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MAXIMA AND MINIMA OF STATIONARY RANDOM SEQUENCES UNDER A LOCAL DEPENDENCE RESTRICTION

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Abstract: In this paper a local mixing condition $\widetilde{D}(u_n, v_n)$ for stationary random sequences satisfying Davis' condition $D(u_n, v_n)$ is introduced. Under these conditions, the asymptotic joint distribution of the maxima and minima can be calculated with the knowledge of the crossing probabilities. An illustrative example of a 2-dependent sequence where the maxima and minima are not asymptotically independent is also given.

1 – Introduction

Let $\{X_n\}$ be a strictly stationary random sequence with marginal distribution function F, let $\{u_n\}$ and $\{v_n\}$ be real sequences and consider the maxima $M_n = \max\{X_1, X_2, ..., X_n\}$ and the minima $W_n = \min\{X_1, X_2, ..., X_n\}$.

It is well known that, if $\{X_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, the maxima and minima, with linear normalization, are asymptotically independent. Davis (1979) gives the sufficient conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$, under which the maxima and minima, both jointly and marginally, behave as though the sequence $\{X_n\}$ was i.i.d.. The condition $D(u_n, v_n)$ is an asymptotic independence condition, weaker than strong mixing, and $D'(u_n, v_n)$ is a local dependence condition which implies the non existence of clustering of high and low values of the sequence $\{X_n\}$ above $\{u_n\}$ and below $\{v_n\}$, respectively.

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Oliveira and Turkman (1992) introduce the local mixing condition $D^*(u_n, v_n)$ which is weaker than $D'(u_n, v_n)$ and generalizes $D''(u_n)$ of Leadbetter and Nandagopalan (1989). If this condition holds along with $D(u_n, v_n)$ the asymptotic joint distribution of the maxima and minima may be computed from the bivariate distribution of two consecutive random variables. Namely, the stationary sequence $\{X_n\}$ satisfies $D(u_n, v_n)$ if for every n and integers $1 \le i_1 < ... < i_p < j_1 ... < j_q \le n$, such that $j_1 - i_p > \ell$,

$$\begin{aligned}(1)\quad &\Big|P\Big(X_{i_1} \le u_n,\, ...,\, X_{i_p} \le u_n,\, X_{j_1} \le u_n,\, ...,\, X_{j_q} \le u_n\Big) - \\ &\quad - P\Big(X_{i_1} \le u_n,\, ...,\, X_{i_p} \le u_n\Big)\, P\Big(X_{j_1} \le u_n,\, ...,\, X_{j_q} \le u_n\Big)\Big| \;\le\; \alpha_{n,\ell}\ ,\end{aligned}$$

$$\begin{aligned}&\Big|P\Big(X_{i_1} > v_n,\, ...,\, X_{i_p} > v_n,\, X_{j_1} > v_n,\, ...,\, X_{j_q} > v_n\Big) - \\ &\quad - P\Big(X_{i_1} > v_n,\, ...,\, X_{i_p} > v_n\Big)\, P\Big(X_{j_1} > v_n,\, ...,\, X_{j_q} > v_n\Big)\Big| \;\le\; \alpha_{n,\ell}\ ,\end{aligned}$$

and

$$\begin{aligned}&\Big|P\Big(v_n < X_{i_1} \le u_n,\, ...,\, v_n < X_{i_p} \le u_n,\, v_n < X_{j_1} \le u_n,\, ...,\, v_n < X_{j_q} \le u_n\Big) - \\ &-P\Big(v_n < X_{i_1} \le u_n,\, ...,\, v_n < X_{i_p} \le u_n\Big)\, P\Big(v_n < X_{j_1} \le u_n,\, ...,\, v_n < X_{j_q} \le u_n\Big)\Big| \le \alpha_{n,\ell}\ ,\end{aligned}$$

where $\lim\limits_{n\to+\infty} \alpha_{n,\ell_n} = 0$ for some ℓ_n such that $\lim\limits_{n\to+\infty} \ell_n/n = 0$.

