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ON A VARIATIONAL INEQUALITY FOR THE NONHOMOGENEOUS DEGENERATED KIRCHHOFF EQUATION WITH A FRICTIONAL DAMPING

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Abstract: We study a unilateral problem for the nonhomogeneous degenerated Kirchhoff equation with a frictional damping. Making use of the penalty method and Galerkin's approximations, we establish global existence and uniqueness theorems.

1 – Introduction

The one-dimensional nonlinear equation of motion of an elastic string

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2\,L}\int_0^L \left|\frac{\partial u}{\partial s}(s,t)\right|^2 ds\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

was proposed by Kirchhoff [8]. Here h is the area of the cross section, L is the length of the string, ρ is the mass density, P_0 is the initial tension and E is Young's modulus of a material. Another model for the motion of an elastic string was given in Carrier [2]. The mixed problem for (1) was studied by a number of authors, see Bernstein [1], Pohozaev [16], Lions [11], Ebihara–Medeiros–Milla Miranda [4], Menzala [14], D'ancona–Spagnolo [3] and their references.

Most authors considered homogeneous case, when ρ_0, P_0, h and E are constants. These restrictions allowed to obtain an a priori estimate independent of t. When the initial data are analytic functions, it is possible to prove a global in t existence theorem, see Bernstein [1], Pohozaev [16], and D'ancona–Spagnolo [3]. If the initial data are only from some Sobolev space, then the presence of a

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frictional damping allows to prove the global existence theorem for small initial data, see Nishihara–Yamada [15].

In [11], Lions proposed to study the problem which modelled a nonhomogeneous material. In this case one does not have such a global estimate and, until now, we know only local existence results, see Frota [5]. Our goal in this paper is to consider a unilateral problem for equation (1), having a frictional damping, in a case of a nonhomogeneous material; that is, ρ, P_0, h and E are functions of x, t; and the initial tension P_0 can be zero. We study the case when $\rho(x,t)$ is only nonnegative. It means that the material is composite and in some places the density of the material is much times smaller then in others, or the string has holes inside.

We formulate our problem as follows: Let Ω be a bounded open set of $\mathbb{R}^n$ with a sufficiently smooth boundary Γ, $Q=\Omega\times(0,T)$, and $\Sigma=\Gamma\times(0,T)$, where T is a finite positive number. We denote by $\mathbb{K}$ the closed convex set of $W_0^{2,2p+2}(\Omega)$, $1\le p<\infty$, defined by

$$\mathbb{K}=\left\{\sigma\in W_0^{2,2p+2}(\Omega)\,;\ |\Delta\sigma(x)|\le 1 \text{ and } \sigma(x)\ge 0 \text{ a.e. on } \Omega\right\}\,. \tag{2}$$

Given $M\colon\ \overline{\Omega}\times[0,T]\times[0,\infty)\to\mathbb{R}$, $\rho\colon\ \overline{\Omega}\times[0,T]\to\mathbb{R}$, $f\colon\ \Omega\times(0,T)\to\mathbb{R}$, $\phi_0\colon\ \Omega\to\mathbb{R}$ and $\phi_1\colon\ \Omega\to\mathbb{R}$, find a function $u=u(x,t)$ satisfying

$$\int_Q\left(\rho\,u_{tt}-M\left(x,\,t,\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial s_i}(s,t)\right|^2ds\right)\Delta u+\alpha\,u_t-f\right)(v-u_t)\,dx\,dt\ge 0\ , \tag{3}$$

$\forall\, v\in\mathbb{K}$,

$$\frac{\partial u}{\partial t}(x,t)\in\mathbb{K}\quad \text{ a.e. on }\ [0,T] \tag{4}$$

and taking the following initial and boundary values

$$u=0\quad \text{ on }\ \Sigma=\Gamma\times(0,T)\ , \tag{5}$$

$$u(x,0)=\phi_0(x)\,,\qquad \frac{\partial u}{\partial t}(x,0)=\phi_1(x)\ \text{ in }\Omega\ . \tag{6}$$

We observe that if $u_t\in\operatorname{int}(\mathbb{K})$ = interior of $\mathbb{K}$, then u is a solution of the equation

$$\rho(x,t)\,\frac{\partial^2u}{\partial t^2}-M\left(x,\,t,\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial s_i}(s,t)\right|^2ds\right)\Delta u+\alpha\,\frac{\partial u}{\partial t}=f,\quad \text{ in }\ Q=\Omega\times(0,T)$$

and from (4) the velocity $\frac{\partial u}{\partial t}$ is positive. Moreover, one can note that the restriction $|\Delta u_t|\le 1$, does not have an explicit physical meaning. Nevertheless, it implies boundedness of $|u_t|$ and $|\nabla u|$ in Q. This, in turn, gives boundedness of the internal energy of an elastic string whose motion is simulated by the above equation.

When a function M is strictly positive and does not depend on x and t, $\alpha = 0$ and $\rho = 1$, the unilateral problem (3)–(6) for $n = 1$, was studied by Lar'kin [9], where global existence and uniqueness theorems were proved. Later this results were generalized in Lar'kin–Medeiros [10], where Ω was a square in $\mathbb{R}^2$, and in Frota [6] for dimensions $n \geq 2$. Local existence and uniqueness theorems for (3)–(6) were proved in Medeiros–Milla Miranda [13].

