PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 4 – 1998

A REMARK ON THE UNIQUENESS OF FUNDAMENTAL SOLUTIONS TO THE p-LAPLACIAN EQUATION, $p > 2$

Ph. Laurençot

Presented by J.F. Rodrigues

Abstract: Uniqueness of fundamental solutions to the p-Laplacian equation is investigated in the class of nonnegative functions taking on their initial data in the sense of bounded measures.

1 – Introduction

In this note, we study the uniqueness of nonnegative solutions to the Cauchy problem

$$u_t - \operatorname{div}\Big(|\nabla u|^{p-2}\, \nabla u\Big) = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ , \tag{1.1}$$

$$u(0) = M\,\delta \ , \tag{1.2}$$

where p and M are positive real numbers, $p > 2$, and δ denotes the Dirac mass centered at $x = 0$. A solution to (1.1)–(1.2) is usually called a fundamental or source-type solution in the litterature.

The problem (1.1)–(1.2) is not a standard Cauchy problem, since the initial data in (1.2) involves a measure. A precise meaning has thus to be given to (1.2). Since $M\,\delta$ is a bounded measure, the natural way to give a sense to (1.2) is to

Received: January 23, 1997; *Revised*: April 23, 1997.
1991 Mathematics Subject Classification: 35K65.
Keywords and Phrases: Degenerate parabolic equation, p-Laplacian, source-type solution, uniqueness.
This work was done while I was visiting the Weierstraß–Institut in Berlin. I thank this institution for its hospitality.

assume that u takes on the initial data $M\delta$ in the sense of bounded measures, i.e.

$$\lim_{t\to 0}\int u(x,t)\,\zeta(x)\ dx = M\,\zeta(0) \tag{1.3}$$

for every $\zeta\in\mathcal{C}_b(\mathbb{R}^N)$. Here, $\mathcal{C}_b(\mathbb{R}^N)$ denotes the space of real-valued bounded and continuous functions on $\mathbb{R}^N$.

Existence of a solution to (1.1)–(1.3) is well-known, and an explicit formula is available for such a solution. Indeed, for $M>0$, define

$$W_M(x,t) = t^{-k}\left(A_M - b\left(|x|t^{-k/N}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}$$

for $(x,t)\in\mathbb{R}^N\times(0,+\infty)$, where z_+ denotes the positive part of the real number z,

$$k=\frac{N}{N(p-2)+p}\ ,\qquad b=\frac{p-2}{p}\left(\frac{k}{N}\right)^{1/(p-1)}\ ,$$

and A_M is a constant depending on M, N and p such that $|W_M(t)|_{L^1}=M$. Then, W_M is a solution to (1.1) and fulfills (1.3).

Our main concern in this paper is the question of uniqueness of nonnegative solutions to (1.1)–(1.3). The starting point of our study is the following result of Kamin and Vázquez ([6]).

Theorem 1.1 ([6, Theorem 1]). *Let M be a positive real number and u be a nonnegative function such that for each $T>0$,*

$$u\in\mathcal{C}((0,T);L^1_{\rm loc}(\mathbb{R}^N))\cap L^1(0,T;W^{1,p-1}_{loc}(\mathbb{R}^N))\ ,$$

and

$$\int_0^T\!\!\int\left(-u\,\varphi_t + |\nabla u|^{p-2}\,\nabla u\cdot\nabla\varphi\right)dx\,dt = 0 \tag{1.4}$$

for every test function $\varphi\in W^{1,\infty}(0,T;L^\infty(\mathbb{R}^N))\cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^N))$ with compact support. Assume further that

$$\lim_{t\to 0}|u(t)|_{\mathcal{C}(K)} = 0 \tag{1.5}$$

for every compact subset K of $\mathbb{R}^N\setminus\{0\}$, and

$$\lim_{t\to 0}\int_{B_R(0)} u(x,t)\,dx = M\ ,\qquad R>0\ , \tag{1.6}$$

where $B_R(x)$ denotes the open ball of $\mathbb{R}^N$ of center x and radius R. Then, $u=W_M$.

