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COMPLEMENTS IN MODULAR AND SEMIMODULAR LATTICES

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Abstract: We study the relations between the complements of a and b when a is covered by b on finite upper-semimodular lattices and when $a<b$ on modular lattices. We give some results that generalize the well known properties of complements in distibutive lattices. From there, we derive a property of semisimple R-modules.

1 – Introduction

In this paper we will only consider lattices that have a least element, denoted by 0, and a greatest element, denoted by 1. Given a lattice L and $a \in L$ we say that $a' \in L$ is a complement of a if $a \wedge a' = 0$ and $a \vee a' = 1$, and we denote the set of complements of a by C_a.

We write $a \prec b$ when b covers a. We recall that a lattice is upper-semimodular if $a \wedge b \prec a \Rightarrow b \prec a \vee b$, $\forall\, a, b \in L$. Following M. Stern [4] we refer to these lattices as semimodular.

Let L be a lattice. Consider a pair $(a,b) \in L^2$ such that $a<b$, $C_a \neq \emptyset$ and $C_b \neq \emptyset$. If L is distributive then $C_a = \{a'\}$, $C_b = \{b'\}$ and $a<b \Leftrightarrow b'<a'$. This property can be generalized in a number of ways:

$P_1 :\ \exists\, (a',b') \in C_a \times C_b\colon\ b'<a'$ $\qquad Q_1 :\ \exists\, (a',b') \in C_a \times C_b\colon\ b' \le a'$

$P_2 :\ \forall\, b' \in C_b,\ \exists\, a' \in C_a\colon\ b'<a'$ $\qquad Q_2 :\ \forall\, b' \in C_b,\ \exists\, a' \in C_a\colon\ b' \le a'$

$P_3 :\ \forall\, a' \in C_a,\ \exists\, b' \in C_b\colon\ b'<a'$ $\qquad Q_3 :\ \forall\, a' \in C_a,\ \exists\, b' \in C_b\colon\ b' \le a'$

We say that a lattice L satisfies P_i, respectively Q_i if every pair $(a,b) \in L^2$ with $a<b$, $\mathrm{C}_a \neq \emptyset$ and $\mathrm{C}_b \neq \emptyset$ satisfies P_i, respectively Q_i.

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If L is a finite modular lattice, we note that Q_2 is a restriction of the well known order between the ideals of a finite poset (see also P. Ribenboim [2]).

Given a pair $(a, b) \in L^2$ such that $a < b$, $C_a \neq \emptyset$ and $C_b \neq \emptyset$, the implications $P_2 \Rightarrow P_1$, $P_3 \Rightarrow P_1$, $Q_2 \Rightarrow Q_1$, $Q_3 \Rightarrow Q_1$ and $P_i \Rightarrow Q_i$, $i = 1, 2, 3$ are valid, for this pair. As they are valid for every pair, they are also valid for lattices. These implications are ilustrated in the next picture.

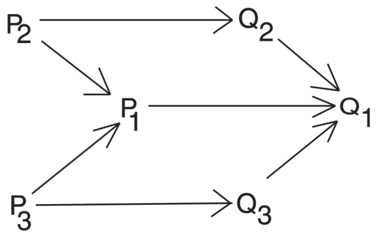

Fig. 1

There are finite complemented lattices where not even Q_1 is satisfied. Here is an example:

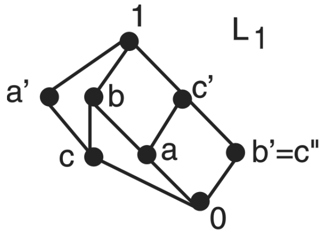

Fig. 2

L_1 is a complemented finite lattice and satisfies the Jordan–Dedekind chain condition. We note that L_1 is not semimodular.

