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REPRESENTATION OF CURVES OF CONSTANT WIDTH IN THE HYPERBOLIC PLANE

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Abstract: If γ is a curve of constant width in the hyperbolic plane $\mathbb{H}^2$, and l is a diameter of γ, the *track function* $x(\theta)$ gives the coordinate of the point of intersection $l(x(\theta))$ of l with the diameter of γ that makes an angle θ with l. We show that $x(\theta)$ determines the shape of γ up to the choice of a constant; this provides a representation of all curves of constant width in $\mathbb{H}^2$. The track function is locally Lipschitz on $(0,\pi)$, satisfies $|x'(\theta)\sin\theta| < 1-\epsilon$ for some $\epsilon > 0$, and, if l is appropriately chosen, has a continuous extension to $[0,\pi]$ such that $x(0) = x(\pi)$; conversely, any function satisfying these three conditions is the track function of some curve of constant width. As a by-product of the representation thus obtained, we prove that each curve of constant width in $\mathbb{H}^2$ can be uniformly approximated by real analytic curves of constant width, and extend to all curves of constant width some results previously established under restrictive smoothness assumptions.

1 – Introduction

A closed convex curve γ in the Euclidean plane is said to have constant width $\mathcal{W}$ if the distance between every two distinct parallel lines of support of Ω is equal to $\mathcal{W}$; equivalently, γ has constant width $\mathcal{W}$ if, for each $p \in \gamma$, the maximum distance from p to other points of γ is equal to $\mathcal{W}$. This latter condition can be taken as the definition of constant width for simple closed curves in arbitrary metric spaces: here we are concerned with such curves in the hyperbolic

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plane with Gaussian curvature -1, denoted throughout by $\mathbb{H}^2$ (an alternative approach to constant width in $\mathbb{H}^2$, based on horocycles, appears in [3]). (A word on terminology: by *lines* or *segments* in $\mathbb{H}^2$ we understand *geodesics* or *segments of geodesic*.)

Let γ be a simple closed curve in $\mathbb{H}^2$ with constant width $\mathcal{W}$: if $p, q \in \gamma$ are such that $|p\,q| = \mathcal{W}$, the segment $\overline{p\,q}$ is called a *diameter* of γ; thus diameters are maximal chords, and each point of γ belongs to at least one diameter. Every diameter is a *double normal* of γ, cutting γ orthogonally at both ends (more precisely, the perpendicular line at each extremity of each diameter is a line of support of γ); and every chord of γ that is orthogonal to γ at one end is a diameter, and therefore a double normal (see [1] and [2]).

It was observed in [2] that any two distinct diameters must intersect each other. Let us now fix a diameter $l(x)$ of γ, where x is the arc-length parameter (thus we are also fixing an orientation of l in the sense of increasing xx): letting p start at the *positive* end of l, the angle θ that the *oriented* diameters with *positive* end p make with l increases strictly and continuously from 0 to 2π as p performs one counterclockwise revolution around γ. (Notice that each corner point of γ is an end of more than one diameter; these diameters spread an angle and must he taken up in succession.) We let $l(x(\theta))$ be the point of intersection of l and the oriented diameter l_θ that cuts l at an angle θ, and let $f(\theta)$ be the distance from $l(x(\theta))$ to the positive end of l_θ: we call $x(\theta)$ the *track function*, and $f(\theta)$ the *intersection function*, of the curve γ (relative, of course, to the fixed diameter l). Thus, both the track and intersection functions of the circle are constant; but we refer the reader to [2, Example 7] for a more instructive example. Below we list some properties of $x(\theta)$ and $f(\theta)$ that follow readily from the definition:

Lemma 1. *Both the track function $x(\theta)$ and the intersection function $f(\theta)$ are continuous on each interval $(0, \pi)$ and $(\pi, 2\pi)$. Furthermore, we have for $\theta \in (0, \pi)$:*

a) $x(\theta + \pi) = x(\theta)$;

b) $0 \leq f(\theta) \leq \mathcal{W}$;

c) $f(\theta) + f(\theta + \pi) = \mathcal{W}$.

We show in Section 2 that it is possible to choose l in such a way that $x(\theta)$ has a continuous extension to $[0, 2\pi]$; then $f(\theta)$ also has a continuous extension to $[0, 2\pi]$, and $x(\theta)$ and $f(\theta)$ are then extended to $\mathbb{R}$ by periodicity, so that a), b), c) remain valid on $\mathbb{R}$.

It is clear that $x(\theta)$ and $f(\theta)$ completely determine the shape of γ, but these functions are not independent of each other: under the assumption that γ is at least C^3, we prove in [2] that

$$f'(\theta) = -x'(\theta)\cos\theta \; ; \tag{1}$$

as a result of (1), we see that $f(\theta)$ determines $x(\theta)$ up to the choice of a constant; since this constant merely corresponds to a translation along the line l, we see that $f(\theta)$ embodies all the information about the shape of γ (this is [2, Remark 9]); in particular, $f(\theta)$ is constant if and only if γ is a circle. On the other hand, $x(\theta)$ also determines $f(\theta)$ up to a constant, and the different possible choices of this constant lead to a family of parallel curves. It is therefore only a matter of convenience which of the functions $f(\theta)$ and $x(\theta)$ do we decide to work with, and convenience suggests that we choose $x(\theta)$.

For general curves of constant width we prove in Section 2 (Proposition 2) that the track function is locally Lipschitz (L.L.) on $(0,\pi)$; and, in Section 4 (Theorem 10), we prove that $f(\theta) = \lambda - \int_0^\theta x'(\phi)\cos\phi\, d\phi$ for some constant λ, thus showing that $f(\theta)$ is also L.L. on $(0,\pi)$ and that (1) is valid for almost every $\theta \in \mathbb{R}$. As a consequence, we obtain (Theorem 10) a general parameterization $\gamma(\theta)$ of curves of constant width, which is L.L. on $\mathbb{R}\backslash\{n\pi : n \in \mathbb{Z}\}$. Using this parameterization, we prove in Theorem 11 that each such curve $\gamma(\theta)$ can be uniformly approximated by analytic curves $\widetilde{\gamma}(\theta)$ of constant width (i.e., $\max_{0\leq\theta\leq 2\pi} |\gamma(\theta) - \widetilde{\gamma}(\theta)|$ can be made as small as we wish). [The analogous result for the Euclidean plane $\mathbb{E}^2$ was established in [7] by Wegner.] These results are then used in Section 5 to generalize results previously known only for differentiable curves (Theorem 12), and to prove some new results (Theorem 13).

Another question is whether any continuous function $x(\theta)$, periodic of period π and L.L. on $(0,\pi)$, is the track function of some curve of constant width. Not all functions will do: it is necessary that $|x'(\theta)\sin\theta|$ be bounded away from 1; but it turns out that this condition is also sufficient. This is proved in Section 3 (Theorem 7) for differentiable functions, and in full generality in Section 4 (Theorem 8).

This paper owes some inspiration to the work of Hammer and his co-authors ([4]–[6]), but the parameterization $\gamma(\theta)$ that we obtain, although also based on the diameters of γ, is *not* the direct analogue for $\mathbb{H}^2$ of the representation Hammer obtains in [5] for curves of constant width in $\mathbb{E}^2$, for the parameterizing angle he uses is different. We believe our approach is justified by the results so far obtained (including the main result in [2] and those in Section 5 here).