Owing to [6, Lemma 3.1], it turns out that (1.5) and (1.6) yield (1.3). However, the opposite assertion is not true in general and the set of assumptions (1.5)–(1.6) is thus stronger than (1.3). It is the purpose of this note to prove that (1.3) is sufficient to obtain uniqueness, provided that u is assumed to be in $L^\infty(0,T;L^1(\mathbb{R}^N))$ for each $T>0$. In the framework of [6], this further requirement is fulfilled as a consequence of (1.5) and [6, Lemma 3.1]. Our result then reads:

Theorem 1.2. *Let M be a positive real number and u be a nonnegative function such that for each $T>0$,*

$$u\in L^\infty(0,T;L^1(\mathbb{R}^N))\cap L^1(0,T;W^{1,p-1}_{loc}(\mathbb{R}^N))$$

and satisfies (1.4) for every test function $\varphi\in W^{1,\infty}(0,T;L^\infty(\mathbb{R}^N))\cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^N))$ with compact support, and (1.3) as well. Then, $u=W_M$.

The basic idea of the proof of Theorem 1.2 is to show that a solution to (1.1)–(1.3) in the sense of Theorem 1.2 is necessarily radially symmetric and nonincreasing with respect to the space variable for each $t>0$. It is then sufficient to notice that (1.3) implies (1.5)–(1.6) for radially symmetric and nonincreasing functions and to use Theorem 1.1 to complete the proof.

2 – Proof of Theorem 1.2

Let M be a positive real number and u be a nonnegative solution to (1.1)–(1.3) in the sense of Theorem 1.2. Since $u(t)\in L^1(\mathbb{R}^N)$ for almost every $t>0$ and $-\operatorname{div}(|\nabla v|^{p-2}\,\nabla v)$ generates a contraction semigroup in $L^1(\mathbb{R}^N)$ ([3], [2]), $u\in \mathcal{C}((0,+\infty);L^1(\mathbb{R}^N))$ and we infer from (1.1) and (1.3) that

$$\int u(x,t)\,dx = M\,, \qquad t>0\ . \tag{2.1}$$

Also, u belongs to $\mathcal{C}(\mathbb{R}^N\times(0,+\infty))$ (see e.g. [1], [4]).

Lemma 2.1. *For every $t>0$, $u(t)$ is radially symmetric and nonincreasing with respect to the space variable.*

Proof: For $\epsilon>0$ and $r>0$, we put

$$u_0^{\epsilon,r}(x)=u(x,\epsilon)\,\chi_{B_r(0)}(x)\,, \qquad x\in\mathbb{R}^N\ ,$$

where $\chi_{B_r(0)}$ denotes the characteristic function of the ball $B_r(0)$. Then, $u_0^{\epsilon,r} \in L^1(\mathbb{R}^N)$ is compactly supported and we denote by $u^{\epsilon,r}$ the solution to (1.1) with initial data $u_0^{\epsilon,r}$.

Let $t>0$. On the one hand, since $u(\epsilon)$ and $u_0^{\epsilon,r}$ are in $L^1(\mathbb{R}^N)$, the L^1-contraction property of (1.1) yields

$$\left|u\,(t+\epsilon) - u^{\epsilon,r}(t)\right|_{L^1} \leq \left|u(\epsilon) - u_0^{\epsilon,r}\right|_{L^1} . \tag{2.2}$$

On the other hand, we claim that

$$\lim_{\epsilon\to 0}\left|u(\epsilon) - u_0^{\epsilon,r}\right|_{L^1} = 0 . \tag{2.3}$$

Indeed, proceeding as in [5, Lemma 4.1], we consider $\zeta \in \mathcal{C}_b(\mathbb{R}^N)$, $0 \leq \zeta \leq 1$ such that $\zeta(x) = 1$ if $|x| \geq r/2$ and $\zeta(0) = 0$. Then,

$$\left|u(\epsilon) - u_0^{\epsilon,r}\right|_{L^1} = \int_{\{|x|\geq r\}} u(x,\epsilon)\,dx \;\leq \int u(x,\epsilon)\,\zeta(x)\,dx \ ,$$

and the right-hand side of the above inequality converges to zero as $\epsilon \to 0$ by (1.3), hence the claim.