Let us now consider the following lattice:

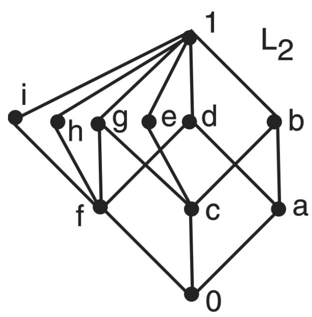

Fig. 3

L_2 is a finite, complemented lattice which is also semimodular. The pair (a, b) satisfies neither Q_3 nor P_2. However, every pair satisfies Q_2. We will prove that finite semimodular complemented lattices, in particular partition lattices, satisfy Q_2, and that all modular lattices satisfy P_2 and P_3. We will also prove that modular complemented lattices satisfy the following property:

$$P: \quad \forall\, a \neq b \in L, \quad \text{if } C_a \neq \emptyset \text{ and } C_b \neq \emptyset \quad \text{then } C_a \neq C_b\,.$$

The lattice L_2 does not satisfy P. In fact $i \neq h$ and $C_i = C_h = \{e, c, a, b\}$.

It is easy to see that the properties P_i, Q_i, $i = 1, 2, 3$, and P are preserved under direct products, that is, if, $\forall\, i \in I$, L_i satisfies one of these properties, then $\prod_{i \in I} L_i$ also satisfies that property.

2 – Finite semimodular lattices

We start with more general results concerning semimodular lattices.

Lemma 1. *Let L be a finite semimodular lattice, $a, b \in L$, $a \prec b$, $b' \in C_b \backslash C_a$.*

i) *$(a \vee b') \wedge b = a$ and $a \vee b'$ is a co-atom.*

ii) *If there exists $c \in L$ such that $b' \prec c$ and $a \vee c = 1$ then $c \in C_a$.*

iii) *If there exists an atom $a_1 \in L$ such that $a_1 \vee (a \vee b') = 1$ then, $a_1 \vee b'$ is a complement of a and $b' \prec a_1 \vee b'$.*

Proof: Let L be a finite semimodular lattice and $a, b \in L$, $a \prec b$, and let $b' \in C_b \backslash C_a$.

i) We have $a \wedge b' = 0$ and $b' \notin C_a$ therefore $a \vee b' < 1$. Since $(a \vee b') \vee b = 1$ we have $(a \vee b') \wedge b < b$. From $a \leq (a \vee b') \wedge b < b$ and $a \prec b$ we conclude $a = (a \vee b') \wedge b$. As L is semimodular $a = (a \vee b') \wedge b \prec b$ implies $a \vee b' \prec a \vee b' \vee b = 1$.

ii) If $a \wedge c > 0$ then there is a_1 such that $0 \prec a_1 \leq a \wedge c$. Also $a_1 \leq a < b$ so $a_1 \wedge b' = 0$ and as $0 \prec a_1$ then $b' \prec a_1 \vee b'$. On the other hand $a_1 < c$ and $b' \prec c$ imply $a_1 \vee b' \leq c$ so $c = a_1 \vee b'$. We conclude $a \vee b' = (a \vee a_1) \vee b' = a \vee (a_1 \vee b') = a \vee c = 1$. But from $a < b$ and $b \wedge b' = 0$ we get $a \wedge b' = 0$ so $b' \in C_a$, which is a contradiction.

iii) We have $a \wedge b' = 0$ and $b' \notin C_a$ so $a \vee b' < 1$. From $a_1 \vee (a \vee b') = 1$ we have $a_1 \not\leq a \vee b'$, and so $a_1 \wedge (a \vee b') = 0$. This implies $a_1 \wedge b' = 0$ so, by the semimodular property, $b' \prec a_1 \vee b'$. Now by using ii) we conclude $a_1 \vee b' \in C_a$. ∎

Theorem 2. *Let L be a finite semimodular lattice, $a, b \in L$ such that $a \prec b$ and $C_b \neq \emptyset$. If $C_a \not\subseteq C_b$ then $\forall\, b' \in C_b \backslash C_a$, $\exists\, c \in C_a\colon\, b' \prec c$.*

Proof: Let L be as stated, and let $a, b \in L$, $a \prec b$ and $a' \in C_a \backslash C_b$. Let $b' \in C_b \backslash C_a$. We have $a' \vee b = 1$ and $a' \notin C_b$ so $0 < a' \wedge b$. Let a_1 be an atom such that $a_1 \leq a' \wedge b$. From $a \wedge a' = 0$ and $a_1 \leq a'$ we have $a \wedge a_1 = 0$, and, as $0 \prec a_1$ and L is semimodular we conclude $a \prec a \vee a_1$. Since $a < b$ and $a_1 \leq b$ we get $a \vee a_1 \leq b$, and, as $a \prec b$, we have $a \vee a_1 = b$.

