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ON A CONJECTURE RELATIVE TO THE MAXIMUM OF HARMONIC FUNCTIONS ON CONVEX DOMAINS: UNBOUNDED DOMAINS *

L.R. Berrone

Abstract: Let u be a harmonic function on a bounded domain Ω which satisfies the mixed boundary conditions $u\big|_{\Gamma_0} = 0$, $\frac{\partial u}{\partial n}\big|_{\Gamma_1} = 1$, where Γ_1 is composed by a finite number of subarcs of $\partial\Omega$, $\Gamma_0 = \partial\Omega \sim \Gamma_1$ and n indicates the outward unit normal. In [2] has been conjectured that if Ω is convex and the subset Γ_1 is made to vary on $\partial\Omega$ while its measure is maintained equal to a constant $C > 0$, then $\sup_{x\in\Omega} u$ attains its maximum value when Γ_1 is a certain connected subarc of measure C. In the present paper, the case of unbounded domains is discussed.

1 – Introduction

Let $\Omega \neq \mathbb{R}^2$ be a plane domain and let Γ_1 denote a relatively open part of the boundary $\partial\Omega$. We denote by Γ_0 the remaining portion of Ω_0; i.e., $\Gamma_0 = \partial\Omega \sim \Gamma_1$. Throughout this paper we will be concerned with the mixed boundary value problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega\ , \\ u(x) = 0, & x \in \Gamma_0\ , \\ \dfrac{\partial u}{\partial n}(x) = 1, & x \in \Gamma_1\ . \end{cases} \tag{1.1}$$

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We shall assume the boundary $\partial\Omega$ is sufficiently regular; in general, boundaries $\mathcal{C}^2$ or piecewise $\mathcal{C}^1$ are to be considered below. Some minor modifications in the statement of (1.1) are needed when Ω is an unbounded domain, which shall be opportunely indicated. Under these assumptions, problem (1.1) has a unique classical solution $u \in \mathcal{C}^2(\Omega) \cup \mathcal{C}^0(\overline{\Omega})$ which, as it is well-known, admits several physical interpretations (see, for example, [8]). To our present purpose it will be illustrative, however, to consider the solution u to problem (1.1) as giving the equilibrium position of an elastic membrane Ω which is submitted to a unitary normal force on Γ_1, while is fixed at zero along the portion Γ_0 of the boundary. To emphasize the dependence on Γ_1 of the solution u to problem (1.1), we will frequently write it in the form $u[\Gamma_1]$ and so, $u[\Gamma_1](x)$ will denote the value of $u[\Gamma_1]$ at a point $x \in \Omega$.

We are interested in the behaviour of the functional $\Gamma_1 \mapsto \sup_{x\in\Omega} u[\Gamma_1](x)$ when the relatively open set Γ_1 varies on $\partial\Omega$ in such a way that its measure $|\Gamma_1|$ is preserved. So, for $C > 0$ we define $\mathcal{F}(C)$ to be the family of relatively open subsets Γ_1 of $\partial\Omega$ with a finite number of components and constant measure $|\Gamma_1| = C$. The following conjecture has been posed in the thesis [2]:

Conjecture 1.1. If Ω is a convex bounded domain with boundary piecewise $\mathcal{C}^1$, then $\sup_{\Gamma_1\in\mathcal{F}(C)}(\sup_{x\in\Omega} u[\Gamma_1](x))$ is realized for a certain *connected* subset $\Gamma_1 \in \mathcal{F}(C)$.

Recalling the above suggested mechanical interpretation for (1.1), Conjecture 1.1 can be stated by saying that if a membrane initially fixed at zero is lifted by an unitary force on portions Γ_1 of its boundary which have a constant total measure $C > 0$; then, the membrane reaches a maximum height when Γ_1 is a certain arc of measure C.

Conjecture 1.1 arose from attempts to estimate the solution to mixed boundary problems through the calculation of sub and supersolutions. Think, for example, on the problem

$$(1.2) \qquad \begin{cases} \Delta u(x) = 0, & x \in \Omega\ , \\ u(x) = \phi(x), & x \in \Gamma_0\ , \\ \dfrac{\partial u}{\partial n}(x) = \psi(x), & x \in \Gamma_1\ , \end{cases}$$

where ϕ and ψ are bounded continuous functions and $\sup_{\Gamma_1} \psi > 0$. It is often useful in practice to find estimates for $\sup_\Omega u$ or $\inf_\Omega u$ and a simple resource to obtain such estimates is the computation of sub or supersolutions to (1.2). When Γ_1 is not connected, the explicit computation of sub or supersolutions to (1.2) usually becomes less simpler (cf. [3]). Now, the general validity of Conjecture 1.1 would facilitate such calculations in some cases. In fact, by the maximum principles ([7], [5]), we have $\sup_\Omega u \le \sup_\Omega v$, where v solves the following problem

$$\begin{cases} \Delta v(x) = 0, & x \in \Omega\ , \\ v(x) = \sup\limits_{\Gamma_0} \phi, & x \in \Gamma_0 \\ \dfrac{\partial u}{\partial n}(x) = \sup\limits_{\Gamma_1} \psi, & x \in \Gamma_1\ . \end{cases} \tag{1.3}$$

If Ω is convex, then Conjecture 1.1 applies to (1.3) providing $\sup_\Omega v \le \sup_\Omega w$, where w satisfies a problem like (1.3) but now with Γ_1 being a connected subset of the boundary. For w we can easily find supersolutions belonging to simple classes of functions ([2], [3]). By denoting with U a supersolution, we deduce $\sup_\Omega u \le \sup_\Omega v \le \sup_\Omega w \le \sup_\Omega U$.

