PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 4 – 1998

ON THE PATHWISE UNIQUENESS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: A sufficient condition for uniqueness of solutions of ordinary differential equations is generalized to the setting of stochastic differential equations driven by brownian motion. The result extends the classical theorem of Ito and is consistent with respect to more recent pathwise uniqueness results.

1. Let W_t, $0 \le t \le T$ $(T \in \Re_+)$ denote an m-dimensional Wiener process defined on the probability space $(\Omega, \Re, P)$. Suppose $\{\Re_t : 0 \le t \le T\}$ is a nonanticipating family of sub-σ-algebras of $\Re$ with respect to the m-dimensional Wiener process W_t.

We are concerned with the uniqueness of solutions for the Ito stochastic differential equation

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t\,, \qquad 0 \le t \le T\ ,$$

with initial data $X_0 = c$ or, in integral form,

$$(1) \qquad X_t = c + \int_0^t f(s, X_s)\, ds + \int_0^t g(x, X_s)\, dW_s\,, \qquad 0 \le t \le T\ .$$

Here $f(t,x)$ and $g(t,x)$ are m-vector, respectively $m \times m$-matrix valued Borel measurable functions on $[0,T] \times \Re^m$ and c is a random variable independent of W_t for $0 \le t$. A solution of (1) on $[0,T]$ is an a.s. continuous, $\Re_t$-adapted stochastic process $(X_t)_{t \in [0,T]}$ such that a.s.

$$\int_0^T |f(s, X_s)|^2\, ds < \infty\,, \qquad \int_0^T |g(s, X_s)|^2\, ds < \infty\ ,$$

Received: June 12, 1997.
1991 Mathematics Subject Classification: 60H10.
Keywords: Stochastic differential equation, pathwise uniqueness.

and equation (1) is satisfied a.s. Equation (1) has the pathwise uniqueness property if any pair of solutions X_t and Y_t with $X_0 = Y_0 = c$ a.s. satisfy

$$P\Big\{\sup_{0\le t\le T} |X_t - Y_t| > 0\Big\} = 0 \ .$$

Pathwise uniqueness implies that solutions are unique in the law sense (have the same distributions) but pathwise uniqueness and law uniqueness are not equivalent (see [9]).

Ito [6] (see also Arnold [1, page 105]) showed that (1) has the pathwise uniqueness property if f and g satisfy a Lipschitz condition in the second variable and a certain growth condition. For more recent pathwise uniqueness results we refer to Constantin [3], Da Prato [5] and Taniguchi [5].

The aim of this note is to give a general theorem for the pathwise uniqueness of the solution of (1) using the method proposed by Athanassov [2] in the deterministic case.

2. The following lemma will be useful:

Lemma [2]. *Let $u(t)$ be a continuous function on $[0,T]$, $u(t) > 0$ for $t > 0$, having derivative $u'(t) \in L_1[0,T]$, $u'(t) > 0$ for $t > 0$. Let $v(t)$ be a continuous, nonnegative function on $[0,T]$ such that*

$$\lim_{t\to 0^+} \frac{v(t)}{u(t)} = 0$$

and

$$v(t) \le \int_{0^+}^{t} \frac{u'(s)}{u(s)}\, v(s)\, ds\,, \qquad 0 \le t \le T\ .$$

Then $v(t) = 0$ for $0 \le t \le T$.

Let L_{2k}^m (here $k = 1, 2$) be the space of all random variables $x\colon \Omega \to R^m$ with finite L_{2k}-norm

$$\|x\| = \Big\{\sum_{i=1}^{m} E|x_i|^{2k}\Big\}^{\frac{1}{2k}}\,, \qquad x = (x_1, ..., x_m) \in L_{2k}^m\ .$$

Theorem. *Let us suppose that there exists a continuous function $u(t)$ on $[0,T]$ with $u(t) > 0$ for $t > 0$, having derivative $u'(t) \in L_1[0,T]$, $u'(t) > 0$ for*

$t > 0$ and $\lim_{t\to 0^+} u'(t) = \infty$ such that for $0 < t \le T$ and $x, y \in R^m$,

$$\Big|f(t,x) - f(t,y)\Big|^2 + \Big|g(t,x) - g(t,y)\Big|^2 \le \frac{u'(t)}{2\,u(t)}\,|x-y|^2 \ , \tag{2}$$

and

$$|f(t,x)|^2 + |g(t,x)|^2 \le L + M\,|x|^2\,, \qquad x \in R^m, \quad t \in [0,T] \ . \tag{3}$$

If $c \in L_4^m$ is independent of W_t for $t \ge 0$ and f, g are continuous, then (1) has the pathwise uniqueness property.