In this paper we prove the existence and uniqueness of a global solution to (3)–(6) without smallness conditions for the initial data and without geometrical restrictions for Ω. We need a positive coefficient α in order to have a sufficient regularity of solutions when ρ is degenerated. It is known that if $\alpha = 0$ and $\rho \geq 0$, then the Cauchy problem even in a linear case is ill-posed. If ρ is strictly positive, α can be equal to zero, see Frota–Lar'kin [7]. To prove a global existence theorem without smallness restrictions for the initial data and f, we use properties of a convex set to which belongs our solution. A choice of this set depends on growth properties of a function $M(x,t,\|u(t)\|^2)$.

Another problem for the degenerate Kirchhoff equation is to prove a global in t uniqueness theorem. In Ebihara–Medeiros–Milla Miranda [4], the local uniqueness theorem was proved. In order to establish a global uniqueness result, we again use properties of a convex set to which belongs our solution.

2 – Penalized problem

Let β be an operator from $W_0^{2,2p+2}(\Omega)$ into the dual space $W^{-2,\frac{2p+2}{2p+1}}(\Omega)$ defined by

$$\langle \beta(u), v \rangle = -\int_\Omega u^- \, v \, dx + \int_\Omega \Big(1 - |\Delta u|^{2p}\Big)^- \, \Delta u \, \Delta v \, dx \ ,$$

where $h^-(x) = \max\{-h(x), 0\}$. We can verify that β is a monotone and hemicontinuous operator,

$$\beta(S) \text{ is bounded for each } S \text{ bounded in } W_0^{2,2p+2}(\Omega) \tag{7}$$

and

$$\beta(u) = 0 \iff u \in \mathbb{K} \ . \tag{8}$$

Under this conditions, we have the following existence result for the penalized problem.

Theorem 1. *Let $\rho \in C^1(\overline{Q})$, $M \in C^1(\overline{Q} \times [0,\infty))$ be real functions and α be a positive real number such that:*

$$0 \le \rho(x,t), \ \ \forall (x,t) \in \overline{Q} \quad \text{with} \ \ 0 < \rho_0 \le \rho(x,0), \ \ \forall x \in \overline{\Omega} \ , \tag{9}$$

$$0 \le M(x,t,\lambda)\,, \qquad \forall (x,t,\lambda) \in \overline{Q} \times [0,\infty) \ , \tag{10}$$

$$M(x,t,\lambda) \le C(1+\lambda^p)\,, \qquad \forall (x,t,\lambda) \in \overline{Q} \times (0,\infty) \ , \tag{11}$$

$$\left|\frac{\partial M}{\partial x_i}(x,t,\lambda)\right| \le C(1+\lambda^p)\,, \qquad \forall (x,t,\lambda) \in Q \times (0,\infty), \ \ i=1,...,n \ ; \tag{12}$$

$$0 < a_0 \le 2\,\alpha - |\rho_t(x,t)|\,, \qquad \forall (x,t) \in Q \ . \tag{13}$$

If $f, f' \in L^2(0,T;L^2(\Omega))$, $\phi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\phi_1 \in \operatorname{int}(\mathbb{K})$, then there exists $\epsilon_0 \in (0,1)$ such that for each $\epsilon \in (0,\epsilon_0)$ and $\nu \in \mathbb{N}$ there is a function $u_{\nu\epsilon}$:

$$\begin{cases} u_{\nu\epsilon} \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)), \\ u'_{\nu\epsilon} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^{2p+2}(0,T;W_0^{2,2p+2}(\Omega)), \\ u''_{\nu\epsilon} \in L^\infty(0,T;L^2(\Omega)) \ , \end{cases} \tag{14}$$

satisfying the following initial boundary value problem

$$\begin{cases} \Big(\rho_\nu(t)\, u''_{\nu\epsilon}(t) - M(t, \|u_{\nu\epsilon}(t)\|^2)\,\Delta u_{\nu\epsilon}(t) + \alpha\, u'_{\nu\epsilon}(t), \ v\Big) + \\ \qquad + \dfrac{1}{\epsilon}\,\langle \beta(u'_{\nu\epsilon}(t)), v\rangle = (f(t), v) \quad \text{ a.e. on } \ [0,T], \ \ \forall v \in W_0^{2,2p+2}(\Omega) \ , \\ u_{\nu\epsilon}(0) = \phi_0(x)\,, \qquad u'_{\nu\epsilon}(0) = \phi_1(x) \ . \end{cases} \tag{15}$$

Here $|\cdot|$,$(\cdot,\cdot)$ and $\|\cdot\|$,$((\cdot,\cdot))$ denote the norm and the inner product in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, $\rho_\nu(x,t) = \frac{1}{\nu} + \rho(x,t)$ and ρ_0, C, a_0 are positive constants.