In this paper we will investigate what occurs when the condition of boundedness is suppressed in Conjecture 1.1. Two of the most simple instances of unbounded domains, the half-plane and the strip, are respectively studied in sections 2 and 3. Concretely, Conjecture 1 is shown to be true for the half-plane and false for the strip. The insight gained by analyzing these simple cases will enable us to discuss additional assumptions which could make a statement like Conjecture 1.1 generally true for unbounded domains. Such discussion and other comments also related to Conjecture 1.1 are informally presented in section 3.

2 – The half-plane

When $\Omega = \mathbb{R}^2_+$, problem (1.1) has to be formulated as follows

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in \mathbb{R}^2_+\ , \\ u(x,0) = 0, & x \in \Gamma_0\ , \\ \dfrac{\partial u}{\partial y}(x,0) = -1, & x \in \Gamma_1\ , \\ \lim u(x,y) = 0 & \text{as } |(x,y)| \to \infty\ ; \end{cases} \tag{2.1}$$

where Γ_1 is an open subset of the real axis and $\Gamma_0 = \mathbb{R} \sim \Gamma_1$. As a consequence of Phragmèn–Lindelöf like theorems which generalize the maximum principle to unbounded domains ([7]), the solution $u[\Gamma_1]$ to problem (2.1) is positive everywhere and reaches its maximum value at a point of Γ_1. Furthermore, if f denotes the boundary value of $u[\Gamma_1]$, then f is a Hölder-continuous function on $\mathbb{R}$ which is $\mathcal{C}^\infty$ on $\Gamma_1 \cup (\Gamma_0)^0$ (see [6], [5]).

We shall consider, for $2A > C > 0$ and $n \in \mathbb{N}$, the family of open sets

$$\mathcal{F}(A,C,n) = \Big\{\Gamma_1 : \Gamma_1 = \bigcup_{k=1}^{n}(a_k, b_k),\ -A \le a_k \le b_k < a_{k+1} \le A, \\ \sum_{k=1}^{n}(b_k - a_k) = C\Big\} , \tag{2.2}$$

and we shall attend to the solutions $u[\Gamma_1]$ to problem (2.1) with $\Gamma_1 \in \mathcal{F}(A,C,n)$. It should be noted that the inequalities $b_k < a_{k+1}$ in (2.2) mean that the components of a $\Gamma_1 \in \mathcal{F}(A,C,n)$ are separated. This restriction is due to the fact that the Dirichlet condition $u(x,0) = 0$ is meaningless when imposed on a single point. We also define

$$\mathcal{F}(C) = \bigcup_{n=1}^{\infty} \mathcal{F}(n,C,n) \ ;$$

i.e., the family open subset of $\mathbb{R}$ with a finite number of separated components and constant measure C.

The main result of this section can be stated as follows.

Theorem 2.1. $\sup_{\Gamma_1 \in \mathcal{F}(C)}(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y))$ *is attained when* Γ_1 *is an open interval of length* C.

The argument below presented to prove Theorem 2.1 follows rather classical and concrete steps. In fact, we begin by proving that there exists a function in the family $\{u[\Gamma_1] : \Gamma_1 \in \mathcal{F}(A,C,n)\}$ $(n \ge A)$, which actually realizes the optimum value of $\sup_{\Gamma_1 \in \mathcal{F}(C)}(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y))$ and then, we show that the Γ_1 corresponding to that optimum must necessarily be connected; that is, an open interval. Later, the case in which the supremum is taken over the family $\mathcal{F}(C)$ is easily reduced to that one.

To begin with we recall that if $f(x)$, $x \in \mathbb{R}$, gives the continuous boundary value of a function $u(x,y)$ which is harmonic in the upper half-plane and goes to zero when $|(x,y)| \to \infty$, then the representation formula (see, for example, [1])

$$u(x,y) = (P_y * f)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{(x-\xi)^2 + y^2}\, f(\xi)\, d\xi \ , \tag{2.3}$$

holds for $(x,y) \in \mathbb{R}^2_+$. In (2.3), $P_y(x) = y/[\pi(x^2+y^2)]$, $y > 0$, is the Poisson kernel for the upper half-plane. This fact is often expressed in a somewhat imprecise way by saying that (2.3) provides the solution to the Dirichlet problem

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in \mathbb{R}^2_+ \ , \\ u(x,0) = f(x), & x \in \mathbb{R} \ , \\ \lim u(x,y) = 0 & \text{as } |(x,y)| \to \infty \ . \end{cases} \tag{2.4}$$

As a first result in order to prove Theorem 2.1, we have the following one which gives, for a harmonic function with continuous boundary values, useful expressions for its normal derivative at boundary points.

