mathcal{T}}(f^{-1}(V)) ,$$

i.e., $f^{-1}(V) \in \mathcal{T}$. ∎

It follows from Propositions 3.4, 3.5 and 3.6 that the category $\mathsf{WNear}_{(\mathsf{WN5})}$ of Weil nearness spaces satisfying (WN5) is isomorphic to the category $\mathsf{R_0Top}$ of symmetric topological spaces. We have now an alternative way of equipping a set with the structure of a symmetric topological space: by prescribing the set of interior entourages. Moreover:

Proposition 3.7. *The category $\mathsf{WNear}_{(\mathsf{WN5})}$ is a bicoreflective subcategory of WNear.*

Proof: Given a Weil nearness space $(X,\mathcal{E})$, let $\mathcal{E}_T$ denote the set of all interior entourages of $(X,\mathcal{T}_{\mathcal{E}})$. We already know that $(X,\mathcal{E}_T)$ is a Weil nearness space satisfying (WN5). Furthermore, for any morphism $f\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$ in WNear, $f\colon (X,\mathcal{T}_{\mathcal{E}}) \to (X',\mathcal{T}_{\mathcal{E}'})$ is continuous so, by Proposition 3.6, $f\colon (X,\mathcal{E}_T) \to (X',\mathcal{E}'_T)$ is also in WNear. We get this way a functor

$$\begin{array}{ccc} T\colon \mathsf{WNear} & \longrightarrow & \mathsf{WNear}_{(\mathsf{WN5})} \\ (X,\mathcal{E}) & \longmapsto & (X,\mathcal{E}_T) \\ \Big((X,\mathcal{E}) \xrightarrow{f} (X',\mathcal{E}')\Big) & \longmapsto & \Big((X,\mathcal{E}_T) \xrightarrow{f} (X',\mathcal{E}'_T)\Big) \; . \end{array}$$

This is the coreflector functor. Since $\mathcal{E} \subseteq \mathcal{E}_T$, $id : (X,\mathcal{E}_T) \longrightarrow (X,\mathcal{E})$ is in WNear. This is the coreflection map for $(X,\mathcal{E})$. ∎

Uniform spaces

We observed in 2.2 (c) that the category UWNear of (Weil) uniform spaces and uniformly continuous maps is a full subcategory of WNear. Furthermore, the following holds:

Proposition 3.8. *The category* UWNear *is bireflective in* WNear.

Proof: For any $(X,\mathcal{E}) \in \mathsf{WNear}$ let

$$\mathcal{E}_U = \left\{ E \in \mathcal{E} \mid \exists\, (E_n)_{n\in\mathbb{N}} \text{ in } \mathcal{E} \text{ such that } E_1 = E \text{ and } E_{n+1}^2 \subseteq E_n \text{ for each } n \in \mathbb{N} \right\} .$$

Obviously, $(X,\mathcal{E}_U)$ is a uniform space and $1_X : (X,\mathcal{E}) \to (X,\mathcal{E}_U)$ is in WNear. This is the bireflection map. In fact, for any $f : (X,\mathcal{E}) \to (X',\mathcal{E}')$ in WNear with $(X',\mathcal{E}') \in \mathsf{UWNear}$, $f\colon (X,\mathcal{E}_U) \to (X',\mathcal{E}')$ is uniformly continuous: for any $E \in \mathcal{E}'$, as $(X',\mathcal{E}')$ is uniform, there is a family $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{E}'$ with $E_1 = E$ and $E_{n+1}^2 \subseteq E_n$ for every $n \in \mathbb{N}$. Take the family

$$\Big((f \times f)^{-1}(E_n)\Big)_{n\in\mathbb{N}}$$

which is in $\mathcal{E}$. This shows that $(f \times f)^{-1}(E) \in \mathcal{E}_U$. ∎

Proximal spaces

Definitions 3.9 (Efremovič [8]; cf. Naimpally and Warrack [17]).

