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THE DIOPHANTINE EQUATIONS $x^2 - k = 2 \cdot T_n\left(\frac{b^2 \pm 2}{2}\right)$

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It is the object of this note to demonstrate that the two equations of the title have only finitely many solutions in positive integers x and n for any given integers b and k, $k \neq \pm 2$. In these equations $(T_n(x))_{n\geq 0}$ is the sequence of the Chebyshev polynomials of the first kind.

1 – Chebyshev polynomials of the first kind, $(T_n(x))_{n\geq 0}$, are defined by the recurrence relation

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)\,, \qquad \forall\, x \in \mathbb{C}, \quad \forall\, n \in \mathbb{N}^* \,, \tag{1.1}$$

where $T_0(x) = 1$ and $T_1(x) = x$, $\mathbb{C}$ being the set of complex numbers.

Also, we have the sequence $(\widetilde{T}_n(x))_{n\geq 0}$ of polynomials "associated" of the Chebyshev polynomials $(T_n(x))_{n\geq 0}$, defined as it follows:

$$\widetilde{T}_{n+1}(x) = 2x \cdot \widetilde{T}_n(x) + \widetilde{T}_{n-1}(x)\,, \qquad \forall\, x \in \mathbb{C}, \quad \forall\, n \in \mathbb{N}^* \,, \tag{1.2}$$

with $\widetilde{T}_0(x) = 1$ and $\widetilde{T}_1(x) = x$.

The connection between the sequence $(\widetilde{T}_n)_{n\geq 0}$ and the sequence $(T_n)_{n\geq 0}$ is given by the simple relations:

$$\begin{cases} \widetilde{T}_k(x) = \dfrac{T_k(i \cdot x)}{i^k}\,, \\[2ex] T_k(x) = \frac{\widetilde{T}_k(i \cdot x)}{i^k}\,, \qquad k \in \mathbb{N}, \quad x \in \mathbb{C}\,, \end{cases} \tag{1.3}$$

where $i^2 = -1$.

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Two important properties of the polynomials $(T_n(x))_{n\geq 0}$ are given by the formulas:

$$T_n(\cos\varphi) = \cos n\,\varphi\,, \qquad \varphi\in\mathbb{C}, \quad n\in\mathbb{N}\;, \tag{1.4}$$

and

$$T_m(T_k(x)) = T_{mk}(x)\,, \qquad \forall\, m,k\in\mathbb{N}, \quad \forall\, x\in\mathbb{C}\;. \tag{1.5}$$

2 – We are going to prove the following lemmas:

Lemma 1. *If $(T_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind, then one has*

$$T_n(a^2-1) = 2\cdot\left(T_n\left(\frac{a}{\sqrt{2}}\right)\right)^2 - 1\,, \qquad \forall\, n\in\mathbb{N}, \quad \forall\, a\in\mathbb{C}\;. \tag{2.1}$$

Proof: Indeed, we have

$$\begin{aligned} T_n(a^2-1) &= T_n\left(2\cdot\left(\frac{a}{\sqrt{2}}\right)^2 - 1\right) = T_n\left(T_2\left(\frac{a}{\sqrt{2}}\right)\right) \\ &= T_2\left(T_n\left(\frac{a}{\sqrt{2}}\right)\right) = 2\cdot\left(T_n\left(\frac{a}{\sqrt{2}}\right)\right)^2 - 1\,, \qquad \text{q.e.d.}\;. \end{aligned}$$

Now, if we put in (2.1) $a=\frac{b}{\sqrt{2}}$, we obtain

$$2\cdot T_n\left(\frac{b^2-2}{2}\right) = z_k^2 - 2\;,$$

where $z_n = 2\cdot T_n(\frac{b}{2})\in\mathbb{Z}, \forall\, n\in\mathbb{N}, \forall\, b\in\mathbb{Z}$.

Thus, we have

$$x^2 - k = z_n^2 - 2\,, \qquad k\neq 2\;,$$

i.e.,

$$x^2 - z_n^2 = k-2\,, \qquad k\neq 2\;. \blacksquare \tag{2.2}$$

Lemma 2. *If $(\widetilde{T}_n(x))_{n\geq 0}$ is the sequence of polynomials "associated" of the Chebyshev polynomials $(T_n(x))_{n\geq 0}$, then one has:*

$$\begin{aligned} &\textbf{a)} \qquad \widetilde{T}_{2n}\left(\frac{b}{2}\right) = T_n\left(\frac{b^2+2}{2}\right), \qquad b\in\mathbb{C}, \quad n\in\mathbb{N}\;; \\ &\textbf{b)} \qquad \widetilde{T}_{2n}\left(\frac{b}{2}\right) = 2\cdot\widetilde{T}_n^2\left(\frac{b}{2}\right) - (-1)^n\,, \qquad b\in\mathbb{C}, \quad n\in\mathbb{N}\;. \end{aligned} \tag{2.3}$$

