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LIFTING OF CERTAIN ISOMORPHIC PROPERTIES TO $X \otimes_\epsilon Y$ *

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Abstract: In this short note we prove that some isomorphic properties, like property (V), (D), (RDP) and the recent (w), lift from certain Banach spaces X, Y to their injective tensor product.

Several papers have been devoted to the question of when certain isomorphic properties are enjoyed by spaces of vector-valued continuous functions (see [1], [2], [12], [5], [6], [7], [19], [14], [17] and references therein), but as far as we know very few results exist about the same properties in the injective tensor product $X \otimes_\epsilon Y$ of two Banach spaces X, Y (see [8], [9], [19], [3]). In this short note we want to study that question in this almost new setting, this way generalizing some known results; but actually to get concrete examples of applications of our Theorem 2, about property (V) or (D) or (RDP) (see below) we need those results about $C(K,Y)$-spaces, so that all of our results may be considered as corollaries of them. Besides Theorem 2 we present some more results on the new property (w). Even in this case to get concrete applications of our theorems, we need previous results on $C(B_{X^{***}}, Y)$.

Before starting we want to recall the necessary definitions

Definition 1 ([16]). A Banach space X has property (V) of Pelczynski if every unconditionally converging operator defined on it is weakly compact.

Definition 2 ([10]). A Banach space X has property (D) of Grothendieck if every weakly completely continuous (or Dieudonné) operator defined on it is weakly compact.

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Definition 3 ([10]). A Banach space X has property (RDP) of Grothendieck if every completely continuous (or Dunford–Pettis) operator defined on it is weakly compact.

Definition 4 ([19]). A Banach space X has property (w) of Saab and Saab if every operator from it into its dual is weakly compact.

Our first result is about property (V). In order to get it we need the following

Lemma 1. *Suppose E, F are two Banach spaces such that there is a norm one quotient map $Q\colon\ E \to F$ for which the range $Q^*(F^*)$ of Q^* is a norm one complemented subspace of E^*. Consider another Banach space Y and suppose that the following assumptions are verified:*

(**a**) *E^* or Y^* has the approximation property.*

(**b**) *E^* or Y^* has the Radon–Nikodym property.*

Then the mapping $Q \otimes_\epsilon id_Y\colon\ E \otimes_\epsilon Y \to F \otimes_\epsilon Y$ is a quotient mapping.

Proof: Thanks to our assumptions (a) and (b) we have the following isometries

$$(E \otimes_\epsilon Y)^* = E^* \otimes_\pi Y^*\,, \qquad (F \otimes_\epsilon Y)^* = F^* \otimes_\pi Y^*$$

(see [4]). The mapping $Q^* \otimes_\pi id_{Y^*}:\ F^* \otimes_\pi Y^* \to Q^*(F^*) \otimes_\pi Y^*$ clearly is an isometry onto; since the range space of it is a closed subspace of $E^* \otimes_\pi Y^*$ (because $Q^*(F^*)$ is norm one complemented into E^*) the mapping $Q^* \otimes_\pi id_{Y^*}$: $F^* \otimes_\pi Y^* \to E^* \otimes_\pi Y^*$ is an isometry into; but it clearly is the conjugate operator of the mapping $Q \otimes_\epsilon id_Y$ that is so a quotient mapping from $E \otimes_\epsilon Y$ onto $F \otimes_\epsilon Y$. We are done. ∎

We are now ready to prove the first result; in it we shall denote by Δ the Cantor set

Theorem 2. *Let X be a Banach space such that X^* is isometric to a L_1-space and Y be a Banach space with property (V) of Pelczynski. Then $X \otimes_\epsilon Y$ has property (V) of Pelczynski.*

Proof: Let $T:\ X \otimes_\epsilon Y \to Z$ be an unconditionally converging operator and (x_n) a bounded sequence in $X \otimes_\epsilon Y$; it will be shown that $(T(x_n))$ allows a weakly convergent subsequence. Since for every separable subspace $E \subset X$ there is a separable L_1-predual subspace $F \subset X$ with $E \subset F$ (see [13], p.232), there is a separable L_1-predual subspace X_0 of X such that (x_n) is contained in

$X_0 \otimes_\epsilon Y$. Because of the density of the (not complete) tensor product $X_0 \otimes Y$ inside $X_0 \otimes_\epsilon Y$, we can find a bounded sequence $z_n \in X_0 \otimes Y$ such that

$$\|x_n - z_n\|_\epsilon \to 0 \ .$$

For each $n \in \mathbb{N}$, choose a finite dimensional Banach subspace Y_n of Y so that $z_n \in X_0 \otimes_\epsilon Y_n$. Now a result in [11] states that there is a norm one quotient mapping Q from $C(\Delta)$ onto X_0. Consider the mappings

$$Q \otimes_\epsilon id_{Y_n} \colon C(\Delta) \otimes_\epsilon Y_n \to X_0 \otimes_\epsilon Y_n \ .$$