(**1**) Let X be a set and let $\ll$ be a binary relation on $\mathcal{P}(X)$. The pair $(X,\ll)$ is a *proximal space* provided that:

- (**P1**) $X \ll X$ and $\emptyset \ll \emptyset$;
- (**P2**) $A \ll B$ implies $A \subseteq B$;
- (**P3**) $A \subseteq B \ll C \subseteq D$ implies $A \ll D$;
- (**P4**) $A \ll C$ and $B \ll C$ imply $A \cup B \ll C$;
- (**P5**) $A \ll B$ and $A \ll C$ imply $A \ll B \cap C$;
- (**P6**) if $A \ll B$ there exists a subset C of X such that $A \ll C \ll B$;
- (**P7**) $A \ll B$ implies $X\backslash B \ll X\backslash A$.

(**2**) Let $(X_1,\ll_1)$ and $(X_2,\ll_2)$ be proximal spaces. A function $f\colon X_1 \to X_2$ is a *proximal map* if $f^{-1}(A) \ll_1 f^{-1}(B)$ whenever $A \ll_2 B$.

(**3**) Proximal spaces and proximal maps are the objects and morphisms of the category Prox.

The relation $\ll$ defined above is usually called a *proximity* on X.

It is well-known that Prox is isomorphic to the category of totally bounded uniform spaces and uniformly continuous maps (which is bireflective in UWNear). Thus, since the considered categories are topological, we have by Proposition 3.8:

Proposition 3.10. Prox *is, up to isomorphism, a bireflective subcategory of* WNear. ∎

Let us present, in the sequel, another way of concluding Proposition 3.10 which yields as a corollary a characterization of proximal spaces in terms of Weil nearnesses.

A straightforward verification shows that, in case $(X,\mathcal{E}) \in \mathsf{UWNear}$, $<_{\mathcal{E}}$ is a proximity on X.

Let us now consider the converse problem of endowing a proximal space with a (functorial) Weil nearness structure.

Lemma 3.11. *Let* $(X,\ll)$ *be a proximal space.*

(**a**) *For every* $A,B,C \subseteq X$, $(E_{A,C} \cap E_{C,B}) \circ (E_{A,C} \cap E_{C,B}) \subseteq E_{A,B}$.

(**b**) *If* $E = \bigcap_{i=1}^{n} E_{A_i,B_i}$, $E' = \bigcap_{i=1}^{n} E_{C_i,D_i}$ *and, for every* $i \in \{1,...,n\}$, $A_i \ll C_i \ll D_i \ll B_i$, *then, for every* $x \in X$, $E'[x] \ll E[x]$.

(**c**) *If* $\bigcap_{i=1}^{n} E_{A_i,B_i} \subseteq E_{A,B}$ *and, for every* $i \in \{1,...,n\}$, $A_i \ll B_i$, *then* $A \ll B$.

Proof: (**a**) Let $(x,y),(y,z) \in E_{A,C} \cap E_{C,B}$ such that $(x,z) \notin X \backslash A \times X \backslash A$. In case $x \in A$, y is necessarily in C, which, in turn, implies that $x \in B$ (since $(x,y) \in E_{C,B}$) and $z \in B$ (since $(y,z) \in E_{C,B}$). Hence $(x,z) \in B \times B \subseteq E_{A,B}$.

The case $z \in A$ can be proved in a similar way.

(**b**) An easy computation shows that, for every $x \in X$, $E_{C,D}[x] \ll E_{A,B}[x]$ whenever $A \ll C \ll D \ll B$. Now a proof by induction on $n \geq 1$ is evident:

If $E = \bigcap_{i=1}^{n+1} E_{A_i,B_i}$ and $E' = \bigcap_{i=1}^{n+1} E_{C_i,D_i}$ with $A_i \ll C_i \ll D_i \ll B_i$ for every $i \in \{1,...,n+1\}$, then, for every $x \in X$,

$$E'[x] = E_{C_1,D_1}[x] \cap \bigcap_{i=2}^{n+1} E_{C_i,D_i}[x] \ .$$

By inductive hypothesis and by the case $n = 1$ already proved, we obtain

$$E'[x] \ll E_{A_1,B_1}[x] \cap \bigcap_{i=2}^{n+1} E_{A_i,B_i}[x] = E[x] \ .$$

(**c**) For any $i \in \{1, ..., n\}$ let C_i be such that $A_i \ll C_i \ll B_i$. An application of (a) yields

$$\bigcap_{i=1}^{n} (E_{A_i,C_i} \cap E_{C_i,B_i})^2 \subseteq \bigcap_{i=1}^{n} E_{A_i,B_i} \subseteq E_{A,B} \ .$$

Let

$$E = \bigcap_{i=1}^{n} (E_{A_i,C_i} \cap E_{C_i,B_i})$$

and define

$$X_1 := \Big\{ x \in X \mid E[x] \cap A = \emptyset \Big\}$$

and

$$X_2 := \Big\{ x \in X \mid E[x] \cap A \neq \emptyset \Big\} \ .$$

Note that $X_2 \neq \emptyset$ whenever $A \neq \emptyset$. Now we have $A \subseteq X \setminus \bigcup_{x \in X_1} E[x]$. For each $i \in \{1, ..., n\}$ consider A'_i, C'_i, C''_i and B'_i such that

$$A_i \ll A'_i \ll C'_i \ll C_i \ll C''_i \ll B'_i \ll B_i \ .$$

From (b) we may conclude that, for every $x \in X$, $E'[x] \ll E[x]$, where E' denotes the entourage

$$\bigcap_{i=1}^{n} (E_{A'_i,C'_i} \cap E_{C''_i,B'_i}) \ .$$

Then we have

$$A \subseteq X \setminus \bigcup_{x \in X_1} E[x] \subseteq X \setminus \bigcup_{x \in X_1} E'[x] \subseteq \bigcup_{x \in X_2} E'[x] \ .$$

It is now easy to conclude that, due to the special form of E', there is a finite subset F_2 of X_2 such that

$$\bigcup_{x \in X_2} E'[x] = \bigcup_{x \in F_2} E'[x] \ .$$

Indeed, since E' is of the form $\bigcap_{j=1}^{2n} E_{A''_j,B''_j}$ we have

$$\bigcup_{x \in X_2} E'[x] = \bigcup_{x \in X_2} \bigcap_{j=1}^{2n} (E_{A''_j,B''_j}[x]) \ ,$$

and it suffices now to form F_2 by choosing exactly one element from each non-empty set of the following 3^{2n} disjoint sets

$$
\begin{gathered}
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap A_{2n}'' \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap (B_{2n}'' \backslash A_{2n}'') \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap (X \backslash B_{2n}'') \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \backslash A_{2n-1}'') \cap A_{2n}'' \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \backslash A_{2n-1}'') \cap (B_{2n}'' \backslash A_{2n}'') \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \backslash A_{2n-1}'') \cap (X \backslash B_{2n}'') \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \backslash B_{2n-1}'') \cap A_{2n}'' \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \backslash B_{2n-1}'') \cap (B_{2n}'' \backslash A_{2n}'') \\
X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \backslash B_{2n-1}'') \cap (X \backslash B_{2n}'') \\
\vdots \\
X_2 \cap (X \backslash B_1'') \cap (X \backslash B_2'') \cap \ldots \cap (X \backslash B_{2n-1}'') \cap (X \backslash B_{2n}'') \ ,
\end{gathered}
$$

whose union is X_2.