Proof: Let $(w_j)_{j\in\mathbb{N}}$ be a basis in $W_0^{2,2p+2}(\Omega)$, orthonormal in $L^2(\Omega)$. For each $m \in \mathbb{N}$ we define approximate solutions

$$u_{\nu\epsilon m}(x,t) = \sum_{j=1}^{m} g_{\nu\epsilon jm}(t)\, w_j(x) \ ,$$

where $g_{\nu\epsilon jm}$ are solutions to the Cauchy problem for the following system:

$$
(16)\quad \begin{cases} \Big(\rho_\nu(t)\, u''_{\nu\epsilon m}(t), w_j\Big) - \Big(M(t, \|u_{\nu\epsilon m}(t)\|^2)\, \Delta u_{\nu\epsilon m}(t),\, w_j\Big) + \\ \qquad + \alpha(u'_{\nu\epsilon m}(t), w_j) + \dfrac{1}{\epsilon}\, \langle \beta(u'_{\nu\epsilon m}(t)), w_j\rangle = (f(t), w_j)\,, \quad 1 \le j \le m\,, \\ u_{\nu\epsilon m}(0) = \displaystyle\sum_{j=1}^{m} (\phi_0, w_j)\, w_j = \phi_{0m}\,, \\ u'_{\nu\epsilon m}(0) = \displaystyle\sum_{j=1}^{m} (\phi_1, w_j)\, w_j = \phi_{1m}\,. \end{cases}
$$

For $\nu > 0$, (16) can be reduced to a normal system of m ordinary differential equations, which has solutions at some interval $[0, T_m]$. This ensures the existence of approximate solutions $u_{\nu\epsilon m}$.

Let $V_m = [w_1, ..., w_m]$ be the m-dimensional subspace of $W_0^{2,2p+2}(\Omega)$ spanned by $w_1, ..., w_m$. Then we have the approximate equation

$$
(17)\qquad \Big(\rho_\nu(t)\, u''_{\nu\epsilon m}(t),\, v\Big) - \Big(M\Big(t, \|u_{\nu\epsilon m}(t)\|^2\Big)\, \Delta u_{\nu\epsilon m}(t),\, v\Big) + \\ + \alpha(u'_{\nu\epsilon m}(t), v) + \frac{1}{\epsilon}\, \langle \beta(u'_{\nu\epsilon m}(t)), v\rangle = (f(t), v)\,, \quad \forall\, v \in V_m\,.
$$

The next step is devoted to deriving a priori estimates of $u_{\nu\epsilon m}$. From now on C_i, for $i \in \mathbb{N}$, denotes a positive constant independent of ν, ϵ, m and t.

A priori estimate 1. Taking $v = 2\, u'_{\nu\epsilon m}(t)$ in (17), we obtain

$$
(18)\quad \frac{d}{dt}\Big(\rho_\nu(t), (u'_{\nu\epsilon m}(t))^2\Big) + 2\,\alpha\, |u'_{\nu\epsilon m}(t)|^2 + \frac{2}{\epsilon}\Big\langle \beta(u'_{\nu\epsilon m}(t)), u'_{\nu\epsilon m}(t)\Big\rangle = \\ = \Big(\rho'_\nu(t), (u'_{\nu\epsilon m}(t))^2\Big) + 2\int_\Omega M\Big(x, t, \|u_{\nu\epsilon m}(t)\|^2\Big)\, \Delta u_{\nu\epsilon m}(x,t)\, u'_{\nu\epsilon m}(x,t)\, dx \\ + 2\Big(f(t), u'_{\nu\epsilon m}(t)\Big)\,.
$$

Using integration by parts, (11) and (12), we get

$$
\left| 2\int_\Omega M\Big(x, t, \|u_{\nu\epsilon m}(t)\|^2\Big)\, \Delta u_{\nu\epsilon m}(x,t)\, u'_{\nu\epsilon m}(x,t)\, dx \right| \le \\ \le C_1 \Big[\|u_{\nu\epsilon m}(t)\|^{2p+2} + \|u'_{\nu\epsilon m}(t)\|^2 + \|u_{\nu\epsilon m}(t)\|^{2p}\, \|u'_{\nu\epsilon m}(t)\|^2 \Big]\,.
$$

On the other hand, by Young's inequality we have

$$\|u_{\nu\epsilon m}(t)\|^{2p}\,\|u'_{\nu\epsilon m}(t)\|^2 \leq \frac{p}{p+1}\,\|u_{\nu\epsilon m}(t)\|^{2p+2} + \frac{1}{p+1}\,\|u'_{\nu\epsilon m}(t)\|^{2p+2}\;,$$

$$\|u_{\nu\epsilon m}(t)\|^2 \leq \frac{p}{p+1} + \frac{1}{p+1}\,\|u_{\nu\epsilon m}(t)\|^{2p+2}\;,$$

$$\|u'_{\nu\epsilon m}(t)\|^2 \leq \frac{p}{p+1} + \frac{1}{p+1}\,\|u_{\nu\epsilon m}(t)\|^{2p+2}\;.$$

Therefore,

$$\tag{19} \begin{aligned} &\left|2\int_\Omega M\Big(x,t,\|u_{\nu\epsilon m}(t)\|^2\Big)\,\Delta u_{\nu\epsilon m}(x,t)\,u'_{\nu\epsilon m}(x,t)\,dx\right| \leq \\ &\qquad\qquad \leq C_2 + C_3\,\|u_{\nu\epsilon m}(t)\|^{2p+2} + C_4\,\|u'_{\nu\epsilon m}(t)\|^{2p+2}\;. \end{aligned}$$