2 – Families of lines in $\mathbb{H}^2$

Consider a curve $\gamma \subseteq \mathbb{H}^2$ of constant width $\mathcal{W}$. Let $\mathcal{L}$ be the set of lines in $\mathbb{H}^2$ that extend the diameters of γ. This set $\mathcal{L}$ possesses the following properties:

i) any two distinct lines in $\mathcal{L}$ intersect each other;

ii) the distance between any two such intersection points is not greater than $\mathcal{W}$;

iii) given a line $l_0 \in \mathcal{L}$ and $\theta \in (0,\pi)$, there exists exactly one line $l_\theta \in \mathcal{L}$ such that the angle from l_0 to l_θ equals θ.

In this section we work in the abstract with a set $\mathcal{L}$ of lines satisfying i)–iii): it is of no consequence how such set originated. We start by fixing a line $l_0(x)$ in $\mathcal{L}$ parameterized by the the arc-length x. As before, the *track function* $x(\theta)$ of the family $\mathcal{L}$ (relative to the line l_0) gives the coordinate in l_0 of the point in $l_0 \cap l_\theta$.

Proposition 2. *The track function $x(\theta)$ is locally Lipschitz on $(0,\pi)$; hence it possesses a derivative almost everywhere. Moreover, the quantity $x'(\theta)\sin\theta$ is bounded.*

Proof: Given $0 < \theta_0 < \frac{\pi}{2}$, we show that $x(\theta)$ satisfies a Lipschitz condition on $[\theta_0, \pi-\theta_0]$. Take θ_1 and θ_2 in this interval: if $x(\theta_1) \neq (\theta_2)$, then the lines l_{θ_1}, l_{θ_2} and l_0 form a triangle whose angles adjacent to l_0 are either θ_1 and $\pi-\theta_2$, or $\pi-\theta_1$ and θ_2; in both cases, the third angle, which we denote by $\alpha(\theta_1,\theta_2)$, is less than $|\theta_1-\theta_2|$. By condition ii) above, no side of this triangle exceeds $\mathcal{W}$. Applying the law of sines for hyperbolic triangles, we have

$$\frac{\sinh\Big(|x(\theta_1)-x(\theta_2)|\Big)}{\sin\Big(\alpha(\theta_1,\theta_2)\Big)} \leq \frac{\sinh\mathcal{W}}{\sin\theta_1} \;,$$

and therefore

$$\Big|x(\theta_1)-x(\theta_2)\Big| \leq \frac{\sinh\mathcal{W}}{\sin\theta_1}\,|\theta_1-\theta_2| \leq \frac{\sinh\mathcal{W}}{\sin\theta_0}\,|\theta_1-\theta_2| \;, \tag{2}$$

which establishes the Lipschitz condition on $[\theta_0, \pi-\theta_0]$. If $x'(\theta_1)$ exists, then the first inequality in (2) also shows that $|x'(\theta_1)\sin\theta_1| \leq \sinh\mathcal{W}$, and this proves our second assertion. ∎

Lemma 3. *For fixed $\theta_1 \in (0,\pi)$, and denoting by $\alpha(\theta_1,\theta) \in [0,\frac{\pi}{2}]$ the smallest of the two angles between l_{θ_1} and l_θ, we have $\theta \to \theta_1$ if and only if $\alpha(\theta_1,\theta) \to 0$.*

Proof: The only if part is obvious, since $\alpha(\theta_1,\theta) \leq |\theta_1 - \theta|$ by the proof of Proposition 2. We now assert that $\theta \mapsto \alpha(\theta_1,\theta)$ is continuous: indeed, and ignoring degenerate cases, the lines l_{θ_1}, l_θ and $l_{\theta'}$, form a triangle $\Delta(\theta_1,\theta,\theta')$ whose sides do not exceed $\mathcal{W}$ and one of whose angles is smaller than $|\theta-\theta'|$; hence, as $\theta' \to \theta$, the area of $\Delta(\theta_1,\theta,\theta')$ becomes arbitrarily small and the sum of its other two angles approaches π, which means that $\alpha(\theta_1,\theta') \to \alpha(\theta_1,\theta)$ and proves our assertion. In conclusion, as $\theta \to \theta_1$ from above or from below, the angle $\alpha(\theta_1,\theta)$ takes on all possible small values; and this proves the *if* part since, by condition iii) above, there are, for each $0 < \varphi \leq \frac{\pi}{2}$, at most two angles θ such that $\alpha(\theta_1,\theta) = \varphi$. ∎

We notice that Proposition 2 also holds for sets of lines in the Euclidean plane $\mathbb{E}^2$ satisfying i)–iii): the proof is virtually the same. (For lines in $\mathbb{E}^2$, this result, with a different proof, is implicit in the work of Hammer and Sobczyk [4].) This fact is used in the proof of our next proposition, which, together with Lemma 3, says that, for almost all choices of the fixed line $l_0 \in \mathcal{L}$, the resulting track function $x(\theta)$ has a continuous extension to $[0,\pi]$ satisfying $x(0) = x(\pi)$. For $\theta \neq \theta'$, we denote by $p(\theta,\theta')$ the intersection point of l_θ and $l_{\theta'}$.

Proposition 4. *There exists $\lim_{\theta'\to\theta} p(\theta,\theta')$ for almost every $\theta \in (0,\pi)$.*

Proof: We make use of the existence of a *geodesic mapping* $\mathcal{H}$ between $\mathbb{H}^2$ and the open unit disk $\mathbb{U} \subseteq \mathbb{E}^2$: $\mathcal{H}$ is a C^∞ diffeomorphism $\mathbb{H}^2 \to \mathbb{U}$ that sends the geodesics of $\mathbb{H}^2$ onto chords of $\mathbb{U}$ (this is of course just a fancy presentation of the so-called Beltrami disk model for hyperbolic geometry). Consider the set $\mathcal{M}$ of straight lines in $\mathbb{E}^2$ that extend the segments $\mathcal{H}(l)$, $l \in \mathcal{L}$. This set $\mathcal{M}$ has the properties i)–iii) listed above: this is obvious for i) and ii) (although the constant in ii) may change), and less obvious for iii); we now prove iii).

Let $r_0(t)$ be the line in $\mathcal{M}$ corresponding to the fixed line $l_0(x)$ in $\mathcal{L}$: here t is the arc-length parameter along r_0, and we assume that the function $t = T(x)$ such that $\mathcal{H}(l_0(x)) = r_0(T(x))$ is monotonous increasing. Now consider the differentiable function $h\colon \mathbb{R} \times [0,\pi] \to [0,\pi]$ defined as follows: if the line l in $\mathbb{H}^2$ intersects l_0 at $l_0(x)$ and the angle from l_0 to l is θ, then the angle from r_0 to $\mathcal{H}(l)$ is $h(x,\theta)$; also $h(x,0) = 0$ and $h(x,\pi) = \pi$. The continuous function $\phi(\theta) = h(x(\theta),\theta)$ then gives the angle from r_0 to $\mathcal{H}(l_\theta)$; and, since $x(\theta)$ is bounded, we have $\lim_{\theta\to 0}\phi(\theta) = 0$ and $\lim_{\theta\to\pi}\phi(\theta) = \pi$, which shows that $\phi(\theta)$ assumes all values in $(0,\pi)$ and proves iii).

The expression for $\varphi = \phi(\theta)$ shows that this function is L.L. on $(0, \pi)$; but the whole argument can be reversed to show that ϕ^{-1} is also L.L. (just observe that, by Proposition 2, the track function $t(\varphi)$ relative to r_0 of the set $\mathcal{M}$, is L.L.). Hence, a set $R \subseteq (0, \pi)$ has measure zero if and only if $\phi(R)$ has measure zero. Denoting by r_φ the line in $\mathcal{M}$ that makes the angle φ with r_0, and by $q(\varphi, \varphi')$ the intersection point of r_φ and $r_{\varphi'}$, it therefore suffices to prove the following:

Claim. *There exists* $\lim_{\varphi' \to \varphi} q(\varphi, \varphi')$ *for almost every* $\varphi \in (0, \pi)$.