Lemma 2.2. *Let u be the solution to problem (2.3) with f continuous. Then:*

i) *By assuming that*

$$\int_{-\infty}^{+\infty} \left| \frac{f(x_0+\xi) + f(x_0-\xi) - 2\,f(x_0)}{\xi^2} \right| d\xi < +\infty \;; \tag{2.5}$$

then, there exists the normal derivative of u at $(x_0,0)$ and it is given by

$$-\frac{\partial u}{\partial y}(x_0,0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\,f(x_0) - \Big(f(x_0+\xi) + f(x_0-\xi)\Big)}{\xi^2}\, d\xi \;. \tag{2.6}$$

ii) *If $\mathrm{supp}(f)$ is compact and x_0 denotes a point of maximum of f where f'' is supposed to exist then, the following expression holds for the normal derivative at $(x_0,0)$*

$$-\frac{\partial u}{\partial y}(x_0,0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(x_0) - f(\xi)}{(x_0-\xi)^2}\, d\xi \;. \tag{2.7}$$

Condition (2.5) resembles a similar one for the circle given by Dini in [4] and it is satisfied, for example, when $\mathrm{supp}(f)$ is compact and $f''(x_0)$ exists. Although formula (2.7) for the normal derivative at a point of maximum holds in a more general setting, that provided in Lemma 2.2 will suffices for our present purpose.

Proof of Lemma 2.2: **i)** By using elementary properties of the Poisson kernel ([1], [5]), from (2.3) we deduce

$$-\frac{\partial u}{\partial y}(x_0,0) = \lim_{y\downarrow 0} \frac{f(x_0) - u(x_0,y)}{y} = \frac{1}{\pi} \lim_{y\downarrow 0} \int_{-\infty}^{+\infty} \frac{f(x_0) - f(\xi)}{(x_0-\xi)^2 + y^2}\, d\xi \;. \tag{2.8}$$

After appropriate changes of variable, the integral of (2.8) can be also expressed as

$$\int_{-\infty}^{+\infty} \frac{f(x_0) - f(\xi)}{(x_0-\xi)^2 + y^2}\, d\xi = \int_{-\infty}^{+\infty} \frac{f(x_0) - f(x_0-\xi)}{\xi^2 + y^2}\, d\xi = \int_{-\infty}^{+\infty} \frac{f(x_0) - f(x_0+\xi)}{\xi^2 + y^2}\, d\xi \;;$$

hence,

$$\lim_{y\downarrow 0} \int_{-\infty}^{+\infty} \frac{f(x_0) - f(\xi)}{(x_0-\xi)^2 + y^2}\, d\xi = \frac{1}{2} \lim_{y\downarrow 0} \int_{-\infty}^{+\infty} \frac{2\,f(x_0) - \Big(f(x_0-\xi) + f(x_0+\xi)\Big)}{\xi^2 + y^2}\, d\xi \;. \tag{2.9}$$

Now, for $y > 0$ we have

$$\left|\frac{2\,f(x_0)-\Big(f(x_0-\xi)+f(x_0+\xi)\Big)}{\xi^2+y^2}\right| \le \left|\frac{2\,f(x_0)-\Big(f(x_0-\xi)+f(x_0+\xi)\Big)}{\xi^2}\right| ,$$

and, taking into account (2.5), the proof of (2.6) follows from (2.8) and from an application to (2.9) of the dominated convergence theorem.

ii) Since f'' exists at the point of maximum x_0, Taylor's formula allow us to write, in a neighbourhood of such a point,

$$f(x) = f(x_0) + f''(x_0)\,(x-x_0)^2/2 + o(|x-x_0|^2) \;;$$

and therefore, the dominated convergence theorem can be applied to the last member of (2.8) to obtain (2.7). ■

As a consequence of the previous lemma, in the next one we show the uniform boundedness of the functions of the family $\{u[\Gamma_1]\colon\ \Gamma_1 \in \mathcal{F}(C)\}$.

Corollary 2.3. *The functions of the family* $\{u[\Gamma_1]\colon\ \Gamma_1 \in \mathcal{F}(C)\}$ *are uniformly bounded on* $\mathbb{R}^2_+$.

Proof: Let us consider the boundary value f of a function $u[\Gamma_1]$, $\Gamma \in \mathcal{F}(C)$. Function f is compactly supported and, as we claim above, it is $\mathcal{C}^\infty$ on Γ_1; therefore, if $x_0 \in \Gamma_1$ is a point of maximum of f, Lemma 2.2 applies to give

$$1 = -\frac{\partial u[\Gamma_1]}{\partial y}(x_0,0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x_0)-f(\xi)}{(x_0-\xi)^2}\,d\xi \;. \tag{2.10}$$