Since $\rho \in C^1(\overline{Q})$, we conclude

$$\tag{20} \left|\Big(\rho_\nu(t),(u'_{\nu\epsilon m}(t))^2\Big)\right| \leq C_5 + C_6\,\|u'_{\nu\epsilon m}(t)\|^{2p+2}$$

and

$$\tag{21} 2\,\Big(f(t),u'_{\nu\epsilon m}(t)\Big) \leq C_7 + |f(t)|^2 + C_8\,\|u'_{\nu\epsilon m}(t)\|^{2p+2}\;.$$

From (18)–(21) we have

$$\begin{aligned} &\frac{d}{dt}\Big(\rho_\nu(t),(u_{\nu\epsilon m}(t))^2\Big) + \alpha\,|u'_{\nu\epsilon m}(t)|^2 + \frac{2}{\epsilon}\,\Big\langle\beta(u'_{\nu\epsilon m}(t)),u'_{\nu\epsilon m}(t)\Big\rangle \leq \\ &\qquad\qquad \leq C_9 + |f(t)|^2 + C_{10}\,|\Delta u_{\nu\epsilon m}(t)|^{2p+2} + C_{11}\,|\Delta u'_{\nu\epsilon m}(t)|^{2p+2} \\ &\qquad\qquad \leq C_{12} + |f(t)|^2 + C_{13}\,|\Delta u'_{\nu\epsilon m}(t)|^{2p+2}\;. \end{aligned}$$

After integration over t, we obtain

$$\tag{22} \begin{aligned} &\Big(\rho_\nu(t),(u'_{\nu\epsilon m}(t))^2\Big) + 2\,\alpha\int_0^t |u'_{\nu\epsilon m}(s)|^2\,ds + \frac{2}{\epsilon}\int_0^t \Big\langle\beta(u'_{\nu\epsilon m}(s)),u'_{\nu\epsilon m}(s)\Big\rangle\,ds \leq \\ &\qquad\qquad \leq C_{14} + C_{13}\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds\;. \end{aligned}$$

Using the identity $h = h^+ - h^-$, we can see that

$$\begin{aligned} \Big(|\Delta u'|^{2p+2} - 1\Big) &= \Big(|\Delta u'|^2 - 1\Big) - |\Delta u'|^2\,\Big(1 - |\Delta u'|^{2p}\Big) \\ &= \Big(|\Delta u'|^2 - 1\Big) - |\Delta u'|^2\,\Big[\Big(1 - |\Delta u'|^{2p}\Big)^+ - \Big(1 - |\Delta u'|^{2p}\Big)^-\Big] \\ &\leq \Big(|\Delta u'|^2 - 1\Big) + \Big(1 - |\Delta u'|^{2p}\Big)^-\,|\Delta u'|^2 \\ &\leq \Big(1 - |\Delta u'|^{2p}\Big)^- + \Big(1 - |\Delta u'|^{2p}\Big)^-\,|\Delta u'|^2 \\ &\leq 2\Big(1 - |\Delta u'|^{2p}\Big)^-\,|\Delta u'|^2\;. \end{aligned}$$

Then

$$\|\Delta u'_{\nu\epsilon m}(t)\|^{2p+2}_{L^{2p+2}(\Omega)} - \mu(\Omega) \le 2\Big\langle \beta(u'_{\nu\epsilon m}(t)), u'_{\nu\epsilon m}(t)\Big\rangle \ . \tag{23}$$

We can rewrite this as follows

$$\left[\frac{1}{2C}\,|\Delta u'_{\nu\epsilon m}(t)|^{2p+2} + \frac{1}{2}\,\|\Delta u'_{\nu\epsilon m}(t)\|^{2p+2}_{L^{2p+2}(\Omega)}\right] - \mu(\Omega) \le 2\Big\langle \beta(u'_{\nu\epsilon m}(t)), u'_{\nu\epsilon m}(t)\Big\rangle \ ,$$

hence, after integration in time from 0 to t, we have

$$\frac{1}{\epsilon}\left\{C_{15}\left[\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds + \int_0^t \|\Delta u'_{\nu\epsilon m}(s)\|^{2p+2}_{L^{2p+2}(\Omega)}\,ds\right] - \mu(\Omega)\int_0^t ds\right\} \le$$
$$\le \frac{2}{\epsilon}\int_0^t \Big\langle \beta(u'_{\nu\epsilon m}(s)), u'_{\nu\epsilon m}(s)\Big\rangle\,ds \ .$$

From this inequality and (22), one can see

$$\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds + \int_0^t \|\Delta u'_{\nu\epsilon m}(s)\|^{2p+2}_{L^{2p+2}(\Omega)}\,ds \le \tag{24}$$
$$\le C_{16} + \epsilon\,C_{17}\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds \ ,$$

so it can be rewritten as follows

$$(1-\epsilon\,C_{17})\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds + \int_0^t \|\Delta u'_{\nu\epsilon m}(s)\|^{2p+2}_{L^{2p+2}(\Omega)}\,ds \le C_{16} \ .$$