(This result is Theorem 2 in [4], but we reproduce the proof here for the reader's convenience.) We assume that the line r_0 is the horizontal axis in $\mathbb{E}^2$ and that $r_0(0)$ is the origin, and put $\boldsymbol{u}(\varphi) = (\cos\varphi, \sin\varphi)$, $\boldsymbol{u}'(\varphi) = (-\sin\varphi, \cos\varphi)$. The line r_φ is then given by

$$r_\varphi(\lambda) = a(\varphi)\, \boldsymbol{u}'(\varphi) + \lambda\, \boldsymbol{u}(\varphi)\,, \qquad \lambda \in \mathbb{R}\,, \tag{3}$$

where $a(\varphi) = -t(\varphi)\sin\varphi$. Notice that, letting $a(0) = a(\pi) = 0$, $a(\varphi)$ is continuous on $[0, \pi]$ and (3) is the equation of r_0 when $\varphi = 0$; also, $a(\varphi)$ is L.L. on $(0, \pi)$. (In fact, $a(\varphi)$ is *uniformly* Lipschitz on $[0, \pi]$, since $a'(\varphi) = t(\varphi)\cos\varphi + t'(\varphi)\sin\varphi$ is bounded by Proposition 2.) Hence $a'(\varphi)$ exists almost everywhere, and we conclude the proof of the claim by showing that $\lim_{\varphi' \to \varphi} q(\varphi, \varphi')$ exists whenever $a'(\varphi)$ does. Indeed, we have $q(\varphi, \varphi') = r_\varphi(\lambda)$, where

$$\begin{aligned}
\lambda &= \frac{-a(\varphi)\left\langle \boldsymbol{u}'(\varphi), \boldsymbol{u}'(\varphi')\right\rangle + a(\varphi')}{\left\langle \boldsymbol{u}(\varphi), \boldsymbol{u}'(\varphi')\right\rangle} \\
&= \frac{-a(\varphi)\left\langle \boldsymbol{u}'(\varphi), \dfrac{\boldsymbol{u}'(\varphi') - \boldsymbol{u}'(\varphi)}{\varphi' - \varphi}\right\rangle + \dfrac{a(\varphi') - a(\varphi)}{\varphi' - \varphi}}{\left\langle \boldsymbol{u}(\varphi), \dfrac{\boldsymbol{u}'(\varphi') - \boldsymbol{u}'(\varphi)}{\varphi' - \varphi}\right\rangle}\,,
\end{aligned}$$

and, if $a'(\varphi)$ exists, this converges to $-a'(\varphi)$ when $\varphi' \to \varphi$. ∎

Now we consider the differentiability properties of the set of lines $\mathcal{L}$: if the track function $x(\theta)$ is C^k $(1 \le k \le \omega)$, we could say by definition that $\mathcal{L}$ is C^k provided we were certain that all track functions of $\mathcal{L}$ (relative to each line of $\mathcal{L}$) were also C^k. That this does indeed happen is more or less obvious, but we think the following indirect argument might be of interest. We first claim that $x(\theta)$ and the track function of $\mathcal{M}$, $t(\varphi)$, are of the same differentiability class: for if $x(\theta)$ is C^k then so is $\phi(\theta) = h(x(\theta), \theta)$; since ϕ^{-1} is Lipschitz, $\phi'(\theta)$ never vanishes and therefore ϕ^{-1} is also C^k; finally, $t(\varphi) = (T \circ x \circ \phi^{-1})(\varphi)$ is also C^k; and, since we can reverse the argument, this proves our claim. Now, we *define* the set of lines $\mathcal{M}$ in $\mathbb{E}^2$ to be C^k if the function $a(\varphi)$ appearing in (3) is C^k — or, more precisely, if the periodic extension of $a(\varphi)$, given by $a(\varphi+\pi)=-a(\varphi)$, is C^k: it is clear that the differentiability class of $a(\varphi)$ is independent of the choice of reference frame. But, since $a(\varphi)=-t(\varphi)\sin\varphi$, we have:

Proposition 5. *$\mathcal{M}$ is of class C^k $(1 \le k \le \omega)$ if and only if the track function $t(\varphi)$ relative to any line is C^k and has bounded k^{th}-derivative on $(0,\pi)$, and has a C^{k-1} periodic extension of period π to $\mathbb{R}$. [If $k=\infty$ or $k=\omega$, then by $k-1$ we understand k.]*

This complicated wording now gives the *definition* for sets of lines in $\mathbb{H}^2$: we say $\mathcal{L}$ is C^k if its track function $x(\theta)$ relative to $l_0 \in \mathcal{L}$ has the properties just listed for $t(\varphi)$; our discussion shows this is independent of the choice of l_0. This definition, however, is not very practical, but this is easily remedied:

$\mathcal{L}$ is of class C^k if and only if the track function relative to each of its lines is C^k on $(0,\pi)$.

For proving the if part, we observe that then all track functions $t(\varphi)$ of $\mathcal{M}$ are C^k on $(0,\pi)$, and this implies that $a(\varphi)$ in (3) is C^k; *hence each* $t(\varphi)$ also satisfies the additional conditions set forth in Proposition 5, and therefore so does *each* $x(\theta)$.

3 – Existence of curves of constant width with given track function

We first carry out the details of the construction on the assumption that $x(\theta)$ is sufficiently smooth: thus $x(\theta)$ is at least C^2, and periodic of period π; and we assume, for a reason that will be clear later on, that $|x'(\theta)\sin\theta|$ is *bounded away from* 1.

We fix any line $l(x)$ in $\mathbb{H}^2$, where x is the arc-length, and let $l_\theta(\rho)$ be the line, again parameterized by arc-length, which starts at $l(x(\theta))$ and makes an angle θ with l: thus we have $l_{\theta+\pi}(\rho)=l_\theta(-\rho)$. Consider the mapping $\Psi(\rho,\theta)=l_\theta(\rho)$, and let $(\boldsymbol{u}_1,\boldsymbol{u}_2)$ be the positively oriented orthonormal moving frame defined by $\boldsymbol{u}_1=\frac{\partial\Psi}{\partial\rho}$. We define the coefficients λ_1, λ_2 by

$$\frac{\partial\Psi}{\partial\theta}=\lambda_1\,\boldsymbol{u}_1+\lambda_2\,\boldsymbol{u}_2\ ; \tag{4}$$

it is proved in [2, Lemma 8] that

$$\lambda_1(\rho,\theta) = x'(\theta)\cos\theta\,, \qquad \lambda_2(\rho,\theta) = -x'(\theta)\sin\theta\cosh\rho + \sinh\rho\ . \tag{5}$$

We now define *geodesic rectangular coordinates* $\Phi(u,v)$ based on l: for each u, the curve $v \mapsto \Phi(u,v)$ is the unit-speed geodesic that cuts l *orthogonally* at $l(u) = \Phi(u,0)$, in such a way that the angle from $l'(u)$ to $\frac{\partial\Phi}{\partial v}(u,0)$ is positive. Thus $\Phi(u,v)$ is a coordinate chart covering the whole of $\mathbb{H}^2$, and $(\frac{\partial\Phi}{\partial u}, \frac{\partial\Phi}{\partial v})$ is a positive orthogonal frame at each point.