Proof: We have:

$$\textbf{a)}\qquad \widetilde{T}_{2n}\left(\frac{b}{2}\right) = \frac{T_{2n}\left(i \cdot \frac{b}{2}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \frac{b}{2}\right)$$
$$= (-1)^n \cdot T_n\left(T_2\left(i \cdot \frac{b}{2}\right)\right) = (-1)^n \cdot T_n\left(2 \cdot \left(\frac{i \cdot b}{2}\right)^2 - 1\right)$$
$$= (-1)^n \cdot T_n\left(-\left(\frac{b^2}{2} + 1\right)\right) = (-1)^{2n} \cdot T_n\left(\frac{b^2+2}{2}\right), \qquad \text{q.e.d.}$$

$$\textbf{b)}\qquad \widetilde{T}_{2n}\left(\frac{b}{2}\right) = \frac{T_{2n}\left(i \cdot \frac{b}{2}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \frac{b}{2}\right)$$
$$= (-1)^n \cdot T_2\left(T_n\left(i \cdot \frac{b}{2}\right)\right) = (-1)^n \cdot \left(2 \cdot T_n^2\left(i \cdot \frac{b}{2}\right) - 1\right)$$
$$= (-1)^n \cdot \left(2 \cdot \left(i^n \cdot \widetilde{T}_n\left(\frac{b}{2}\right)\right)^2 - 1\right)$$
$$= (-1)^n \cdot \left(2 \cdot (-1)^n \cdot \widetilde{T}_n^2\left(\frac{b}{2}\right) - 1\right)$$
$$= 2 \cdot \widetilde{T}_n^2\left(\frac{b}{2}\right) - (-1)^n, \qquad \text{q.e.d.}\ .$$

Now, from Lemma 2 we obtain:

$$2 \cdot T_n\left(\frac{b^2+2}{2}\right) = 2 \cdot \widetilde{T}_{2m}\left(\frac{b}{2}\right) = 2 \cdot \left(2 \cdot \widetilde{T}_n^2\left(\frac{b}{2}\right) - (-1)^n\right)$$
$$= \left(2 \cdot \widetilde{T}_n\left(\frac{b}{2}\right)\right)^2 - 2(-1)^n$$
$$= \widetilde{z}_n^2 - (-1)^n \cdot 2\ ,$$

where $\widetilde{z}_n = 2 \cdot \widetilde{T}_n(\frac{b}{2}) \in \mathbb{Z}, \forall\, b \in \mathbb{Z}, \forall\, n \in \mathbb{N}$.

Thus, we have

$$x^2 - k = \widetilde{z}_n^2 - 2(-1)^n\ , \tag{2.4}$$

or, equivalently,

$$x^2 - \widetilde{z}_n^2 = k \pm 2\,, \qquad k \neq \pm 2\ . \tag{2.5}$$

It will be observed that for given $k \in \mathbb{Z}$, $k \neq \pm 2$, the set of values of x satisfying equations (2.2) and (2.5) is finite and, accordingly, there are finitely many values of n satisfying the equations $x^2 - k = 2 \cdot T_n(\frac{b^2 \pm 2}{2})$, $n \in \mathbb{N}$, $b \in \mathbb{Z}$.

Thus, for each given $k \in \mathbb{Z}$, $k \neq \pm 2$, there are finitely many possible values of x, $n \in \mathbb{N}$, satisfying the equation $x^2 - k = 2 \cdot T_n(\frac{b^2+2}{2})$, $b \in \mathbb{Z}$. This concludes the proof of the result of this paper. ∎

Remarks.

α) For $b = 1$ in $x^2 - k = 2 \cdot T_n(\frac{b^2+2}{2})$, we obtain the equation

$$x^2 - k = L_{2n}\,, \quad n \in \mathbb{N}\,, \tag{2.6}$$

where $(L_n)_{n \geq 0}$ is the sequence of the Lucas numbers, defined as it follows:

$$L_{n+1} = L_n + L_{n-1}\,, \quad L_0 = 2, \quad L_1 = 1\,.$$

Clearly, in (2.6) we utilized the identity

$$T_n\left(\frac{3}{2}\right) = \frac{1}{2} \cdot L_{2n}\,, \quad \forall\, n \in \mathbb{N}\,. \tag{2.7}$$

β) If we put $k = 0$ in (2.6) one obtains (see (2.5)) that the numbers L_{2n}, $n \in \mathbb{N}$, are not perfect squares.

γ) For $b = 4$ in $x^2 - k = 2 \cdot T_n(\frac{b^2+2}{2})$, we obtain the equation

$$x^2 - k = \sqrt{5 \cdot F_{6n}^2 + 4}\,, \quad n \in \mathbb{N}\,, \tag{2.8}$$

where $(F_n)_{n \geq 0}$ is the sequence of the Fibonacci numbers:

$$F_{n+1} = F_n + F_{n-1}\,, \quad F_0 = 0, \quad F_1 = 1\,.$$

In (2.8) we utilized the identities

$$F_{6n} = 8 \cdot U_{n-1}(9)\,, \quad \forall\, n \in \mathbb{N}\,,$$

and

$$T_n^2(x) - (x^2 - 1) \cdot U_{n-1}^2(x) = 1\,, \quad \forall\, x \in \mathbb{C}, \quad \forall\, n \in \mathbb{N}^*\,, \tag{2.10}$$

where $(U_n(x))_{n \geq 0}$ is the sequence of Chebyshev polynomials of the second kind

$$U_{n+1}(x) = 2x \cdot U_n(x) - U_{n-1}(x)\,, \quad x \in \mathbb{C}, \quad n \in \mathbb{N}^*\,, \tag{2.11}$$

with $U_0(x) = 1$ and $U_1(x) = 2x$.

δ) If we put $k = 0$ in (2.8) one obtains (see (2.5)) that the equation

$$x^4 - 5 \cdot F_{6n}^2 = 4 \tag{2.12}$$

has not solutions $(x, n) \in \mathbb{Z} \times \mathbb{N}$.

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