Let $c := a_1 \vee b'$. Then $a \vee c = a \vee (a_1 \vee b') = (a \vee a_1) \vee b' = b \vee b' = 1$. By the third part of Lemma 1, we conclude $c \in C_a$. ∎

Corollary 3. *Let L be a finite semimodular lattice. If $a, b \in L$ are such that $a \prec b$, $C_b \neq \emptyset$ and $C_a \not\subseteq C_b$ then (a, b) satisfies Q_2.*

Proof: Let L, $a, b \in L$ be as stated. Let $b' \in C_b$. If $b' \in C_b \backslash C_a$ then by the theorem $\exists\, a' \in C_a\colon\, b' \prec a'$ in particular $b' \leq a'$. If $b' \in C_a$ then take $a' = b'$. ∎

Corollary 4.

i) *Finite complemented semimodular lattices satisfy Q_2.*

ii) *Let L be a finite semimodular lattice, $a, b \in L$ such that $C_a \neq \emptyset$, $C_b \neq \emptyset$ and $a \prec b$. Then (a, b) satisfies Q_1.*

Proof: **i**) Let L be a finite complemented semimodular lattice and $(a, b) \in L^2$. First we note that on a finite complemented lattice if we show that Q_2 holds when $a \prec b$ then this property is also valid when $a < b$. Let $a \prec b$. The property holds when $b' \in C_a$. Supose $b' \in C_b \backslash C_a$. By the first part of Lemma 1, the element $a \vee b'$ is a co-atom. It is easy to see that there is a complement, a_1, of $a \vee b'$ which is an atom. So, by Lemma 1, iii), $b' \prec b' \vee a_1$ and Q_2 is satisfied.

ii) Let L and $a, b \in L$ be as stated. If $C_a \not\subseteq C_b$ then, by Corollary 3, (a, b) satisfies Q_2 and therefore satisfies Q_1. If $C_a \subseteq C_b$ then choose $a' \in C_a$ and consider the pair $(a', a') \in C_a \times C_b$. ∎

There are finite semimodular lattices which do not satisfy Q_2. Here is an example:

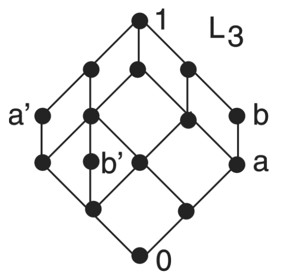

Fig. 4

3 – The case of modular lattices

Note that, given a modular lattice L, and $a, b \in L$, if $C_a \neq \emptyset$ and $C_b \neq \emptyset$ then C_a and C_b are antichains. The next theorem tells us how they are related.

Theorem 5. *Modular lattices satisfy P_2 and P_3.*

Proof: It is enough to show P_2 because this implies P_3 by duality. Let L be a modular lattice. Consider $a,\ b \in L$ such that $a < b,\ C_a \neq \emptyset$ and $C_b \neq \emptyset$. For all $b' \in C_b$ choose $a' \in C_a$. We will prove that $a'' := (a' \wedge b) \vee b'$ is a complement of a greater than b'. In fact, $a \vee a'' = (a \vee (a' \wedge b)) \vee b' = ((a \vee a') \wedge b) \vee b' = b \vee b' = 1$. Also, $a \wedge ((a' \wedge b) \vee b') = a \wedge (((a' \wedge b) \vee b') \wedge b) = a \wedge ((a' \wedge b) \vee (b' \wedge b)) = a \wedge a' \wedge b = 0$. We have $b' \leq a''$ and if $b' = a''$ then we would have $a < b,\ a \wedge b' = b \wedge b' = 0$ and $a \vee b' = b \vee b' = 1,$ which contradicts the modularity of L. ∎

Corollary 6. *In a complemented modular lattice, for each ascending chain*

$$0 < a_1 < \cdots < a_n < \cdots < 1$$

there exists a descending chain

$$1 > a'_1 > \cdots > a'_n > \cdots > 0$$

such that a'_i is a complement of a_i in L.