Proof. Since $c \in L_4^m$ and f, g are continuous and satisfy (3), there is at least one solution to (1) on some interval $[0,T]$ with $T > 0$, cf. [7].

Let X, Y denote continuous solutions of (1) on an interval $[0,t_0]$ which, for simplicity, is assumed to be contained in the unit interval. For $n \in N$, let

$$\tau_n = \inf\Big\{0 \le t\colon\ |X_t| > n \text{ or } |Y_t| > n\Big\} \ \wedge\ t_0 \ ,$$

and define

$$v_n(t) = E\Big(\sup_{s\le t}\Big|(X-Y)_s \ \wedge\ \tau_n\Big|^2\Big)\,, \qquad 0 \le t \le t_0 \ .$$

Using condition (2) we can write for $0 \le t \le t_0$

$$\begin{aligned} v_n(t) &\le 2\Big[E\Big(\int_0^{\tau_n\wedge t}\Big|f(s,X_s) - f(s,Y_s)\Big|^2\,ds + \int_0^{\tau_n\wedge t}\Big|g(s,X_s) - g(s,Y_s)\Big|^2\,ds\Big)\Big] \\ &\le \int_0^{t_0}\frac{u'(s)}{u(s)}\,E\Big(\sup_{r\le s\wedge\tau_n}|X_r - Y_r|^2\Big)\,ds = \int_0^{t_0}\frac{u'(s)}{u(s)}\,v_n(s)\,ds \ . \end{aligned}$$

Now let $\varepsilon > 0$. Choose $\gamma > 0$ such that

$$|f(t,x)|^2 + |g(t,x)|^2 \le \frac{\varepsilon}{8}\,u'(t)\,, \qquad 0 < t \le \gamma, \quad |x| \le n \ .$$

Since for $a, b \in R^m$ the inequality

$$|a-b|^2 = \sum_{i=1}^m (a_i - b_i)^2 \le 2\Big(\sum_{i=1}^m a_i^2 + \sum_{i=1}^m b_i^2\Big) = 2\,|a|^2 + 2\,|b|^2$$

holds, we deduce that

$$\begin{aligned} v_n(t) &\le 2\Big[E\Big(\int_0^{\tau_n\wedge t}\Big|f(s,X_s) - f(s,Y_s)\Big|^2\,ds + \int_0^{\tau_n\wedge t}\Big|g(s,X_s) - g(s,Y_s)\Big|^2\,ds\Big)\Big] \\ &\le 4\,E\Big(\int_0^{\tau_n\wedge t}|f(s,X_s)|^2\,ds + \int_0^{\tau_n\wedge t}|f(s,Y_s)|^2\,ds \\ &\quad + \int_0^{\tau_n\wedge t}|g(s,X_s)|^2\,ds + \int_0^{\tau_n\wedge t}|g(s,Y_s)|^2\,ds\Big) \\ &\le \varepsilon\int_0^t u'(s)\,ds = \varepsilon\,u(t)\,, \qquad 0 < t \le \gamma \ . \end{aligned}$$

We hence may apply the lemma and conclude that $v_n = 0$. Since n is arbitrary we have that $X_t = Y_t$ a.s. for every fixed $t \in [0, t_0]$ and hence for a countable dense set S in $[0, t_0]$. By the continuity of X and Y we have that coincidence in S implies coincidence throughout the entire interval $[0, t_0]$ and hence

$$P\Big\{ \sup_{0 \le t \le t_0} |X_t - Y_t| > 0 \Big\} = 0 \ .$$

This completes the proof of the theorem. ∎

Corollary 1. *Assume there exists a constant $K > 0$ such that*

$$\Big| f(t,x) - f(t,y) \Big| + \Big| g(t,x) - g(t,y) \Big| \le K\,|x-y|\,, \qquad 0 \le t \le T, \quad x, y \in R^m \ ,$$

$$|f(t,x)|^2 + |g(t,x)|^2 \le K^2 (1 + |x|^2)\,, \qquad 0 \le t \le T, \quad x \in R^m \ .$$

If f and g are continuous on $[0,T] \times R^m$ and $c \in L_4^m$ is independent of W_t for $t \ge 0$, then (1) has the pathwise uniqueness property.