The integral in (2.10) can be split as follows

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(x_0)-f(\xi)}{(x_0-\xi)^2}\,d\xi &= \frac{1}{\pi}\int_{\Gamma_1}\frac{f(x_0)-f(\xi)}{(x_0-\xi)^2}\,d\xi + \frac{1}{\pi}\int_{\Gamma_0}\frac{f(x_0)-f(\xi)}{(x_0-\xi)^2}\,d\xi \\ &= \frac{1}{\pi}\int_{\Gamma_1}\frac{f(x_0)-f(\xi)}{(x_0-\xi)^2}\,d\xi + \frac{1}{\pi}\,f(x_0)\int_{\Gamma_0}\frac{1}{(x_0-\xi)^2}\,d\xi \;.\end{aligned} \tag{2.11}$$

Since $f(x_0) \ge f(x)$, $x \in \mathbb{R}$, the first integral in the last member of (2.7) is non-negative and so, from (2.10) and (2.11) we deduce

$$f(x_0) \le \pi\left(\int_{\Gamma_0}\frac{1}{(x_0-\xi)^2}\,d\xi\right)^{-1} . \tag{2.12}$$

Now, denote by (a,b) the component of Γ_1 that contains x_0. A glance to the behaviour of the function $\xi \mapsto (x_0-\xi)^{-2}$, $\xi \in \Gamma_0$, $(x_0 \in \Gamma_1)$, shows that

$$\int_{\Gamma_0} \frac{1}{(x_0-\xi)^2}\, d\xi \geq \int_{-\infty}^{a} \frac{1}{(x_0-\xi)^2}\, d\xi + \int_{a+|\Gamma_1|}^{+\infty} \frac{1}{(x_0-\xi)^2}\, d\xi$$
$$= \frac{1}{x_0-a} + \frac{1}{|\Gamma_1|+a-x_0} > \frac{2}{|\Gamma_1|}\,,$$

which together with (2.12), gives

$$f(x_0) < \frac{\pi}{2}\,|\Gamma_1| = \frac{\pi}{2}\,C\,. \tag{2.13}$$

Finally we note that by the maximum principle, the maximum value M of $u[\Gamma_1]$ on $\overline{\mathbb{R}^2_+}$ coincides with the maximum of f on Γ_1, so that (2.11) provides $M < \pi\, C$. ∎

Let Γ_1 belong to $\mathcal{F}(A,C,n)$. A particular way of denoting $\sup u[\Gamma_1]$ will be useful in the sequel. Namely, in view of the structure of Γ_1, to emphasize the dependence on the endpoints $a_1, b_1, ..., a_n, b_n$ of $\sup u[\Gamma_1]$ we write

$$F_n(a_1,b_1,...,a_n,b_n) = \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y)\,.$$

For each $n \in \mathbb{N}$, functions F_n are naturally defined on the set

$$X_n = \Big\{(a_1,b_1,...,a_n,b_n) \in \mathbb{R}^{2n} \colon\ a_1 \leq b_1 \leq ... \leq a_n \leq b_n\Big\}\,.$$

Since the application $\Gamma_1 \mapsto u[\Gamma_1]$ is continuous in the sense that given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|u[\Gamma_1^*] - u[\Gamma_1]\|_\infty < \epsilon$ whenever $|\Gamma_1^*\,\Delta\Gamma_1| < \delta$, we realize that functions F_n are continuous on X_n. Moreover, the set defined by

$$K_n(A,C) = \Big\{(a_1,b_1,...,a_n,b_n) \in X_n \colon\ -A \leq a_1,\ b_n \leq A,\ \sum_{k=1}^{n}(b_k-a_k) = C\Big\}\,,$$

is obviously a compact subset of X_n and therefore, the function F_n attains its maximum value on K_n. Then we have proved the following lemma.

Lemma 2.4. *For each $n \in \mathbb{N}$, $\sup_{\Gamma_1\in\mathcal{F}(A,C,n)}(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y))$ is realized by a certain Γ_1 that belongs to $\mathcal{F}(A,C,n)$.*

Proof: See the previous discussion. ∎

Next, let us choose a function $u[\Gamma_1]$ with Γ_1 belonging to $\mathcal{F}(A,C,n)$, $n > 1$. In what follows, we will show that when Γ_1 is not connected, then there exists another function $u[\Gamma_1^*]$ with Γ_1^* also belonging to $\mathcal{F}(A, C, n)$ such that

$$\sup_{(x,y)\in\mathbb{R}_+^2} u[\Gamma_1^*](x,y) > \sup_{(x,y)\in\mathbb{R}_+^2} u[\Gamma_1](x,y) \ .$$

With this purpose in mind, let us denote by f the boundary values of $u[\Gamma_1]$ and define, for each $k = 1, 2, ..., n$,

$$f_k(x) = \begin{cases} f(x), & x \in (a_k, b_k) \ , \\ 0, & x \in \mathbb{R} \sim (a_k, b_k) \ . \end{cases}$$