If we choose $\epsilon_0 = \frac{1}{C_{17}}$, then, for $0 < \epsilon < \epsilon_0$, we have $(1-\epsilon\,C_{17}) > 0$, and consequently

$$\int_0^t |\Delta u'_{\nu\epsilon m}(s)|^{2p+2}\,ds \le C_{18} \quad\text{and}\quad \int_0^t \|\Delta u'_{\nu\epsilon m}(s)\|^{2p+2}_{L^{2p+2}(\Omega)}\,ds \le C_{19} \ . \tag{25}$$

We note that

$$\begin{aligned}
|\Delta u(t)|^{2p+2} &= |\Delta u(0)|^{2p+2} + \int_0^t \frac{d}{ds}\Big[|\Delta u(s)|^2\Big]^{p+1}\,ds \\
&\le \left[|\Delta u(0)|^{2p+2} + \int_0^T |\Delta u'(t)|^{2p+2}\,dt\right] + (2p+1)\int_0^t |\Delta u(s)|^{2p+2}\,ds \\
&= C_{20} + (2p+1)\int_0^t |\Delta u(s)|^{2p+2}\,ds \ .
\end{aligned}$$

By Gronwall's Lemma

$$|\Delta u(t)|^{2p+2} \le |\Delta u(0)|^{2p+2}\,e^{(2p+2)T} + e^{(2p+2)T}\int_0^t |\Delta u'(t)|^{2p+2}\,dt \ ,$$

therefore, we have from (25)

$$(26)\qquad |\Delta u_{\nu\epsilon m}(t)| \le C_{21}\ ,$$

$$(27)\quad \Big(\rho_\nu(t), (u'_{\nu\epsilon m}(t))^2\Big) + \alpha\int_0^t |u'_{\nu\epsilon m}(s)|^2\,ds + \frac{2}{\epsilon}\int_0^t \Big\langle \beta(u'_{\nu\epsilon m}(s)), u'_{\nu\epsilon m}(s)\Big\rangle\,ds \le C_{22}\ ,$$

$$(28)\qquad \|u_{\nu\epsilon m}(t)\|^2 \le C_{23} \quad \text{and} \quad \|u'_{\nu\epsilon m}(t)\|^2 \le C_{24}\ ,$$

$$(29)\qquad \|\beta(u'_{\nu\epsilon m})\|_{L^{\frac{2p+2}{2p+1}}(0,T;W^{-2,\frac{2p+2}{2p+1}}(\Omega))} \le C_{25}\ .$$

A priori estimate 2. Differentiating (17) with respect to t, taking $v = 2\,u''_{\nu\epsilon m}(t)$ and using monotonicity of β, we come to the inequality

$$(30)\qquad \Big(\rho_\nu(t), (u''_{\nu\epsilon m}(t))^2\Big) + \int_0^t\int_\Omega (2\alpha + \rho'(x,s))\,(u''_{\nu\epsilon m}(x,s))^2\,dx\,ds \le$$

$$\le \Big(\rho_\nu(0), (u''_{\nu\epsilon m}(0))^2\Big) + 2\int_0^t \Big(M'(s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds$$

$$+\, 2\int_0^t (f'(s), u''_{\nu\epsilon m}(s))\,ds + 2\int_0^t \Big(\frac{\partial M}{\partial\lambda}(s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds$$

$$+\, 2\int_0^t \Big(M(s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u'_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds\ .$$

Now we choose a real number γ such that

$$(31)\qquad 0 < \gamma < \frac{a_0}{4}\ ,$$

where a_0 is given by (13). Taking into account the first estimate, we can see

$$(32)\qquad 2\int_0^t \Big(M'(s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds \le$$

$$\le \int_0^t\int_\Omega \frac{1}{\gamma}\Big[M'(x,s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u_{\nu\epsilon m}(x,s)\Big]^2\,dx\,ds + \gamma\int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds$$

$$\le C_{26} + \gamma\int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds\ ,$$

and similarly,

$$(33)\qquad 2\int_0^t \Big(\frac{\partial M}{\partial\lambda}(s, \|u_{\nu\epsilon m}(s)\|^2)\,\Delta u_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds \le$$

$$\le C_{27} + \gamma\int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds\ ,$$

$$(34) \qquad 2\int_0^t \Big(M(s,\|u_{\nu\epsilon m}(s)\|^2)\,\Delta u'_{\nu\epsilon m}(s),\, u''_{\nu\epsilon m}(s)\Big)\,ds \le$$
$$\le C_{28} + \gamma \int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds \ ,$$

$$(35) \qquad 2\int_0^t \Big(f'(s), u''_{\nu\epsilon m}(s)\Big)\,ds \le C_{29} + \gamma \int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds \ .$$

Since $\phi_1 \in \operatorname{int}(\mathbb{K})$, we have $\langle \beta(\phi_{1m}), \varphi\rangle = 0$, $\forall\,\varphi \in W_0^{2,2p+2}(\Omega)$ and for m sufficiently large. Then taking $t=0$ in (17), we get

$$\Big(\rho_\nu(0), (u''_{\nu\epsilon m}(0))^2\Big) \le C_{30}\,|u''_{\nu\epsilon m}(0)| \ .$$