Lemma 6. *For all $\rho \in \mathbb{R}$ and $0 < \theta < \pi$ (resp. $\pi < \theta < 2\pi$), we have*

$$\left\langle \boldsymbol{u}_2(\rho,\theta), \frac{\partial\Phi}{\partial u}\right\rangle < 0 \qquad (\textit{resp.}\ \left\langle \boldsymbol{u}_2(\rho,\theta), \frac{\partial\Phi}{\partial u}\right\rangle > 0)\ .$$

Proof: Since $\boldsymbol{u}_2(\rho,\theta+\pi) = -\boldsymbol{u}_2(-\rho,\theta)$, it suffices to prove the lemma for $0 < \theta < \pi$. It is clear that the desired inequality holds for $\rho = 0$; and we can never have $\langle \boldsymbol{u}_2(\rho,\theta), \frac{\partial\Phi}{\partial u}\rangle = 0$ when $0 < \theta < \pi$, since the line $\Psi(\cdot,\theta)$ can cut no line $\Phi(u,\cdot)$ at right angles. ∎

We now look for a curve of constant width $\gamma(\theta)$ in the form $\Psi(f(\theta),\theta)$, and such that the lines $\Psi(\cdot,\theta)$ are the (extended) diameters of γ: the track function of such a curve relative to the diameter $l(x)$ is obviously $x(\theta)$. Since $\gamma'(\theta)$ is to be orthogonal to $\Psi(\cdot,\theta)$, and hence collinear with $\boldsymbol{u}_2$, equations (4) and (5) give

$$f'(\theta) = -x'(\theta)\cos\theta\ , \tag{6}$$

$$\gamma'(\theta) = \Big\{-x'(\theta)\sin\theta\cosh(f(\theta)) + \sinh(f(\theta))\Big\}\,\boldsymbol{u}_2\ . \tag{7}$$

Put $f_\lambda(\theta) = \lambda - \int_0^\theta x'(\varphi)\cos\varphi\,d\varphi$ and $\gamma_\lambda(\theta) = \Psi(f_\lambda(\theta),\theta)$; notice that $f_\lambda(\theta) + f_\lambda(\theta+\pi)$ is constant, and γ_λ has period 2π. We now show that $\gamma_\lambda(\theta)$ has constant width for λ large enough:

Theorem 7. *If λ is such that*

$$f_\lambda(\theta) \geq 0 \quad \textit{and} \quad -x'(\theta)\sin\theta\cosh(f_\lambda(\theta)) + \sinh(f_\lambda(\theta)) \geq 0 \tag{8}$$

for all $\theta \in [0,2\pi]$, then $\gamma_\lambda(\theta)$ is a curve of constant width $\mathcal{W} = 2\lambda - \int_0^\pi x'(\theta)\cos\theta\,d\theta$.

A few remarks are in order. First, if both inequalities (8) degenerate for all $\theta \in [0,\pi]$ then $\lambda = 0$ and $x(\theta)$ is constant; hence γ_0 reduces to a point. In all other cases, when $x(\theta)$ is not constant and (8) holds, γ_λ is a curve of constant width in the proper sense. Second, it follows from (8) that

$$\Big|x'(\theta)\sin\theta\Big| \le \max\Big\{\tanh(f(\theta)),\,\tanh(f(\theta+\pi))\Big\} \le \tanh\mathcal{W}\ ,$$

which shows that there exists λ satisfying (8) if and only if $|x'(\theta)\sin\theta|$ is bounded away from 1. Finally, the set of λ's that satisfy (8) is an interval of the form $[\lambda_0,+\infty)$, with $\lambda_0 \ge 0$ (the second inequality may be rewritten as $\tanh(f_\lambda(\theta)) \ge x'(\theta)\sin\theta$, and tanh is an increasing function), and $(\gamma_\lambda)_{\lambda\ge\lambda_0}$ is a family of parallel curves.

Now we prove Theorem 7. We start by assuming that $\lambda > \lambda_0$: hence, both inequalities (8) are strict for all θ. We first prove that $\gamma_\lambda(\theta)$ is a simple closed curve: if $0 \le \theta_1 < \theta_2 < 2\pi$ then $\gamma(\theta_1) \ne \gamma(\theta_2)$. It is clear that if $0 < \theta_1 < \pi$ and $\pi < \theta_2 < 2\pi$ then $\gamma(\theta_1) \ne \gamma(\theta_2)$, since these points are in opposite sides of the line l. Hence it suffices to show that the restriction of γ to each of the intervals $[0,\pi]$ and $[\pi,2\pi]$ is injective. Put $\gamma(\theta) = \Phi(u(\theta),v(\theta))$: Lemma 6, together with (7) and (8), shows that $u'(\theta) < 0$ on $(0,\pi)$ and $u'(\theta) > 0$ on $(\pi,2\pi)$; therefore γ is indeed injective on $[0,\pi]$ and on $[\pi,2\pi]$. It remains to prove that γ_λ has constant width. Let $\overline{pq}$ be a diameter of γ_λ: then $\overline{pq}$ is a double normal of γ_λ (see [1, Claim 1], for instance), and it follows that $p = \gamma_\lambda(\theta_0)$, $q = \gamma_\lambda(\theta_0+\pi)$ for some θ_0; since

$$\Big|\gamma_\lambda(\theta)\,\gamma_\lambda(\theta+\pi)\Big| = f_\lambda(\theta) + f_\lambda(\theta+\pi) = 2\,\lambda - \int_0^\pi x'(\theta)\cos\theta\,d\theta = \mathcal{W}$$

for all θ, we see that γ_λ has constant width $\mathcal{W}$.

For $\lambda = \lambda_0$, we take a sequence $(\lambda_n)_{n\ge1}$ decreasing to λ_0, and notice that γ_{λ_0} is the *uniform* limit of $(\gamma_{\lambda_n})_{n\ge1}$: a straightforward argument shows that γ_{λ_0} has constant width $2\,\lambda_0 - \int_0^\pi x'(\theta)\cos\theta\,d\theta$, equal to the limit of the widths $2\,\lambda_n - \int_0^\pi x'(\theta)\cos\theta\,d\theta$ of the curves γ_{λ_n}. (We remark that, letting $\gamma_{\lambda_0}(\theta) = \Phi(u(\theta),v(\theta))$, the same argument as above shows that $u(\theta)$ is *non-increasing* on $[0,\pi]$ and *non-decreasing* on $[\pi,2\pi]$; from this and the fact that γ_{λ_0} has constant width we deduce that γ_{λ_0} is a simple curve, in the sense that if $\gamma_{\lambda_0}(\theta_1) = \gamma_{\lambda_0}(\theta_2)$ with $0 \le \theta_1 < \theta_2 < 2\pi$, then γ_{λ_0} constant on $[\theta_1,\theta_2]$.)

4 – The Lipschitz case

It follows from Proposition 4 that the track function $x(\theta)$ of a curve of constant width is, at the worst, L.L. on $(0,\pi)$; and, by Proposition 4, we can always assume

$x(\theta)$ has a continuous extension of period π to the whole real line. In this section we prove the converse: any function satisfying these conditions, and such that $|x'(\theta)\sin\theta|$ is bounded away from 1, is the track function of some curve of constant width.