Taking into account the properties already stated for f, we realize that f_k is a Hölder-continuous and positive function on $\mathbb{R}$ which is $\mathcal{C}^\infty$ on $(-\infty, a_k) \cup (a_k, b_k) \cup (b_k, +\infty)$. The function f can be now written through the f_k's:

$$f(x) = \sum_{k=1}^{n} f_k(x) \,, \qquad x \in \mathbb{R} \ . \tag{2.14}$$

For $0 < \delta < \min_{1\le k\le n-1}(a_{k+1} - b_k)$, let us consider the solution $\widetilde{u}(x,y)$ to the Dirichlet problem (2.4) with the function

$$\widetilde{f}(x) = f_1(x) + \sum_{k=2}^{n} f_k(x+\delta) \,, \qquad x \in \mathbb{R} \ , \tag{2.15}$$

as boundary data. From a graphical viewpoint, the function f_k looks like a "bump" supported by the interval (a_k, b_k) and so, the function $\widetilde{f}$ corresponds to an approaching of the bumps f_k, $k = 2, ..., n$, to the leftmost bump f_1. By making

$$\widetilde{\Gamma}_1 = (a_1, b_1) \cup \bigcup_{k=2}^{n} (a_k - \delta, b_k - \delta) \ , \tag{2.16}$$

the following lemma shows that the normal derivative of $\widetilde{u}$ is less than 1 at every point of $\widetilde{\Gamma}_1$.

Lemma 2.5. $-\frac{\partial \widetilde{u}}{\partial y}(x,0) < 1$ *for* $x \in \widetilde{\Gamma}_1$.

Proof: First of all we observe that $\frac{\partial \widetilde{u}}{\partial y}(x,0)$ there exists in every point of $\widetilde{\Gamma}_1$. Indeed, $\widetilde{f}$ is a compactly supported function that is $\mathcal{C}^\infty$ on $\widetilde{\Gamma}_1$ and therefore, the existence of $\frac{\partial \widetilde{u}}{\partial y}(x,0)$, $x \in \widetilde{\Gamma}_1$, follows from Lemma 2.2 i). Now, by using expression (2.8) we will compare the normal derivatives of $u[\Gamma_1]$ and $\widetilde{u}$ in corresponding points of Γ_1 and $\widetilde{\Gamma}_1$. In this way, if $x \in (a_1, b_1)$; then $f(x) = \widetilde{f}(x)$ and we can write

$$
\begin{aligned}
1 + \frac{\partial \widetilde{u}}{\partial y}(x,0) &= -\frac{\partial u[\Gamma_1]}{\partial y}(x,0) + \frac{\partial \widetilde{u}}{\partial y}(x,0) \\
&= \frac{1}{\pi} \lim_{y \downarrow 0} \left\{ \int_{-\infty}^{+\infty} \frac{f(x) - f(\xi)}{(x-\xi)^2 + y^2} \, d\xi + \int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - \widetilde{f}(x)}{(x-\xi)^2 + y^2} \, d\xi \right\} \\
&= \frac{1}{\pi} \lim_{y \downarrow 0} \int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - f(\xi)}{(x-\xi)^2 + y^2} \, d\xi \ .
\end{aligned}
\tag{2.17}
$$

If the expressions for f and $\widetilde{f}$, respectively given by (2.14) and (2.15), are replaced in the last integral of (2.17), then we obtain

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - f(\xi)}{(x-\xi)^2 + y^2} \, d\xi &= \sum_{k=2}^{n} \int_{-\infty}^{+\infty} \frac{f_k(\xi+\delta) - f_k(\xi)}{(x-\xi)^2 + y^2} \, d\xi \\
&= \sum_{k=2}^{n} \int_{-\infty}^{+\infty} f_k(\xi) \left(\frac{1}{(x-\xi+\delta)^2 + y^2} - \frac{1}{(x-\xi)^2 + y^2} \right) d\xi \\
&= \sum_{k=2}^{n} \int_{a_k}^{b_k} f_k(\xi) \, \frac{2\,\delta\,(\xi - x - \delta/2)}{\Big((x-\xi+\delta)^2 + y^2\Big)\Big((x-\xi)^2 + y^2\Big)} \, d\xi \ .
\end{aligned}
\tag{2.18}
$$

But for $x \in (a_1, b_1)$, $\xi \in (a_k, b_k)$, $k = 2, ..., n$, and $0 < y \le 1$, we have

$$
\frac{2\,\delta\,(\xi - x - \delta/2)}{\Big((x-\xi+\delta)^2 + y^2\Big)\Big((x-\xi)^2 + y^2\Big)} \ge \frac{\delta^2}{(4\,A^2 + 1)^2} \ ,
$$

which together with (2.18) provides

$$
\int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - f(\xi)}{(x-\xi)^2 + y^2} \, d\xi \ge \frac{\delta^2}{(4\,A^2 + 1)^2} \sum_{k=2}^{n} \int_{a_k}^{b_k} f_k(\xi) \, d\xi \ , \qquad 0 < y \le 1 \ .
\tag{2.19}
$$