Thus, by (b),

$$A \subseteq \bigcup_{x \in F_2} E'[x] \ll \bigcup_{x \in F_2} E[x] \ .$$

Now, if $y \in E[x]$ for some $x \in F_2$, there is $a \in A$ with $(x, a) \in E$. Since E is symmetric,

$$(a, y) \in E^2 \subseteq \bigcap_{i=1}^{n} (E_{A_i, C_i} \cap E_{C_i, B_i})^2 \subseteq E_{A,B}$$

and, consequently, $y \in B$. Hence $\bigcup_{x \in F_2} E[x] \subseteq B$ and $A \ll B$. ∎

Proposition 3.12. *Suppose $(X, \ll)$ is a proximal space. Then*

$$\Big\{ E_{A,B} \mid A, B \subseteq X \text{ and } A \ll B \Big\}$$

is a subbase for a Weil uniformity $\mathcal{E}(\ll)$ on X. Furthermore, the proximity $<_{\mathcal{E}(\ll)}$ induced by $\mathcal{E}(\ll)$ coincides with $\ll$.

Proof: It is obvious that $\mathcal{E}(\ll)$ is a non-empty family of entourages of X. Since $\ll$ interpolates, Lemma 3.11 (a) ensures us that $\mathcal{E}(\ll)$ is a Weil uniformity on X.

The non-trivial part of the equivalence of the binary relations $<_{\mathcal{E}(\ll)}$ and $\ll$ is an immediate consequence of Lemma 3.11 (c). ∎

For any $(X,\mathcal{E}) \in \mathsf{UWNear}$ satisfying

$$(\mathbf{WN6})\ \forall E\in\mathcal{E}\ \exists A_1,B_1,...,A_n,B_n\subseteq X\colon\ \Big(\bigcap_{i=1}^{n} E_{A_i,B_i}\subseteq E \text{ and } \bigcap_{i=1}^{n} E_{A_i,B_i}\in\mathcal{E}\Big),$$

the Weil nearness $\mathcal{E}(<_\mathcal{E})$ induced by $<_\mathcal{E}$ coincides with $\mathcal{E}$. Thus, the proximal spaces may be identified as the Weil uniform spaces satisfying (WN6). The same happens for morphisms:

Proposition 3.13. *Let $(X_1,\ll_1)$ and $(X_2,\ll_2)$ be proximal spaces. A map $f\colon X_1\to X_2$ is a proximal map from $(X_1,\ll_1)$ to $(X_2,\ll_2)$ if and only if it is a Weil nearness map from $(X_1,\mathcal{E}(\ll_1))$ to $(X_2,\mathcal{E}(\ll_2))$.*

Proof: Suppose $E\in\mathcal{E}(\ll_2)$ and let $A_1,B_1,...,A_n,B_n\subseteq X_2$ such that $\bigcap_{i=1}^{n} E^{X_2}_{A_i,B_i}\subseteq E$ and $A_i\ll_2 B_i$ for every $i\in\{1,...,n\}$. Then, for each i, $f^{-1}(A_i)\ll_1 f^{-1}(B_i)$ and, therefore,

$$\bigcap_{i=1}^{n} E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)}\in\mathcal{E}(\ll_1)\ .$$

To prove that $(f\times f)^{-1}(E)\in\mathcal{E}(\ll_1)$ it suffices now to check that it contains

$$\bigcap_{i=1}^{n} E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)}\ ,$$

which is straightforward since each $E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)}$ is equal to $(f\times f)^{-1}(E^{X_2}_{A_i,B_i})$.

Conversely, suppose that $A\ll_2 B$. Then $E^{X_2}_{A,B}\in\mathcal{E}(\ll_2)$ and, consequently,

$$E^{X_1}_{f^{-1}(A),f^{-1}(B)}=(f\times f)^{-1}(E^{X_2}_{A,B})\ \in\ \mathcal{E}(\ll_1)\ .$$

By Lemma 3.11, $f^{-1}(A)\ll_1 f^{-1}(B)$. ∎

Then immediately:

Corollary 3.14. *The categories* Prox *and* $\mathsf{UWNear}_{(\mathsf{WN6})}$ *are isomorphic.* ∎