We consider the mapping $\Psi(\rho,\theta)$ defined as in Section 3. Explicitly, we have

$$\Psi(\rho,\theta) = \exp_{l(x(\theta))}(\rho\,\boldsymbol{u}(\theta)) \ , \tag{9}$$

where $\boldsymbol{u}(\theta)$ is the vector making an angle θ with l. From (9) we see that, whenever $x'(\theta)$ exists, $\frac{\partial\Psi}{\partial\theta}(\rho,\theta)$ exists for all ρ. Defining $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, λ_1 and λ_2 as in Section 3, we claim that formulas (5) still hold (almost everywhere). We fix an interval $[\theta_0,\pi-\theta_0]$ with $0<\theta_0<\frac{\pi}{2}$: on this interval $x(\theta)$ is uniformly Lipschitz, and therefore $x'(\theta)$ is bounded and belongs to $L^1([\theta_0,\pi-\theta_0])$. We take a C^2 sequence $(y_n)_{n\geq 1}$ defined on $[\theta_0,\pi-\theta_0]$, converging in L^1-norm to $x'(\theta)$: by taking a subsequence, we may assume that, for almost all θ, $(y_n(\theta))_{n\geq 1}$ converges to $x'(\theta)$. Define

$$x_n(\theta) = x(\theta_0) + \int_{\theta_0}^{\theta} y_n(\varphi)\,d\varphi\,, \qquad \Psi_n(\rho,\theta) = \exp_{l(x_n(\theta))}(\rho\,\boldsymbol{u}(\theta)) \ ; \tag{10}$$

then $(x_n)_{n\geq 1}$ converges uniformly to x, and $(\Psi_n)_{n\geq 1}$ converges uniformly to Ψ when ρ is restricted to some bounded interval. For each θ such that $y_n(\theta)\to x'(\theta)$, and for all ρ, we see that $\frac{\partial\Psi_n}{\partial\theta}(\rho,\theta)\to\frac{\partial\Psi}{\partial\theta}(\rho,\theta)$; since formulas (5) hold for Ψ_n, it follows that they also hold for Ψ at each such θ — that is, at almost all $\theta\in[\theta_0,\pi-\theta_0]$. Since θ_0 is arbitrary, this proves (5) for a.e. $\theta\in[0,\pi]$, and by periodicity for a.e. $\theta\in\mathbb{R}$.

With the same notation as before, let $f_\lambda(\theta)=\lambda-\int_0^\theta x'(\varphi)\cos\varphi\,d\varphi$ and $\gamma_\lambda(\theta)=\Psi(f_\lambda(\theta),\theta)$: then $f_\lambda(\theta)$ is L.L. on $(0,\pi)$ and $f_\lambda(\theta)+f_\lambda(\theta+\pi)$ is constant. We now prove Theorem 7 in full generality:

Theorem 8. *If λ is such that*

$$\begin{aligned} &f_\lambda(\theta)\geq 0 \ \ \textit{for all } \theta \quad \textit{and} \\ &-x'(\theta)\sin\theta\cosh(f_\lambda(\theta))+\sinh(f_\lambda(\theta))\geq 0 \ \ \textit{for a.e. } \theta \ , \end{aligned} \tag{11}$$

then $\gamma_\lambda(\theta)$ is a curve of constant width $\mathcal{W}=2\lambda-\int_0^\pi x'(\theta)\cos\theta\,d\theta$.

Proof: Inequalities (11) hold for a set of a parameters $[\lambda_0,+\infty]$; and, as before, it suffices to prove the theorem for $\lambda>\lambda_0$. Thus there exists $\epsilon(\lambda)>0$ such that

$$\begin{aligned} &f_\lambda(\theta)\geq\epsilon(\lambda) \quad \text{for all } \ \theta, \quad \text{and} \\ &\tau(\theta,\lambda) := -x'(\theta)\sin\theta\cosh(f_\lambda(\theta))+\sinh(f_\lambda(\theta))\geq\epsilon(\lambda) \quad \text{for a.e. } \ \theta \ . \end{aligned} \tag{12}$$

Now the argument used in the proof of Theorem 7 also shows that $\gamma_\lambda(\theta)$ is a simple closed curve, and it follows from (5) that $\gamma'_\lambda(\theta)$ is orthogonal to the line $\Psi(\cdot,\theta)$ for a.e. $\theta \in \mathbb{R}$. We shall now prove that in fact $\Psi(\cdot,\theta)$ cuts γ_λ orthogonally *for all* θ: this proves our theorem, since the proof can then be finished as that of Theorem 7.

For convenience, we prove slightly more than what is needed: namely, that if the second inequality (12) holds in some interval $[\theta_1,\theta_2]$, then $\Psi(\cdot,\theta)$ cuts $\gamma_\lambda\big|_{[\theta_1,\theta_2]}$ orthogonally for all $\theta \in [\theta_1,\theta_2]$. Since γ_λ is L.L., its arc-length is given by

$$\mathcal{S}(\theta) = \int_{\theta_1}^{\theta} |\gamma'_\lambda(\varphi)|\, d\varphi = \int_{\theta_1}^{\theta} \tau(\varphi,\lambda)\, d\varphi \ ,$$

and, since $\tau(\theta,\lambda)$ is bounded above by some constant $k(\lambda)$, we have by (12) that

$$\epsilon(\lambda)\, |\theta-\theta'| \le |\mathcal{S}(\theta) - \mathcal{S}(\theta')| \le k(\lambda)\, |\theta - \theta'| \ .$$

Hence both $\mathcal{S}$ and $\mathcal{S}^{-1}$ are Lipschitz. Letting $\alpha_\lambda(s) = \gamma_\lambda(\mathcal{S}^{-1}(s))$ be the parameterization of γ_λ by arc-length, it follows that, letting $\theta = \mathcal{S}^{-1}(s)$, we have $\alpha'_\lambda(s) = \boldsymbol{u}_2(f_\lambda(\theta),\theta)$ for a.e. s ($\boldsymbol{u}_2$ is as in Section 3). Thus we have the following situation: $\alpha_\lambda(s)$, $s \in [0,s_0]$, is a Lipschitz curve, and there is a set $R \subseteq [0,s_0]$ of measure zero such that, for every $s \in [0,s_0]$, there exists

$$\lim_{t\to s;\, t\notin R} \alpha'_\lambda(t) \ . \tag{13}$$

We claim that then $\alpha'_\lambda(s)$ exists for every s and is given by the above limit. Since in our case the limit (13) is $\boldsymbol{u}_2$, we see that $\Psi(\cdot,\theta)$ is orthogonal to γ_λ, as we wished to prove. It clearly suffices to prove the claim for curves in $\mathbb{E}^2$; and, by considering its component functions, the claim follows directly from the lemma below, which also concludes the proof of Theorem 8.