The remaining case in which $x \in (a_j - \delta,\, b_j - \delta)$ for $2 \le j \le n$ can be treated in a similar way. In fact, in this case we have $f(x+\delta) = \widetilde{f}(x)$ and then

$$
\begin{aligned}
1 + \frac{\partial \widetilde{u}}{\partial y}(x,0) &= -\frac{\partial u[\Gamma_1]}{\partial y}(x+\delta,\, 0) + \frac{\partial \widetilde{u}}{\partial y}(x,0) \\
&= \frac{1}{\pi} \lim_{y \downarrow 0} \left\{ \int_{-\infty}^{+\infty} \frac{f(x+\delta) - f(\xi)}{(x+\delta-\xi)^2 + y^2} \, d\xi + \int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - \widetilde{f}(x)}{(x-\xi)^2 + y^2} \, d\xi \right\} \\
&= \frac{1}{\pi} \lim_{y \downarrow 0} \left\{ \int_{-\infty}^{+\infty} \frac{f(x+\delta) - f(\xi+\delta)}{(x-\xi)^2 + y^2} \, d\xi + \int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - \widetilde{f}(x)}{(x-\xi)^2 + y^2} \, d\xi \right\} =
\end{aligned}
\tag{2.20}
$$

$$= \frac{1}{\pi} \lim_{y \downarrow 0} \int_{-\infty}^{+\infty} \frac{\widetilde{f}(\xi) - f(\xi+\delta)}{(x-\xi)^2 + y^2} \, d\xi = \frac{1}{\pi} \lim_{y \downarrow 0} \int_{-\infty}^{+\infty} \frac{f_1(\xi) - f_1(\xi+\delta)}{(x-\xi)^2 + y^2} \, d\xi$$
$$= \frac{1}{\pi} \lim_{y \downarrow 0} \int_{a_1}^{b_1} f_1(\xi) \left(\frac{1}{(x-\xi)^2 + y^2} - \frac{1}{(x-\xi+\delta)^2 + y^2} \right) d\xi$$
$$= \frac{1}{\pi} \lim_{y \downarrow 0} \int_{a_1}^{b_1} f_1(\xi) \, \frac{2\,\delta\,(x-\xi+\delta/2)}{\Big((x-\xi+\delta)^2 + y^2\Big)\Big((x-\xi)^2 + y^2\Big)} \, d\xi \; .$$

Since $x \in (a_j - \delta,\, b_j - \delta)$, $2 \le j \le n$, and $\xi \in (a_1, b_1)$, and $0 < y \le 1$, we can write

$$\int_{a_1}^{b_1} f_1(\xi) \, \frac{2\,\delta\,(x-\xi+\delta/2)}{\Big((x-\xi+\delta)^2+y^2\Big)\Big((x-\xi)^2+y^2\Big)} \, d\xi \ge \frac{3\,\delta^2}{(4\,A^2+1)^2} \int_{a_1}^{b_1} f_1(\xi)\, d\xi \; . \tag{2.21}$$

Finally, from (2.17)–(2.21) we deduce $1 + \frac{\partial \widetilde{u}}{\partial y}(x,0) > 0$ for every point $x \in \widetilde{\Gamma}_1$ as we have claimed. ∎

From Lemma 2.5 we derive the following one.

Lemma 2.6. *Let $u[\Gamma_1]$ be a solution to problem (2.1) with $\Gamma_1 \in \mathcal{F}(A,C,n)$, $n > 1$. If Γ_1 is not connected, then there exists another $\Gamma_1^* \in \mathcal{F}(A,C,n)$ such that the solution to (2.1) corresponding to Γ_1^* satisfies*

$$\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^*](x,y) > \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y) \; . \tag{2.22}$$

Proof: Assume $\Gamma_1 \in \mathcal{F}(A,C,n)$ and define $\Gamma_1^* = \widetilde{\Gamma}_1$, where $\widetilde{\Gamma}_1$ is given by (2.16). We will see that (2.22) is satisfied by $u[\Gamma_1^*]$. In fact, by Lemma 2.5 we have $-\frac{\partial u[\Gamma_1^*]}{\partial y}(x,0) = 1 > -\frac{\partial \widetilde{u}}{\partial y}(x,0)$, $x \in \Gamma_1^*$, and $u[\Gamma_1^*](x,0) = 0 = \widetilde{u}(x,0)$, $x \in \mathbb{R} \sim \Gamma_1^*$. Since both $u[\Gamma_1^*]$, $\widetilde{u}$ are harmonic and bounded functions on $\mathbb{R}^2_+$, the maximum principle and the theorem on the sign of the normal derivative in a point of maximum ([7], [5]) imply that

$$u[\Gamma_1^*](x,y) > \widetilde{u}(x,y) \, , \qquad (x,y) \in \mathbb{R}^2_+ \cup \{(x,0)\colon\, x \in \Gamma_1^*\} \; ,$$

whence

$$\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^*](x,y) > \sup_{(x,y)\in\mathbb{R}^2_+} \widetilde{u}(x,y) = \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y) \; ,$$

which finishes the proof. ∎

Now, we are in condition to prove a restricted version of Theorem 2.1. Namely, we will show that $\sup_{\Gamma_1\in\mathcal{F}(A,C,n)}(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y))$ occurs when Γ_1 is connected.

Theorem 2.7. $\sup_{\Gamma_1\in\mathcal{F}(A,C,n)}(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y))$ *is attained when* Γ_1 *is an open interval of length* C.