From this and (9), we have

$$\rho_0\,|u''_{\nu\epsilon m}(0)|^2 \le C_{31}\,|u'_{\nu\epsilon m}(0)| \ ,$$

and, consequently,

$$(36) \qquad |u''_{\nu\epsilon m}(0)| \le C_{32} \ .$$

From (30), (32)–(36), we obtain

$$\Big(\rho_\nu(t), (u''_{\nu\epsilon m}(t))^2\Big) + \int_0^t\int_\Omega \Big(2\,\alpha + \rho'(x,s)\Big)\,(u''_{\nu\epsilon m}(x,s))^2\,dx\,ds \le$$
$$\le C_{33} + 4\,\gamma \int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds \ .$$

This inequality and (13) give

$$(a_0 - 4\,\gamma)\int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds \le C_{34} \ ,$$

hence, from (38), we conclude

$$(37) \qquad \int_0^t |u''_{\nu\epsilon m}(s)|^2\,ds \le C_{35} \ .$$

The a priori estimates obtained imply the existence of a subsequence of $(u_{\nu\epsilon m})_{m\in\mathbb{N}}$, which we still denote in the same way, and functions $u_{\nu\epsilon}$ and $\chi_{\nu\epsilon}$ such that

$$\begin{cases} u_{\nu\epsilon m} \overset{*}{\rightharpoonup} u_{\nu\epsilon} & \text{in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \ , \\ u'_{\nu\epsilon m} \rightharpoonup u'_{\nu\epsilon} & \text{in } L^{2p+2}(0,T;W_0^{2,2p+2}(\Omega)) \ , \\ u'_{\nu\epsilon m} \overset{*}{\rightharpoonup} u'_{\nu\epsilon} & \text{in } L^\infty(0,T;H_0^1(\Omega)) \ , \\ u''_{\nu\epsilon m} \rightharpoonup u''_{\nu\epsilon} & \text{in } L^2(0,T;L^2(\Omega)) \ , \\ \beta(u'_{\nu\epsilon m}) \rightharpoonup \chi_{\nu\epsilon} & \text{in } L^{\frac{2p+2}{2p+1}}(0,T;W^{-2,\frac{2p+2}{2p+1}}(\Omega)) \ . \end{cases}$$

Moreover,

$$u_{\nu\epsilon m} \to u_{\nu\epsilon} \quad \text{in } \ L^2(0,T;H_0^1(\Omega)) \ ,$$
$$u'_{\nu\epsilon m} \to u'_{\nu\epsilon m} \quad \text{in } \ L^2(0,T;L^2(\Omega)) \ .$$

Letting m tend to ∞, we obtain

$$\Big(\rho_\nu(t)\, u''_{\nu\epsilon}(t) - M(t,\|u_{\nu\epsilon}(t)\|^2)\,\Delta u_{\nu\epsilon}(t) + \alpha\, u'_{\nu\epsilon}(t),\ v\Big) + \langle \chi_{\nu\epsilon}(t), v\rangle = (f(t), v),$$
$$\text{a.e. on } \ [0,T], \quad \forall\, v \in W_0^{2,2p+2}(\Omega) \ .$$

Obviously, $u_{\nu\epsilon}(t)$ satisfies the initial data (15). Using monotonicity and hemicontinuity of β, we can prove (see Lions [12]) that

$$\chi_{\nu\epsilon}(t) = \beta(u'_{\nu\epsilon}(t)) \quad \text{ in } \ L^{\frac{2p+2}{2p+1}}(0,T;W^{-2,\frac{2p+2}{2p+1}}(\Omega)) \ .$$

This completes the proof of Theorem 1. ∎

3 – The main result

Theorem 2. *Under conditions of Theorem 1, there exists a function u:*

$$\text{(38)} \qquad \begin{cases} u \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \ , \\ u' \in L^\infty(0,T;H_0^1(\Omega)) \cap L^{2p+2}(0,T;W_0^{2,2p+2}(\Omega)) \ , \\ u'(t) \in \mathbb{K} \quad \text{a.e. on } \ [0,T] \ , \\ u'' \in L^2(0,T;L^2(\Omega)) \ , \end{cases}$$

which is a solution to the following problem

$$\text{(39)} \qquad \begin{cases} \Big(\rho(t)\, u''(t) - M(t,\|u(t)\|^2)\,\Delta u(t) + \alpha\, u'(t) - f(t),\ v - u'(t)\Big) \ge 0 \ , \\ \forall\, v \in \mathbb{K}, \quad \text{a.e. on } \ [0,T] \ , \\ u(0) = \phi_0 \,, \quad u'(0) = \phi_1 \ . \end{cases}$$

Moreover, if

$$\text{(40)} \qquad \phi_0(x) \ge 0 \ \text{ a.e. on } \Omega \quad \text{with } \ \|\phi_0\| > 0 \ ,$$

$$\text{(41)} \qquad M(x,t,0) = 0 \ \text{ and } \ M(x,t,\lambda) > 0 \,, \quad (x,t,\lambda) \in \Omega\times[0,T]\times(0,\infty) \ ,$$

then this solution is uniquely defined.