Lemma 9. *Let $g\colon [a,b] \to \mathbb{R}$ be an absolutely continuous function. If there exists a null set $R \subseteq [a,b]$ such that $\lim_{t\to s; t\notin R} g'(t)$ exists for each s, then $g'(s)$ exists for all s and is given by the above limit.*

Proof: Since, for $t \neq s$, we have $g(t) - g(s) = \int_s^t g'(u)\, du = \int_{[s,t]\setminus R} g'(u)\, du$, it follows that

$$\inf_{u\in[s,t]\setminus R} g'(u) \le \frac{g(t)-g(s)}{t-s} \le \sup_{u\in[s,t]\setminus R} g'(u) \ ,$$

from which the lemma is obvious. ∎

An important question is whether Theorem 8 describes all curves of constant width. The (affirmative) answer is given by our next theorem:

Theorem 10. *Let γ be a curve of constant width, and let $x(\theta)$ and $f(\theta)$ be its track and intersection functions relative to some fixed diameter l_0 (chosen so that $x(\theta)$ has a continuous periodic extension to $\mathbb{R}$). Then $x(\theta)$ and $f(\theta)$ are L.L. on $\mathbb{R}\backslash\{n\,\pi\colon\ n\in\mathbb{Z}\}$, and there exists some $\lambda\in\mathbb{R}$ such that*

$$f(\theta) = f_\lambda(\theta) := \lambda - \int_0^\theta x'(\phi)\cos\phi\, d\phi\ . \tag{14}$$

Proof: Only the last assertion needs proof. By substituting an exterior parallel for γ if necessary, we may assume that all intersections of distinct diameters are *inside* γ and at a distance at least $\delta>0$ from it. For $0<\delta'\leq\delta$, let $U(\gamma,\delta')$ be the open set containing γ and its exterior, and also the points inside γ swept by the (half-open) line segments of length δ' and orthogonal to γ. Each point of $U(\gamma,\delta)$ is of the form $\Psi(\rho,\theta)$ for a unique (ρ,θ) with $\rho>0$ and $-\pi<\theta\leq\pi$; hence the vector field $\boldsymbol{u}_2(\rho,\theta)$, with $\rho>0$, is well-defined on $U(\gamma,\delta)$; and, since $x(\theta)$ is L.L. on $(0,\pi)$, $\boldsymbol{u}_2$ is L.L. on $U(\gamma,\delta)\backslash l_0$. By the proof of Theorem 8 each arc of $\gamma_\lambda(\theta)$ inside $U(\gamma,\delta)$ is a trajectory of $\boldsymbol{u}_2$ provided that, for each $0<\delta'<\delta$, $\epsilon(\delta')>0$, there exists $\epsilon(\delta')>0$ such that

$$\tau(\theta,\lambda)\geq\epsilon(\delta') \tag{15}$$

whenever $\gamma_\lambda(\theta)\in U(\gamma,\delta')$. Assuming this is so, it then follows by the uniqueness of trajectories of $\boldsymbol{u}_2$ through each point of $U(\gamma,\delta)\backslash l_0$ that there exist λ_1 and λ_2 such that $\gamma(\theta)=\gamma_{\lambda_1}(\theta)$ for $0<\theta<\pi$ and $\gamma(\theta)=\gamma_{\lambda_2}(\theta)$ for $-\pi<\theta<0$; and of course $\lambda_1=\lambda_2$ for otherwise the two portions of γ would not fit together. This proves Theorem 10, subject to the proof below.

Proof of (15): The inequality $\tau(\theta,\lambda)\geq 0$ is equivalent to $\tanh(f_\lambda(\theta))\geq x'(\theta)\sin\theta$; and, since $\lambda\mapsto\tanh(f_\lambda(\theta))$ is strictly increasing by a rate independent of θ, it follows that if $\tau(\theta,\lambda)$ were not bounded away from zero on $U(\gamma,\delta')$ for $\delta'<\delta$, then it would assume negative values at some point in $U(\gamma,\delta)$. Thus we only have to prove that $\tau(\theta,\lambda)<0$ is impossible when $\gamma_\lambda(\theta)\in U(\gamma,\delta)$. Assume this is not so, and that $\tau(\theta_0,\lambda)<0$ for some $0<\theta_0<\pi$ (the case $-\pi<\theta_0<0$ is similar). Then, for some $\epsilon>0$ and all $\theta_0<\theta<\theta_0+\epsilon$, the point $\gamma_\lambda(\theta)$ and the half-line $l_0(x)$, $x>x(\theta_0)$, are on the same side of the line $\Psi(\cdot,\theta_0)$, and it follows that $x(\theta)>x(\theta_0)$; hence the segment $\Psi(\rho,\theta)$, $0\leq\rho\leq f_\lambda(\theta)$, does not intersect $\Psi(\cdot,\theta_0)$, and neither does, for obvious geometric reasons, the half-line $\Psi(\rho,\theta)$,

$\rho < 0$. Thus the two lines must intersect at a point $\Psi(\rho,\theta)$ with $\rho > f_\lambda(\theta)$ — that is, at a point in $U(\gamma,\delta)$, contradicting the definition of this set. ∎

Theorem 11. *Every curve of constant width can be uniformly approximated by analytic curves of constant width.*

Proof: Let $x(\theta)$ be the track function of the given curve, which we assume to be of the form γ_λ with $\lambda > \lambda_0$. We have to show that for each $\epsilon > 0$ we can find an analytic function $\widetilde{x}(\theta)$, with associated curve of constant width $\widetilde{\gamma}_\lambda(\theta) = \exp_{l(\widetilde{x}(\theta))}(\widetilde{f}_\lambda(\theta)\,\boldsymbol{u}(\theta))$ [where $\widetilde{f}_\lambda(\theta) = \lambda - \int_0^\theta \widetilde{x}'(\varphi)\cos\varphi\,d\varphi$], such that $|\gamma_\lambda(\theta)\,\widetilde{\gamma}_\lambda(\theta)| < \epsilon$ for all θ.

In each step of the proof we approximate $x(\theta)$ by a better behaved $\widetilde{x}(\theta)$ so that $\widetilde{\gamma}_\lambda$ is close to γ_λ. We call $\widetilde{x}(\theta)$ a *good approximation* of $x(\theta)$ if it satisfies, *for the same* λ, both (strict) inequalities (11), with $\widetilde{f}_\lambda(\theta)$ replacing $f_\lambda(\theta)$; this ensures $\widetilde{\gamma}_\lambda$ also has constant width. Since in general $\widetilde{x}'(\theta)$ is *not* uniformly close to $x'(\theta)$, some care is needed to obtain good approximations.

Step 1: *there are good approximations $\widetilde{x}(\theta)$ of $x(\theta)$ which are (uniformly) Lipschitz.*

Since $\lambda > \lambda_0$, there exists $\eta > 0$ such that

$$\tanh(f_\lambda(\theta)) \geq \eta \quad \text{for all } \theta, \quad \text{and} \tag{16}$$

$$\tanh(f_\lambda(\theta)) \geq x'(\theta)\sin\theta + \eta \quad \text{for a.e. } \theta\ . \tag{17}$$

We now take a small $\theta_0 > 0$ (to be specified later), let $\alpha = \frac{x(\theta_0)-x(\pi-\theta_0)}{\pi}$ and define

$$\begin{cases} \widetilde{x}'(\theta) = \alpha & \text{if } 0 \leq \theta \leq \theta_0 \text{ or } \pi-\theta_0 \leq \theta \leq \pi\ , \\ \widetilde{x}'(\theta) = x'(\theta) + \alpha & \text{if } \theta_0 < \theta < \pi - \theta_0\ ; \end{cases}$$

and also put $\widetilde{x}'(\theta+\pi) = \widetilde{x}'(\theta)$. [Notice that $\widetilde{x}'(\theta)$ is only defined almost everywhere.] The number α is chosen so that $\int_0^\pi \widetilde{x}'(\theta)\,d\theta = 0$; hence $\widetilde{x}(\theta) = x(0) + \int_0^\theta \widetilde{x}'(\varphi)\,d\varphi$ is periodic of period π; and, since $x(\theta)$ is uniformly Lipschitz on $[\theta_0, \pi-\theta_0]$, it follows that $\widetilde{x}(\theta)$ is uniformly Lipschitz on $[0,\pi]$. From the definitions of $f_\lambda(\theta)$ and $\widetilde{f}_\lambda(\theta)$ we can check that

$$|\widetilde{f}_\lambda(\theta) - f_\lambda(\theta)| \leq \max_{-\theta_0\leq\varphi\leq\theta_0} |f_\lambda(\varphi) - f_\lambda(0)| + \max_{\pi-\theta_0\leq\varphi\leq\pi+\theta_0} |f_\lambda(\varphi) - f_\lambda(\pi)| + |\alpha|\ ;$$

and also

$$|\widetilde{x}(\theta) - x(\theta)| \leq \max_{-\theta_0\leq\varphi\leq\theta_0} |x(\varphi) - x(0)| + \max_{\pi-\theta_0\leq\varphi\leq\pi+\theta_0} |x(\varphi) - x(\pi)| + |\alpha\,\pi|\ .$$