Proof: From Lemma 2.5 we deduce

$$\sup_{\Gamma_1\in\mathcal{F}(A,C,n)}\Big(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y)\Big) = \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^*](x,y) \ , \tag{2.23}$$

for certain $u[\Gamma_1^*] \in \mathcal{F}(A,C,n)$. If we suppose that Γ_1^* is not connected, then Lemma 2.6 applies furnishing another function $u[\Gamma_1^{**}] \in \mathcal{F}(A,C,n)$ such that

$$\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^{**}](x,y) > \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^*](x,y) \ ,$$

thus contradicting (2.23). Therefore, Γ_1^* must be a connected open set of length C. ∎

In order to prove Theorem 2.1, we will momentarily return to the language and notation introduced above. Concretely, we have a sequence of non negative continuous functions F_n, respectively defined on compact sets $K_n(A,C)$. By considering the natural inclusions $K_n(n,C) \subseteq K_{n+1}(n+1,C)$, $n\in\mathbb{N}$, $2n>C$, we can look at the family $\{K_n(n,C)\}$ as an expanding one and so, the equality $F_{n+1}\Big|_{K_n(n,C)} = F_n$, holds for every $n\in\mathbb{N}$. Moreover, we obviously have $\mathcal{F}(C) = \bigcup_{2n>C} K_n(n,C)$. Let us consider the function $F\colon \mathcal{F}(C)\to\mathbb{R}$, such that $F\Big|_{K_n(n,C)} = F_n$. It is a simple matter to prove that

$$\sup_{\mathcal{F}(C)} F = \sup_{2n>C}\ \sup_{K_n(n,C)} F_n \ .$$

Furthermore, whatever be $n\in\mathbb{N}$, Theorem 2.8 ensures that $\sup_{K_n(n,C)} F_n = F_1(a,b)$, with $(a,b)\subseteq[-A,A]$ and $b-a=C$. Thus, we can conclude that $\sup_{\mathcal{F}(C)} F = F_1(a,b)$ or, by abuse of notation,

$$\sup_{\mathcal{F}(C)} F = F(a,b) \ . \tag{2.24}$$

Proof of Theorem 2.1: From definitions given above, equality (2.24) can be rewritten as

$$\sup_{\Gamma_1\in\mathcal{F}(C)}\Big(\sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1](x,y)\Big) = \sup_{(x,y)\in\mathbb{R}^2_+} u[\Gamma_1^*](x,y) \ ,$$

where $\Gamma_1^* = (a,b)$, $b-a=C$. This finishes the proof. ∎

3 – The strip

Along this section we will explore what happens when the strip $S=\mathbb{R}\times(-1,1)$ is taken as the domain Ω in Conjecture 1.1. Correspondingly, we shall consider the mixed boundary value problem

$$\begin{cases} \Delta u(x,y)=0, & (x,y)\in S\ , \\ u(x,y)=0, & (x,y)\in\Gamma_0\ , \\ \dfrac{\partial u}{\partial n}(x,y)=1, & (x,y)\in\Gamma_1\ , \\ \lim u(x,y)=0 & \text{as } |(x,y)|\to\infty\ ; \end{cases} \tag{3.1}$$

where Γ_1 is a relatively open subset of the boundary $\partial S=(\mathbb{R}\times\{-1\})\cup(\mathbb{R}\times\{1\})$. We will show that a property like that stated in Conjecture 1.1 does not hold for S. The procedure we will employ essentially consists of a comparison, via the calculation of suitable sub and supersolutions, of the maximum values of two solutions of (3.1), each one corresponding to an appropriate choice of Γ_1. Let us begin by introducing some useful notation. For $a>0$ we define $\Gamma_1(a)=[-2a,2a]\times\{-1\}$, $\Gamma_1^*(a)=([-a,a]\times\{-1\})\cup([-a,a]\times\{1\})$, and let u and u^* be the solutions to problem (3.1) for $\Gamma_1=\Gamma_1(a)$ and $\Gamma_1=\Gamma_1^*(a)$, respectively. Note that $|\Gamma_1(a)|=|\Gamma_1^*(a)|=4\,a$. We also define two harmonic functions in $\mathbb{R}^2$, v_a and w_a, by

$$v_a(x,y)=1-y\ ,\qquad w_a(x,y)=(y^2-x^2+a^2-1)/2\ .$$

Lemma 3.1. *The following inequalities are satisfied by the functions u and u^**

$$u(x,y)\le v_a(x,y)\ ,\qquad (x,y)\in\overline{S}\ , \tag{3.2}$$

$$w_a(x,y)\le u^*(x,y)\ ,\qquad (x,y)\in[-a,a]\times[-1,1]\ . \tag{3.3}$$

Proof: We first show that v_a is a supersolution in S to problem (3.1) with $\Gamma_1=\Gamma_1(a)$. In fact, we have

$$v_a(x,1)=0,\quad v_a(x,-1)=2>0\ \text{ and }\ -\frac{\partial v_a}{\partial y}(x,-1)=1\ ,\qquad x\in\mathbb{R}\ .$$

Now, we prove that w_a is a subsolution in $(-a,a)\times(-1,1)$ to problem (3.1) with $\Gamma_1=\Gamma_1^*(a)$. To this end, it suffices to observe that

$$\frac{\partial w_a}{\partial y}(x,1)=1=-\frac{\partial w_a}{\partial y}(x,-1)\ ,\qquad x\in\mathbb{R}\ ,$$

and

$$w_a(\pm a, y) = (y^2 - 1)/2 \le 0\,, \qquad |y| \le 1\ .$$

By the maximum principle, we have $u^*(\pm a, y) > 0$, $|y| < 1$, so that a comparison between u^* and w_a in $(-a, a) \times (-1, 1)$ throws inequality (3.3). ∎

Theorem 3.2. *If* $a > 2$, *then* $\sup_{(x,y)\in S} u(x,y) < \sup_{(x,y)\in S} u^*(x,y)$.