Proof: For each $\nu \in \mathbb{N}$ and $\epsilon \in (0, \epsilon_0)$, we have a function $u_{\nu\epsilon}$ given by Theorem 1. Therefore, $u_{\nu\epsilon}$ satisfies (14)–(15), which imply the following estimates

$$|\Delta u_{\nu\epsilon}(t)| \le \kappa_1 \ , \tag{42}$$

$$\|u_{\nu\epsilon}(t)\| \le \kappa_2 \ , \tag{43}$$

$$\|u'_{\nu\epsilon}(t)\| \le \kappa_3 \ , \tag{44}$$

$$\int_0^t \|\Delta u'_{\nu\epsilon}(s)\|^{2p+2}_{L^{2p+2}(\Omega)}\, ds \le \kappa_4 \ , \tag{45}$$

$$\int_0^t |u''_{\nu\epsilon}|^2\, ds \le \kappa_5 \ , \tag{46}$$

where κ_i, for $i = 1, ..., 5$, are positive constants independent of ν, ϵ and t.

If we fix $\nu < \infty$, letting ϵ tend to zero, taking into account (42)–(46) and the monotonicity of β, then we find a function u_ν, as a limit of a subsequence $(u_{\nu\epsilon})_{\epsilon\in(0,\epsilon_0)}$, such that

$$\begin{cases} u_\nu \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \ , \\ u'_\nu \in L^\infty(0,T; H_0^1(\Omega)) \cap L^{2p+2}(0,T; W_0^{2,2p+2}(\Omega)) \ \text{ and } \ u'_\nu(t) \in \mathbb{K} \ \text{ a.e. on } [0,T] \ , \\ u''_\nu \in L^2(0,T; L^2(\Omega)) \ , \end{cases}$$

and

$$\begin{cases} \Big(\rho_\nu(t)\, u''_\nu(t) - M(t, \|u_\nu(t)\|^2)\, \Delta u_\nu(t) + \alpha\, u'_\nu(t) - f(t),\ v - u'_\nu(t)\Big) \ge 0 \\ \qquad \forall v \in \mathbb{K}, \qquad \text{a.e. on } \ [0,T] \ . \\ u(0) = \phi_0\,, \qquad u'(0) = \phi_1 \ . \end{cases}$$

The sequence $(u_\nu)_{\nu\in\mathbb{N}}$ has the estimates (42)–(46). Thus, taking the limit in a subsequence as ν tend to ∞, we find a function u which satisfies (38) and (39). This completes the proof of the existence. To prove uniqueness, let u and z be two distinct functions satisfying (38) and (39). Then $\psi = u - z$ satisfies the following inequality

$$\begin{aligned} &\Big(\rho(t)\, (\psi'(t))^2\Big) - 2\int_0^t \Big(M(s, \|u(s)\|^2)\, \Delta\psi(s),\ \psi'(s)\Big)\, ds + 2\,\alpha \int_0^t |\psi'(s)|^2\, ds \le \\ &\le \int_0^t \Big(\rho'(s), (\psi'(s))^2\Big)\, ds + 2\int_0^t \Big(\Big[M(t, \|u(s)\|^2) - M(t, \|z(s)\|^2)\Big]\, \Delta z(s),\ \psi'(s)\Big)\, ds \, . \end{aligned}$$

On the other hand, using integration by parts, we have

$$-2\int_0^t \Big(M(s,\|u(s)\|^2)\,\Delta\psi(s),\,\psi'(s)\Big)\,ds =$$

$$= 2\int_0^t \sum_{i=1}^n \int_\Omega M(x,s,\|u(s)\|^2)\,\frac{\partial\psi}{\partial x_i}(x,s)\,\frac{\partial\psi'}{\partial x_i}(x,s)\,dx\,ds$$

$$+\,2\int_0^t \sum_{i=1}^n \int_\Omega \frac{\partial M}{\partial x_i}(x,s,\|u(s)\|^2)\,\frac{\partial\psi}{\partial x_i}(x,s)\,\psi(x,s)\,dx\,ds\ ,$$

and

$$2\int_0^t \sum_{i=1}^n \int_\Omega M(x,s,\|u(s)\|^2)\,\frac{\partial\psi}{\partial x_i}(x,s)\,\frac{\partial\psi'}{\partial x_i}(x,s)\,dx\,ds =$$

$$= \sum_{i=1}^n \int_\Omega \int_0^t M(x,s,\|u(s)\|^2)\,\frac{\partial}{\partial s}\Big(\frac{\partial\psi}{\partial x_i}(x,s)\Big)^2\,ds\,dx$$

$$= \sum_{i=1}^n \Big(M(t,\|u(t)\|^2),\Big(\frac{\partial\psi}{\partial x_i}(t)\Big)^2\Big) - \sum_{i=1}^n \int_0^t \Big(M'(s,\|u(s)\|^2),\Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,ds$$

$$-\sum_{i=1}^n \int_0^t \Big(\frac{\partial M}{\partial\lambda}(s,\|u(s)\|^2),\Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,2\Big(\Big(u'(s),u(s)\Big)\Big)\,ds\ .$$