From these inequalities we see that we can choose θ_0 so that $\widetilde{\gamma}_\lambda(\theta)$ is as close to $\gamma_\lambda(\theta)$ as we wish. We may also choose $|\widetilde{f}_\lambda(\theta) - f_\lambda(\theta)|$ and α to be so small that

$$\tanh(\widetilde{f}_\lambda(\theta)) \geq \tanh(f_\lambda(\theta)) - \frac{\eta}{4} \quad \text{and} \quad |\alpha \sin\theta| \leq \frac{\eta}{4} \quad \text{for all} \quad \theta \ :$$

with this choices we check that inequalities (16) and (17) hold when we replace $f_\lambda(\theta)$, $x'(\theta)$ and η by $\widetilde{f}_\lambda(\theta)$, $\widetilde{x}'(\theta)$ and $\frac{\eta}{2}$ respectively. This ensures that both inequalities (11) hold strictly for $\widetilde{x}(\theta)$ and this λ (and also for parameter values slightly smaller than λ), and therefore $\widetilde{x}(\theta)$ is a good approximation of $x(\theta)$.

Step 2: *there are good approximations of $x(\theta)$ which are piecewise linear.*

By Step 1, we may assume that $x(\theta)$ is Lipschitz; hence there exists K such that $|x'(\theta)| \leq K$ for all θ. We may also assume that inequalities (16) and (17) hold. We choose $\delta > 0$ so that

$$\tanh(f_\lambda(\theta)) \geq \tanh(f_\lambda(\phi)) - \frac{\eta}{4} \quad \text{and} \quad K|\sin\theta - \sin\phi| \leq \frac{\eta}{4}$$

whenever $|\theta - \phi| \leq \delta$. Then, for every interval J with length $\leq \delta$, we have, for all $\theta, \phi \in J$,

$$\tanh(f_\lambda(\theta)) \geq x'(\phi)\sin\theta + \frac{\eta}{2} \ ,$$

and therefore, letting $a_J = \inf_{\phi\in J} x'(\phi)$ and $b_J = \sup_{\phi\in J} x'(\phi)$,

$$\tanh(f_\lambda(\theta)) \geq y\sin\theta + \frac{\eta}{2} \quad \text{for all} \quad \theta \in J \quad \text{and} \quad y \in [a_J, b_J] \ . \tag{18}$$

Choose a step function $y(\theta)$ such that $y(\theta) = y(\theta+\pi)$ and $\int_0^\pi |y - x'|\, d\theta \leq \epsilon$, where $\epsilon > 0$ is to be specified later. We consider a partition $0 = \theta_0 < \theta_1 < \dots < \theta_k = \pi$ of $[0,\pi]$ with diameter less than δ and such that $y(\theta)$ is constant, equal to y_i, on each interval $J_i = [\theta_{i-1}, \theta_i)$. We may assume that $y_i \in [a_{J_i}, b_{J_i}]$, for if not we replace y_i by either of a_{J_i} or b_{J_i}, whichever is closest to y_i: this can only decrease the value of $\int_0^\pi |y - x'|\, d\theta$. It then follows from (18) that

$$\tanh(f_\lambda(\theta)) \geq y(\theta)\sin\theta + \frac{\eta}{2} \quad \text{for all} \quad \theta \in \mathbb{R} \ . \tag{19}$$

We let $\alpha = -\frac{1}{\pi}\int_0^\pi y\, d\theta$: then we have $|\alpha| \leq \frac{\epsilon}{\pi}$, and define $\widetilde{x}(\theta) = x(0) + \int_0^\theta \{y(\phi) + \alpha\}\, d\phi$; thus $\widetilde{x}(\theta)$ is piecewise linear, periodic of period π, and we have the following estimates:

$$|\widetilde{x}'(\theta) - y(\theta)| = |\alpha| < \epsilon \ , \tag{20}$$

$$
\begin{aligned}
|\widetilde{f}_\lambda(\theta) - f_\lambda(\theta)| &= \Big|\int_0^\theta \{\widetilde{x}'(\phi) - x'(\phi)\}\cos\phi\, d\phi\Big| \\
&\le \Big|\int_0^\theta \{y(\phi) - x'(\phi)\}\cos\phi\, d\phi\Big| + \Big|\int_0^\theta \alpha\cos\phi\, d\phi\Big| \\
&\le \epsilon + |\alpha| < 2\,\epsilon\ ,
\end{aligned} \tag{21}
$$

$$
|\widetilde{x}(\theta) - x(\theta)| \le 2\,\epsilon\ . \tag{22}
$$

Since ϵ is arbitrarily small, it follows that $\widetilde{\gamma}_\lambda(\theta)$ can be arbitrarily close to $\gamma_\lambda(\theta)$. It also follows from (19), (20) and (21) that we may choose ϵ so small that inequalities (16) and (17) hold when we replace $f_\lambda(\theta)$, $x'(\theta)$ and η by $\widetilde{f}_\lambda(\theta)$, $\widetilde{x}'(\theta)$ and $\frac{\eta}{4}$, respectively. This completes Step 2.

Step 3: *there are good approximations of $x(\theta)$ which are C^1.*

By Step 2, we may assume that $x(\theta)$ is piecewise linear and that (16) and (17) hold. Given $\epsilon > 0$, we find $y(\theta)$ *continuous*, periodic of period π, and such that $\int_0^\pi |y - x'|\, d\theta \le \epsilon$; furthermore, we require that, for each θ, $y(\theta)$ and $x'(\theta)$ be of the same sign and $|y(\theta)| \le |x'(\theta)|$. It follows easily that (17) still holds when $x'(\theta)$ is replaced by $y(\theta)$: this means we have

$$
\tanh(f_\lambda(\theta)) \ge y(\theta)\sin\theta + \eta \quad \text{for all } \ \theta\ . \tag{23}
$$

Now, just like in Step 2, we define $\widetilde{x}(\theta) = x(0) + \int_0^\theta \{y(\phi) + \alpha\}\, d\phi$ so that $\widetilde{x}$ is periodic and $|\alpha| \le \frac{\epsilon}{\pi}$. The estimates (20)–(22) still hold; and therefore, in view of (23), we may choose ϵ so that $\widetilde{x}(\theta)$ satisfies (16) and (17) with $\frac{\eta}{2}$ substituted for η. This completes Step 3.

Step 4: *there are good approximations of $x(\theta)$ which are analytic.*

We assume, as we may by Step 3, that $x(\theta)$ is C^1; therefore we can find an *analytic* periodic function $y(\theta)$ (a trigonometric polynomial, say) such that $\max_{0\le\theta\le\pi} |y(\theta) - x'(\theta)| \le \epsilon$. Then, letting as usual $\widetilde{x}(\theta) = x(0) + \int_0^\theta \{y(\phi) + \alpha\}\, d\phi$, the functions f_λ, x and x' are *uniformly* close to $\widetilde{f}_\lambda$, $\widetilde{x}$ and $\widetilde{x}'$, respectively, which ensures that (16) and (17) hold for the latter functions (with $\frac{\eta}{2}$ instead of η) when ϵ is small enough. This completes Step 4 and the proof of Theorem 11. ∎

5 – Applications

The representation given in Theorem 10, combined with Theorem 11, allows us to extend results previously known only for differentiable curves to all curves of constant width. In this section we give some examples of this sort of extension.