Proof. The hypothesis of Corollary 1 implies

$$\Big| f(t,x) - f(t,y) \Big|^2 + \Big| g(t,x) - g(t,y) \Big|^2 \le K^2\,|x-y|^2\,, \qquad 0 \le t \le T, \quad x, y \in R^m \ .$$

We can apply our theorem with $u(t) = \exp(4K^2 \sqrt{T}\sqrt{t})$, $0 \le t \le T$. ∎

Corollary 2. *Assume there exists a constant $0 < \alpha < \frac{1}{2}$ such that*

$$\Big| f(t,x) - f(t,y) \Big|^2 + \Big| g(t,x) - g(t,y) \Big|^2 \le \frac{\alpha}{t}\,|x-y|^2\,, \qquad 0 < t \le T, \quad x, y \in R^m \ ,$$

and constants $K > 0$ such that

$$|f(t,x)|^2 + |g(t,x)|^2 \le K + L\,|x|^2\,, \qquad 0 < t \le T, \quad x \in R^m \ .$$

If f and g are continuous on $[0,T] \times R^m$ and $c \in L_4^m$ is independent of W_t for $t \ge 0$, then (1) has the pathwise uniqueness property.

Proof. We apply our theorem with $u(t) = t^{2\alpha}$, $0 \le t \le T$. ∎

Remark 1. Our theorem is more general with respect to the problem of pathwise uniqueness than the result given in [3].

We provide now a concrete example of a stochastic differential equation where our result can be applied and the classical Lipschitz condition is not satisfied. We also cannot apply Corollary 1, nor Proposition 1 from [8].

Example 1. Let $f, g\colon [0,1] \times R \to R$ be given by

$$f(t,x) = g(t,x) = \begin{cases} \dfrac{x}{3\sqrt{t}}, & t > 0, \quad 0 < x \le t, \\ \dfrac{\sqrt{t}}{3}, & x > t, \quad t > 0, \\ 0, & t = 0, \\ 0, & x \le 0\ . \end{cases}$$

The hypotheses of Corollary 2 are satisfied for $\alpha = \frac{2}{9}$, but the classical Lipschitz condition is not satisfied: if it would hold, we would have

$$\Big|f(t,x) - f(t,y)\Big|^2 \le K\,|x-y|^2\,, \qquad t \ge 0, \quad x, y \in R\ ,$$

and for $x = \frac{t}{2}$, $y = 0$, we would obtain $\frac{t}{6} = |f(t, \frac{t}{2}) - f(t,0)|^2 \le K\,\frac{t^2}{4}$ for $t > 0$ which is impossible (let $t \to 0$). ∎

Remark 2. If $f(t,x) = f(t)$, we can consider for our theorem less strong hypotheses: we have the following

Proposition. *In the same hypotheses of the theorem, for $f(t,x) = f(t)$ and*

$$\Big|g(t,x) - g(t,y)\Big|^2 \le \frac{u'(t)}{u(t)}\,|x-y|^2\,, \qquad x, y \in R^m, \quad t \in [0,T]\ ,$$

instead of condition (2), the conclusion of the theorem holds.

Proof. Let X, Y denote continuous solutions of (1) on an interval $[0, t_0]$ which, for simplicity, is assumed to be contained in the unit interval and define v_n for $n \in N$ as in the proof of the theorem. We have

$$\begin{aligned} E\Big(\Big|X(t) - Y(t)\Big|^2\Big) &= E\Big(\int_0^t \Big|g(s,X_s) - g(s,Y_s)\Big|^2\,ds\Big) \le \\ &\le \int_0^T \frac{u'(s)}{u(s)}\,E\Big(\sup|X_s - Y_s|^2\Big)\,ds = \int_0^T \frac{u'(s)}{u(s)}\,v_n(s)\,ds \end{aligned}$$

and further we follow the same lines as for the proof of the theorem. ∎

Example 2. Let us consider a continuous function $u(t)$ on $[0,1]$ with $u(t) > 0$ for $t > 0$, having derivative $u'(t) \in L_1[0,1]$, $u'(t) > 0$ for $t > 0$ and $\lim_{t\to 0^+} \frac{u(t)}{u'(t)} = \infty$.

We define $f(t,x) = t$ for $t \in [0,1]$ and $x \in R$ and let

$$g\colon [0,1] \times R \to R, \quad g(t,x) = \begin{cases} \sqrt{\dfrac{u'(t)}{u(t)}}\, x, & t > 0, \quad 0 \le x \le \dfrac{u(t)}{u'(t)}, \\ \sqrt{\dfrac{u(t)}{u'(t)}}, & t > 0, \quad x > \dfrac{u(t)}{u'(t)}, \\ 0, & t = 0, \quad x \in R, \\ 0, & t \ge 0, \quad x < 0\ . \end{cases}$$

One can easily verify that the Proposition shows that (1) with the above choices of f, g has pathwise unique solutions. ∎

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