Proof: The maximum principle and simple considerations of symmetry show that $\sup_{(x,y)\in S} u(x,y) = u(0,-1)$ and $\sup_{(x,y)\in S} u^*(x,y) = u^*(0,1) = u^*(0,-1)$. But, from Lemma 3.1 we obtain

$$u(0,-1) \le v_a(0,-1) = 2\,, \qquad u^*(0,-1) \ge w_a(0,-1) = a^2/2\ ;$$

and therefore, the inequality

$$u(0,-1) < u^*(0,-1)\ ,$$

holds for $a^2/2 > 2$; that is, for $a > 2$. ∎

In the following theorem, a scaled version of Theorem 3.2 is presented. We denote with S_ω the strip $\mathbb{R} \times (0,\omega)$ and define the subsets of its boundary $\Gamma_1(\omega, a) = [-2a, 2a] \times \{0\}$ and $\Gamma_1^*(\omega, a) = ([-a, a] \times \{0\}) \cup ([-a, a] \times \{\omega\})$. We consider a mixed boundary problem analogous to (3.1) for S_ω and, as before, we denote with u and u^* the solutions to this problem for $\Gamma_1 = \Gamma_1(a,\omega)$ and $\Gamma_1 = \Gamma_1^*(a,\omega)$, respectively.

Theorem 3.3 *If* $a/\omega > 1$, *then* $\sup_{(x,y)\in S_\omega} u(x,y) < \sup_{(x,y)\in S_\omega} u^*(x,y)$.

Proof: If the functions $v_{a,\omega}(x,y) = \omega - y$ and $w_{a,\omega}(x,y) = ((y-\omega/2)^2 - x^2 + a^2 - \omega^2/4)/\omega$ are respectively taken instead of $v_a(x,y)$ and $w_a(x,y)$, the arguments of Lemma 3.1 and Theorem 3.2 can be easily translated to prove this one. ∎

4 – Concluding remarks

The example of the strip, which has been developed in the previous section, clearly shows the inadequacy of Conjecture 1.1 for general unbounded domains. Now, two ways at least are suggested by the anterior analysis to modify the conjecture so that it may be true for every unbounded domain. The first one arises from further reflections on the differences existing among the half-plane and the strip. Presumably, the conjecture fails for the strip due to certain "strip effect"; that is, due to the presence of parts of the boundary which are in front of other parts. In the following conjecture, this "strip effect" is avoided by introducing auxiliary assumptions on the structure of the boundary. We employ the same notation as in the Introduction.

Conjecture 4.1. Let $\Omega \subseteq \mathbb{R}^2$ be a convex unbounded domain such that $\partial\Omega = J \cup \{\infty\}$, where J is a piecewise $\mathcal{C}^1$ Jordan curve. Then $\sup_{\Gamma_1 \in \mathcal{F}(C)} (\sup_{x\in\Omega} u[\Gamma_1](x))$ is realized for a certain *connected* subset $\Gamma_1 \in \mathcal{F}(C)$.

The second way to adapt Conjecture 1.1 has to do with a localization of the condition $|\Gamma_1| = C$ and the resulting conjecture could hopefully be true for every convex domain, bounded or not. This local version of Conjecture 1.1 can be stated as follows.

Conjecture 4.2. If $\Omega \subseteq \mathbb{R}^2$ is a convex domain with boundary piecewise $\mathcal{C}^1$, then there exists $\delta = \delta(\Omega) > 0$ such that $\sup_{\Gamma_1 \in \mathcal{F}(C)} (\sup_{x\in\Omega} u[\Gamma_1](x))$ is realized for a certain *connected* subset $\Gamma_1 \in \mathcal{F}(C)$ provided that $C < \delta$.

Unlike what happens with bounded domains, convexity can not be suppressed from Conjecture 4.2 when Ω is unbounded. In fact, recalling Theorem 3.3 we can convince ourselves that the "strip effect" appears at every scale in the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : \, y + 1/|x| > 0\}$. Furthermore, it can be easily realized that $\sup_{\Gamma_1 \in \mathcal{F}(C)} (\sup_{x\in\Omega} u[\Gamma_1](x)) = +\infty$ in this case. Convexity neither can be eliminated of Conjecture 1.1. A counter-example showing this and a general proof of Conjecture 4.1 based on perturbative techniques will be the subject matter of a forthcoming paper.

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Lucio R. Berrone,
Instituto de Matemática "Beppo Levi",
Av. Pellegrini 250, 2000 Rosario – ARGENTINA
E-mail: `lrblevi@bibfei.edu.ar`