We consider an arbitrary curve $\gamma(\theta)$ of constant width with associated functions $x(\theta)$ and $f(\theta)$. We have

$$|\gamma'(\theta)| = -x'(\theta)\sin\theta\cosh(f(\theta)) + \sinh(f(\theta)) \quad \text{for a.e. } \theta \; ; \tag{24}$$

and, since $\gamma(\theta)$ is L.L., its length is $\mathfrak{L}(\gamma) = \int_0^{2\pi} |\gamma'(\theta)|\, d\theta$. It then follows from (24) and the proof of Theorem 11 that γ is the uniform limit of a sequence γ_n of analytic curves such that $\mathfrak{L}(\gamma_n) \to \mathfrak{L}(\gamma)$. (This is obvious if $x'(\theta)$ is bounded, for then we find analytic functions x_n such that $x_n' \to x'$ in $L^1([0,\pi])$; otherwise we check that in Step 1 of Theorem 11, where we construct good approximations $\widetilde{x}$ of x with bounded derivative, the difference $|\mathfrak{L}(\gamma) - \mathfrak{L}(\widetilde{\gamma})|$ can be made as small as we like.) It is also clear that, denoting by $A(\gamma_n)$ the area of the region bounded by γ, we have $A(\gamma_n) \to A(\gamma)$, and that the widths of γ_n converge to the width $\mathcal{W}$ of γ. From [1, Theorem B] we then obtain, this time without any restrictive assumptions:

Theorem 12. *If γ is a curve of constant width $\mathcal{W}$ in $\mathbb{H}^2$ having perimeter $\mathfrak{L}$ and enclosing a region of area A, then $\mathfrak{L} = \tanh(\frac{\mathcal{W}}{2})\,(2\pi + A)$.*

The differentiable version of Theorem 12 is the essential tool for proving the main result in [2]: *the Reuleaux triangle encloses a smaller area than any other (at least piecewise C^3) curve in $\mathbb{H}^2$ of the same constant width.* Now that we have Theorem 10, this result can easily be extended to include all curves of constant width: the proof in [2] goes through with almost no changes; but we will not go into the details.

We finish this article with a result whose differentiable version was also known in part, but which has the additional interest of requiring an entirely different proof.

Theorem 13. *If γ is a curve of constant width in $\mathbb{H}^2$, then the two following conditions are equivalent:*

i) *every diameter of γ bisects the area enclosed by γ;*

ii) *every diameter of γ bisects the perimeter of γ.*

Furthermore, each of these conditions implies that γ is a circle.

The analogous result for the Euclidean plane is known (see [6]). Also, we have established in [1, Theorem D] that if ii) is verified (and if γ is sufficiently differentiable) then γ is a circle. Otherwise, Theorem 13 seems to be new.

Proof: 1^{st} *part*: ii) $\Rightarrow \gamma$ *is a circle*.

Our hypothesis says that the length $\mathfrak{L}(\theta)$ of $\gamma\big|_{[\theta,\theta+\pi]}$ is constant, equal to $\frac{\mathfrak{L}}{2}$. By differentiating $\mathfrak{L}(\theta) = \int_\theta^{\theta+\pi} |\gamma'(\varphi)|\, d\varphi$, we obtain, using (24) and Lemma 1,

$$-x'(\theta)\sin\theta\cosh(f(\theta)) + \sinh(f(\theta)) = x'(\theta)\sin\theta\cosh(\mathcal{W} - f(\theta)) + \sinh(\mathcal{W} - f(\theta))$$

for a.e. θ. This can be rewritten as

$$x'(\theta)\sin\theta = \frac{\sinh(f(\theta)) - \sinh(\mathcal{W} - f(\theta))}{\cosh(f(\theta)) + \cosh(\mathcal{W} - f(\theta))} = \frac{\sinh\Big(f(\theta) - \dfrac{\mathcal{W}}{2}\Big)}{\cosh\Big(f(\theta) - \dfrac{\mathcal{W}}{2}\Big)} \ ;$$

multiplying both sides by $(f(\theta) - \frac{\mathcal{W}}{2})\cos\theta$, we obtain, using (14),

$$f'(\theta)\cosh\left(f(\theta) - \frac{\mathcal{W}}{2}\right)\sin\theta + \sinh\left(f(\theta) - \frac{\mathcal{W}}{2}\right)\cos\theta = 0 \ ,$$

which means that the derivative of the L.L. function $G(\theta) = \sinh(f(\theta) - \frac{\mathcal{W}}{2})\sin\theta$ vanishes almost everywhere, and therefore $G(\theta)$ is constant. Such a constant can only be zero, and therefore $f(\theta) = \frac{\mathcal{W}}{2}$ for all θ, which means that γ is a circle.

2^{nd} *part*: i)$\Leftrightarrow$ii).

Let $A(\theta)$ be the area bounded by $\gamma\big|_{[\theta,\theta+\pi]}$ and by the diameter $\overline{\gamma(\theta)\,\gamma(\theta+\pi)}$. We have to prove that $\mathfrak{L}(\theta)$ is constant if and only if $A(\theta)$ is constant; this follows at once from the formula

$$A(\theta) = \tanh\left(\frac{\mathcal{W}}{2}\right)\mathfrak{L}(\theta) + \left\{\frac{\mathfrak{L}}{\sinh\mathcal{W}} - \pi\right\} . \tag{25}$$

Since, for each θ, both sides of (25) behave well under (uniform) limits, it is enough to prove the formula for differentiable curves: thus we parameterize γ by the arc-length s, and assume $\gamma(s)$ is at least C^3. The point diametrically opposite to $\gamma(s)$ is given by $\gamma(h(s))$, where $h : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism satisfying $h(s + \mathfrak{L}) = h(s) + \mathfrak{L}$ and $s < h(s) < s + \mathfrak{L}$; and, by [1, (16)], we have

$$h'(s) = k_{\mathrm{g}}(s)\sinh\mathcal{W} - \cosh\mathcal{W} \ , \tag{26}$$

where $k_{\mathrm{g}}(s)$ is the geodesic curvature of γ at $\gamma(s)$. Now let $A(s) = A(\theta(s))$. Using the Gauss–Bonnet Theorem and (26), we have

$$\begin{aligned}A(s) &= \int_s^{h(s)} k_{\rm g}(t)\,dt - \pi\\ &= \frac{1}{\sinh\mathcal{W}}\Big\{\int_s^{h(s)} h'(t)\,dt\Big\} + \frac{\cosh\mathcal{W}}{\sinh\mathcal{W}}(h(s)-s) - \pi\\ &= \frac{1}{\sinh\mathcal{W}}\Big\{h(h(s)) - h(s)\Big\} + \frac{\cosh\mathcal{W}}{\sinh\mathcal{W}}(h(s)-s) - \pi\\ &= \frac{\cosh\mathcal{W}-1}{\sinh\mathcal{W}}(h(s)-s) + \left\{\frac{\mathfrak{L}}{\sinh\mathcal{W}} - \pi\right\}\ ;\end{aligned}$$

and, since $h(s) - s = \mathfrak{L}(\theta(s))$, this proves (25). ∎

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