Hence,

$$\Big(\rho(t),(\psi'(t))^2\Big) + \sum_{i=1}^n \Big(M(t,\|u(t)\|^2),\Big(\frac{\partial\psi}{\partial x_i}(t)\Big)^2\Big) + 2\,\alpha\int_0^t |\psi'(s)|^2\,ds \le$$

$$\le -2\int_0^t \sum_{i=1}^n \int_\Omega \frac{\partial M}{\partial x_i}(x,s,\|u(s)\|^2)\,\frac{\partial\psi}{\partial x_i}(x,s)\,\psi'(x,s)\,dx\,ds$$

$$+\,2\int_0^t \Big(\Big[M(s,\|u(s)\|^2) - M(s,\|z(s)\|^2)\Big]\,\Delta z(s),\,\psi'(s)\Big)\,ds$$

$$+\sum_{i=1}^n \int_0^t \Big(M'(s,\|u(s)\|^2),\,\Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,ds + \int_0^t \Big(\rho'(s),(\psi'(s))^2\Big)\,ds$$

$$+\sum_{i=1}^n \int_0^t \Big(\frac{\partial M}{\partial\lambda}(s,\|u(s)\|^2),\,\Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,2\Big(\Big(u'(s),u(s)\Big)\Big)\,ds\ .$$

Assumption (40) implies that all the solutions u of (38) and (39) have the property

$$\|u(t)\| > 0\,, \qquad \forall\, t \in [0,T]\ , \tag{47}$$

see Frota–Lar'kin [7]. By virtue of (47) and (41), we find

$$0 < m_0 = \min\Big\{M(x,t,\|u(t)\|^2);\ (x,t) \in \overline{Q}\Big\}\ ,$$

from which it follows

$$m_0\,\|\psi(t)\|^2 \le \sum_{i=1}^n \left(M(t,\|u(t)\|^2),\ \left(\frac{\partial\psi}{\partial x_i}(t)\right)^2\right), \qquad \forall\, t \in [0,T]\ .$$

Then

$$\begin{aligned}
\Big(\rho(t),(\psi'(t))^2\Big) &+ m_0\,\|\psi(t)\|^2 + \int_0^t\!\int_\Omega \Big[2\,\alpha - \rho'(x,s)\Big]\,(\psi'(x,s))^2\,dx\,ds \le \\
&\le 2\int_0^t \Big(\Big[M(s,\|u(s)\|^2) - M(s,\|z(s)\|^2)\Big]\,\Delta z(s),\ \psi'(s)\Big)\,ds \\
&\quad - 2\int_0^t \sum_{i=1}^n \int_\Omega \frac{\partial M}{\partial x_i}(x,s,\|u(s)\|^2)\,\frac{\partial\psi}{\partial x_i}(x,s)\,\psi'(x,s)\,ds \\
&\quad + \sum_{i=1}^n \int_0^t \left(M'(s,\|u(s)\|^2),\ \left(\frac{\partial\psi}{\partial x_i}(s)\right)^2\right) ds \\
&\quad + \sum_{i=1}^n \int_0^t 2\Big(\Big(u'(s),u(s)\Big)\Big)\left(\frac{\partial M}{\partial\lambda}(s,\|u(s)\|^2),\ \left(\frac{\partial\psi}{\partial x_i}(s)\right)^2\right) ds\ .
\end{aligned}$$

Let us analize the right-hand side of this inequality. We choose θ such that

$$0 < \theta < \frac{a_0}{2}\ . \tag{48}$$

Therefore,

$$\begin{aligned}
2\int_0^t \Big(\Big[M(s,\|u(s)\|^2) - M(s,\|z(s)\|^2)\Big]\,\Delta z(s),\ \psi'(s)\Big)\,ds &\le \\
\le \frac{C}{\theta}\int_0^t \|\psi(s)\|^2\,ds + \theta\int_0^t |\psi'(s)|^2\,ds\ ,&
\end{aligned} \tag{49}$$

$$2\int_0^t \sum_{i=1}^n \int_\Omega \frac{\partial M}{\partial x_i}(x,s,\|u(s)\|^2)\,\frac{\partial \psi}{\partial x_i}(x,s)\,\psi'(x,s)\,dx\,ds \le \tag{50}$$
$$\le \frac{C}{\theta}\int_0^t \|\psi(s)\|^2\,ds + \theta\int_0^t |\psi'(s)|^2\,ds\ ,$$

$$\sum_{i=1}^n \int_0^t \Big(M'(s,\|u(s)\|^2),\, \Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,ds \le C\int_0^t \|\psi(s)\|^2\,ds\ , \tag{51}$$

$$2\sum_{i=1}^n \int_0^t \Big(\big(u'(s),u(s)\big)\Big)\,\Big(\frac{\partial M}{\partial \lambda}(s,\|u(s)\|^2),\, \Big(\frac{\partial\psi}{\partial x_i}(s)\Big)^2\Big)\,ds \le \tag{52}$$
$$\le C\int_0^t \|\psi(s)\|^2\,ds\ ,$$

where C is a positive constant.

From (13) and (49)–(52), we obtain

$$\Big(\rho(t),(\psi'(t))^2\Big) + m_0\,\|\psi(t)\|^2 + (a_0-2\,\theta)\int_0^t |\psi'(s)|^2\,ds \;\le\; C\int_0^t \|\psi(s)\|^2\,ds\ .$$

This inequality, (48) and Gronwall's Lemma give $\|\psi(t)\| \le 0$ that implies $u=z$. This completes the proof of Theorem 2. ∎

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