Since X_0^* and $C(\Delta)^*$ are L_1-spaces, the range of the isometry (into) $Q^* \colon X_0^* \to C(\Delta)^*$ is a norm one complemented subspace of $C(\Delta)^*$ (this is a consequence of a result going back to T. Ando; see [13], p.162); this and the finite dimensionality of each Y_n allow us to apply Lemma 1: the mappings $Q \otimes_\epsilon id_{Y_n}$ are quotient mappings. We may find a bounded sequence $(f_n) \subset C(\Delta, Y)$ with each f_n belonging to the corresponding $C(\Delta) \otimes_\epsilon Y_n$ (a subspace of $C(\Delta, Y)$) such that $(Q \otimes_\epsilon id_{Y_n})(f_n) = z_n$ for all $n \in \mathbb{N}$; hence $(Q \otimes_\epsilon id_Y)(f_n) = z_n$ for all $n \in \mathbb{N}$. By assumption $T \circ (Q \otimes_\epsilon id_Y)$ is unconditionally converging and hence weakly compact, since $C(\Delta, Y)$ has Pelczynski property (V) by a result of Randriantoanina ([17]), so that also $[T \circ (Q \otimes_\epsilon id_Y)](f_n) = T(z_n)$ is a relatively weakly compact sequence. We are done, thanks to the choice of (z_n). ∎

Similar results, with similar proofs, can be obtained for the other two properties (D) and (RDP). Applying the results in [2], [12], [5], [6], [7] we have concrete examples of spaces Y for which $X \otimes_\epsilon Y$ has one of the first three properties. We underline that among the results quoted at the beginning that one in the paper [3] is obtained with Y reflexive (a more restrictive assumption than that considered in [2]) but with X a $\mathcal{L}_\infty$-space, instead of a space with dual isometric to a L_1-space.

In the next results, we turn our attention to property (w); a first result is the following

Theorem 3. *Let X be a $\mathcal{L}_\infty$-space and Y be a Banach space such that $C(B_{X^{***}}, Y)$ has property (w). Then $X \otimes_\epsilon Y$ has property (w).*

Proof: Let $T \colon X \otimes_\epsilon Y \to (X \otimes_\epsilon Y)^*$ be. We consider the restriction $\widetilde{T}$ of T^{**} to $X^{**} \otimes_\epsilon Y$ that is a closed subspace of $(X \otimes_\epsilon Y)^{**}$, as proved in [8]; hence $\widetilde{T}$ takes $X^{**} \otimes_\epsilon Y$ into $(X \otimes_\epsilon Y)^{***}$. As in [8], there is a norm one projection R of $C(B_{X^{***}}, Y)$ onto $X^{**} \otimes_\epsilon Y$, so that $\widetilde{T} \circ R \colon C(B_{X^{***}}, Y) \to (X \otimes_\epsilon Y)^{***}$. Now, let us consider the canonical projection P of $(X \otimes_\epsilon Y)^{***}$ onto $(X \otimes_\epsilon Y)^*$, that in turn is isometrically embedded into $(X^{**} \otimes_\epsilon Y)^*$ (see [4], Ch.VIII) by an isometric embedding i. So the operator $F = R^* \circ i \circ P \circ \widetilde{T} \circ R$ takes $C(B_{X^{***}}, Y)$ into $(C(B_{X^{***}}, Y))^*$. Thanks to our assumption on Y such an F is weakly compact. Now, let us consider an arbitrary element z in $B_{X \otimes_\epsilon Y}$, the unit ball of $X \otimes_\epsilon Y$. It is clear that $(\widetilde{T} \circ R)(z) = T(z)$. Furthermore, $T(z) \in (X \otimes_\epsilon Y)^*$ and so $P[T(z)] = T(z)$, for all $z \in B_{X \otimes_\epsilon Y}$. These remarks imply that $F(z) = (R^* \circ i \circ T)(z)$ for all $z \in B_{X \otimes_\epsilon Y}$, from which it follows that $(R^* \circ i \circ T)(B_{X \otimes_\epsilon Y}) \subset F(B_{X \otimes_\epsilon Y})$. Since both R^* and i are isometries, the relative weak compactness of $F(B_{X \otimes_\epsilon Y})$ gives that of $T(B_{X \otimes_\epsilon Y})$. We are done. ∎

Corollary 4. *Let X be a $\mathcal{L}_\infty$-space and let Y be a Banach space. Then $X \otimes_\epsilon Y$ has property (w) if Y has either of the following properties:*

(**a**) *Y has property (u) ([15]) and contains no subspace isomorphic to l_1.*

(**b**) *Y is complemented in a Banach lattice and has property (w).*

(**c**) *Y is a closed subspace of an order continuous Banach lattice and has property (w).*

(**d**) *Y^* and Y^{**} have the Radon–Nikodym property and Y has property (w).*

Then $X \otimes_\epsilon Y$ has property (w).

Proof: It is enough to note that under either of the cases (a), (b), (c) or (d) the space $C(B_{X^{***}}, Y)$ has property (w). As a matter of fact, case (a) follows from [2], cases (b) and (c) follow from [19] and case (d) follows from [14]. ∎

This Corollary 4 allows us to extend Proposition 12 in [19] obtained for X^* isometric to a L_1-space and Y a Banach lattice.

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