Note that the category $\mathsf{UWNear}_{(\mathsf{WN6})}$ is a bireflective subcategory of UWNear: Given $(X,\mathcal{E})$ in UWNear we already know that $(X,\mathcal{E}(<_\mathcal{E}))$ belongs to $\mathsf{UWNear}_{(\mathsf{WN6})}$. Since $\mathcal{E}(<_\mathcal{E})\subseteq\mathcal{E}$, $1_X:(X,\mathcal{E})\to(X,\mathcal{E}(<_\mathcal{E}))$ is in UWNear. This is the bireflective map of $(X,\mathcal{E})$ in $\mathsf{UWNear}_{(\mathsf{WN6})}$; indeed, if

$$f\colon (X,\mathcal{E})\to(X',\mathcal{E}')$$

belongs to UWNear, with $(X',\mathcal{E}')\in\mathsf{UWNear}_{(\mathsf{WN6})}$,

$$f\colon (X,\mathcal{E}(<_\mathcal{E}))\to(X',\mathcal{E}')$$

is also in UWNear: for any $E \in \mathcal{E}'$ we may write $\bigcap_{i=1}^{n} E^{X'}_{A_i,B_i} \subseteq E$ where each $E^{X'}_{A_i,B_i} \in \mathcal{E}'$. Therefore

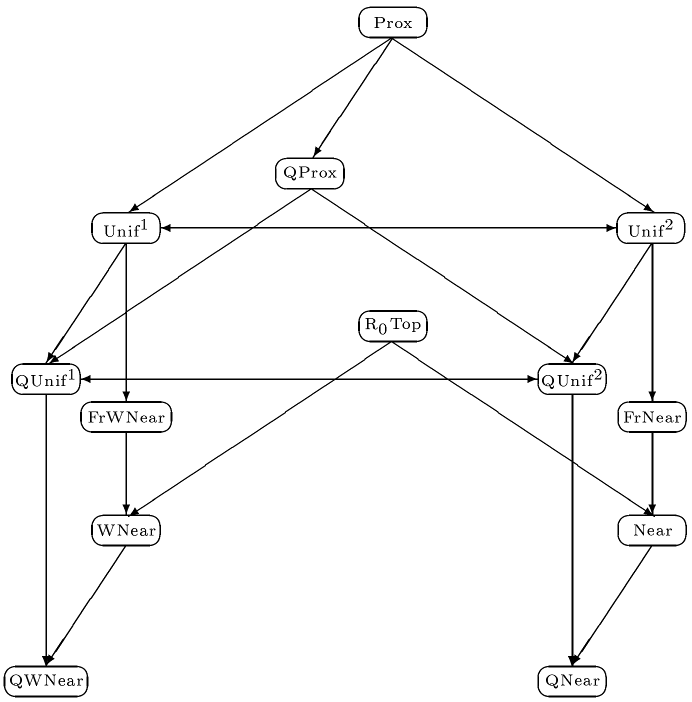
$$E^{X}_{f^{-1}(A_i),f^{-1}(B_i)} = (f \times f)^{-1}(E^{X'}_{A_i,B_i})$$

belongs to $\mathcal{E}$ and, since

$$\bigcap_{i=1}^{n} E^{X}_{f^{-1}(A_i),f^{-1}(B_i)} \subseteq (f \times f)^{-1}(E) \ ,$$

$(f \times f)^{-1}(E) \in \mathcal{E}(<_{\mathcal{E}})$.

As a conclusion, we may now use Weil entourages as a base for the left part of the following diagram, which summarizes the hierarchy of spatial nearness structures in the senses of Tukey and Weil ($\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \leftrightarrow \mathcal{B}$ mean, respectively, that category $\mathcal{A}$ is fully embeddable in category $\mathcal{B}$ and that categories $\mathcal{A}$ and $\mathcal{B}$ are isomorphic; for each $\mathcal{A}$, Q$\mathcal{A}$ denotes the category of the corresponding non-symmetric structures):



[1]in the sense of Weil [25]
[2]in the sense of Tukey [24]

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