The following example shows that modular lattices do not have to satisfy P.

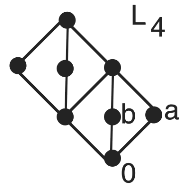

Fig. 5

In fact, we have $a \neq b$ and $C_a = C_b \neq \emptyset$. We note that $a \vee b$ does not have a complement. We will see that, in a modular lattice, if $a \vee b$ and $a \wedge b$ have complements, then P holds.

Theorem 7. *In a modular lattice L, if the set of complemented elements forms a sublattice, then L satisfies P.*

Proof: Let L be a modular lattice, and let $a \neq b \in L$. Supose a and b have complements. If $a < b$ then, let $b' \in C_b$. We know, by Theorem 5, that there is $a' \in C_a$ such that $b' < a'$. Therefore $a' \notin C_b$. Supose that $\{a, b\}$ is an antichain. We will show that, if $a \vee b$ and $a \wedge b$ have complements, then $C_a \neq C_b$.

Let c and d be complements of $a \vee b$ and $a \wedge b$, respectively, such that $c < d$. We have:

$$c \wedge (a \vee b) = 0; \quad c \vee (a \vee b) = 1; \quad d \wedge (a \wedge b) = 0; \quad d \vee (a \wedge b) = 1 \ .$$

We will show that $(b \wedge d) \vee c \in C_a \backslash C_b$:

$$(b \wedge d) \vee c \vee a = \Big((a \wedge b) \vee (b \wedge d)\Big) \vee c \vee a = \bigg(\Big((a \wedge b) \vee d\Big) \wedge b\bigg) \vee c \vee a = b \vee c \vee a = 1 \ ;$$

$$\Big((b \wedge d) \vee c\Big) \wedge a = \bigg(\Big((b \wedge d) \vee c\Big) \wedge (a \vee b)\bigg) \wedge a = \bigg((b \wedge d) \vee \Big(c \wedge (a \vee b)\Big)\bigg) \wedge a = b \wedge d \wedge a = 0 \ ;$$

$$(b \wedge d) \vee c \vee b = \Big((a \wedge b) \vee (b \wedge d)\Big) \vee c \vee b = \bigg(\Big((a \wedge b) \vee d\Big) \wedge b\bigg) \vee c \vee b = b \vee c < 1 \ ,$$

because L is modular. ∎

As an immediate consequence we have the following theorem:

Corollary 8. *Modular complemented lattices satisfy the property P.*

Corollary 9. *In a modular lattice, if two elements a and b are such that $a \vee b$ and $a \wedge b$ are complemented, then a and b also are complemented.*

Proof: The case of a and b being comparable, is trivial. If a and b are not comparable, let c and d be complements of $a \vee b$ and $a \wedge b$, respectively, such that $c < d$. From the proof of Theorem 7 we get $(b \wedge d) \vee c \in C_a \backslash C_b$ and $(a \wedge d) \vee c \in C_b \backslash C_a$. ∎

We conclude with an application of Corollary 6 to the theory of R-modules over a ring:

Corollary 10. *In a semisimple R-module, M, there exists an infinite ascending chain*

$$\{0\} \subset M_1 \subset \cdots \subset M_n \subset \cdots$$

if and only if there exists an infinite descending chain

$$M \supset M'_1 \supset \cdots \supset M'_n \supset \cdots$$

such that M is the direct sum of M'_i and M_i.

Proof: Note that the lattice of submodules of a semisimple R-module is modular and complemented. If we have an infinite ascending chain $\{0\} \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ of submodules of M then, by Corollary 6, we can build an infinite descending chain $M \supset M'_1 \supset M'_2 \supset \cdots \supset M'_n \supset \cdots$, therefore M is not artinian.

The proof of the other implication is analogous, using the dual of Corollary 6. ∎

From this corollary, it follows the well known result that a semisimple R-module, M, is noetherian if and only if it is artinian.

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