Combining (2.2) and (2.3), we obtain, since $u \in \mathcal{C}((0,+\infty); L^1(\mathbb{R}^N))$,

$$\lim_{\epsilon\to 0}\left|u(t) - u^{\epsilon,r}(t)\right|_{L^1} = 0 . \tag{2.4}$$

Next, since $u_0^{\epsilon,r}$ is compactly supported with support in $B_r(0)$, we infer from [6, Lemma 5.1] that, for any (x_1,x_2) in $\mathbb{R}^N \times \mathbb{R}^N$,

$$|x_1| \geq r \ \text{ and } \ |x_2| \geq |x_1| + 2r \implies u^{\epsilon,r}(x_1,t) \geq u^{\epsilon,r}(x_2,t) . \tag{2.5}$$

We then let $\epsilon \to 0$ in (2.5) and use (2.4) and the continuity of $u(t)$ in $\mathbb{R}^N$ to obtain

$$|x_1| \geq r \ \text{ and } \ |x_2| \geq |x_1| + 2r \implies u(x_1,t) \geq u(x_2,t) . \tag{2.6}$$

Since (2.6) is valid for any $r > 0$, Lemma 2.1 follows from the continuity of $u(t)$ in $\mathbb{R}^N$ by letting $r \to 0$ in (2.6). ∎

Having shown that $u(t)$ is radially symmetric and nonincreasing for each $t > 0$, we now prove that u satisfies (1.5) and (1.6).

We first check (1.5). We consider $R > 0$ and a function $\zeta \in \mathcal{C}_b(\mathbb{R}^N)$ such that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if $|x| \geq R$ and $\zeta(x) = 0$ if $|x| \leq R/2$. Since u is radially symmetric and nonincreasing, we have

$$u(x,t)\,\text{meas}\,(B_{2R}(0)\backslash B_R(0)) \leq \int u(y,t)\,\zeta(y)\,dy \tag{2.7}$$

for $|x| \geq 2R$. We then let $t \to 0$ in (2.7) and use (1.3) to obtain (1.5). Next, (1.6) follows from (1.3) by approximating the characteristic function of $B_R(0)$ by bounded continuous functions.

Therefore, u fulfills the assumptions of Theorem 1.1, hence $u = W_M$.

REFERENCES

[1] ALIKAKOS, N.D. and ROSTAMIAN, R. – Gradient estimates for degenerate diffusion equations II, *Proc. Roy. Soc. Edinburgh Sect. A,* 91 (1992), 335–346.

[2] ATTOUCH, H. and DAMLAMIAN, A. – Application des méthodes de convexité et monotonie à l'étude de certaines équations quasi-linéaires, *Proc. Roy. Soc. Edinburgh Sect. A,* 79 (1977), 107–129.

[3] BÉNILAN, PH. – *Opérateurs accrétifs et semi-groupes dans les espaces L^p ($1 \leq p \leq \infty$),* in "Functional Analysis and Numerical Analysis", Japan–France Seminar, (H. Fujita, Ed.), Japan Society for the Promotion of Science, 1978, pp. 15–53.

[4] DIBENEDETTO, E. and HERRERO, M.A. – On the Cauchy problem and initial traces for a degenerate parabolic equation, *Trans. Amer. Math. Soc.,* 314 (1989), 187–224.

[5] ESCOBEDO, M., VÁZQUEZ, J.L. and ZUAZUA, E. – Asymptotic behaviour and source-type solutions for a diffusion-convection equation, *Arch. Rational Mech. Anal.,* 124 (1993), 43–65.

[6] KAMIN, S. and VÁZQUEZ, J.L. – Fundamental solutions and asymptotic behaviour for the p-Laplacian equation, *Rev. Mat. Iberoamericana,* 4 (1988), 339–354.

Philippe Laurençot,
Institut Elie Cartan - Nancy, Université de Nancy I,
BP 239, F-54506 Vandœuvre les Nancy cedex – FRANCE