Furthermore, $D^*(u_n, v_n)$ is satisfied by $\{X_n\}$ if $\lim\limits_{k\to+\infty} \limsup\limits_{n\to+\infty} S^*_{n,k} = 0$ where

$$\begin{aligned}S^*_{n,k} = n \sum_{j=1}^{[n/k]} \Big\{&P\Big(X_1 > u_n,\, X_j \le u_n < X_{j+1}\Big) + P\Big(X_1 < v_n,\, X_j \ge v_n > X_{j+1}\Big) \\ &+ P\Big(X_1 > u_n,\, X_j \ge v_n > X_{j+1}\Big) + P\Big(X_1 < v_n,\, X_j \le u_n < X_{j+1}\Big)\Big\}\ .\end{aligned}$$

For stationary sequences satisfying $D(u_n, v_n)$ and $D^*(u_n, v_n)$, Oliveira and Turkman (1992) consider high and low levels, u_n and v_n, verifying $\lim\limits_{n\to+\infty} P(X_2 \le u_n/X_1 > u_n) = \theta_1$, $\lim\limits_{n\to+\infty} P(X_2 > v_n/X_1 \le v_n) = \theta_2$, $\lim\limits_{n\to+\infty} nP(X_1 > u_n) = \tau_1(x)$ and $\lim\limits_{n\to+\infty} nP(X_1 < v_n) = \tau_2(y)$, with θ_1, θ_2 in $]0,1]$ and $\tau_1(x), \tau_2(y)$ in $]0,+\infty[$. The limit

$$\lim_{n\to+\infty} P\Big(M_n \le u_n,\, W_n > v_n\Big) = e^{-(\theta_1\tau_1(x) + \theta_2\tau_2(y))}$$

is obtained and hence, the maxima and minima, are yet asymptotically independent.

The constant θ_1 is called the extremal index of the stationary sequence $\{X_n\}$ and $\theta = (\theta_1, \theta_2)$ is the extremal index of $\{X_n, -X_n\}$. The definition of multivariate extremal index for multivariate stationary sequences can be found in Nandagopalan (1990). As we already said before, if $\{X_n\}$ satisfies $D(u_n, v_n)$ and $D'(u_n, v_n)$ we easily deduce $\theta_1 = \theta_2 = 1$.

Dealing with the asymptotic behavior of the exceedance point process for stationary sequences satisfying Leadbetter's condition $D(u_n)$, defined by (1), Ferreira (1996) introduce another mixing condition $\widetilde{D}^{(k)}(u_n)$, which also generalizes $D''(u_n)$. The condition $\widetilde{D}^{(k)}(u_n)$ is satisfied by $\{X_n\}$ if k is the minimum positive integer for which there exists a sequence of positive integers $\{k_n\}$, with

$$\lim_{n\to+\infty} k_n = +\infty, \quad \lim_{n\to+\infty} k_n \frac{\ell_n}{n} = 0, \quad \lim_{n\to+\infty} k_n\, \alpha_{n,\ell_n} = 0, \quad \lim_{n\to+\infty} k_n(1-F(u_n)) = 0$$

and

$$s_n^{(k)} = n \sum_{2\le j_1<j_2<\ldots<j_k\le[\frac{n}{k_n}]-1} P\Big(X_1 > u_n, \bigcap_{i=1}^{k} \Big\{X_{j_i} \le u_n < X_{j_i+1}\Big\}\Big) \to 0, \;\; n \to +\infty\ .$$

The condition $D''(u_n)$ is obtained for $k = 1$. The author of $\widetilde{D}^{(k)}(u_n)$ has proven that, if $\{X_n\}$ satisfies $\widetilde{D}^{(2)}(u_n)$ and $\lim\limits_{n\to+\infty} nP(X_1 \le u_n < X_2) = \nu$, with ν in $[0, +\infty[$, then

$$\lim_{n\to+\infty} P(M_n \le u_n) = e^{-\nu+\beta}\,, \qquad \beta \ge 0\ ,$$

if and only if

$$\lim_{n\to+\infty} k_n \sum_{1\le i<j\le[\frac{n}{k_n}]-1} P\Big(X_i \le u_n < X_{i+1},\, X_j \le u_n < X_{j+1}\Big) = \beta\ .$$

In this paper we introduce a local mixing restriction, condition $\widetilde{D}(u_n, v_n)$, which generalizes $\widetilde{D}^{(2)}(u_n)$ and is weaker than $D^*(u_n, v_n)$. Under $D(u_n, v_n)$ and $\widetilde{D}(u_n, v_n)$ the joint limit distribution of the maxima and minima can be computed from the mean number of four kinds of crossings of the considered levels: upcrossings in a cluster of high values; downcrossings in a cluster of low values; paired upcrossings, paired downcrossings and pairs with one upcrossing and one downcrossing in representative clusters.

It should be noticed that under $D(u_n, v_n)$ and $\widetilde{D}(u_n, v_n)$ the maxima and minima are not necessarily asymptotically independent.

2 – Main result

As we said before we consider strictly stationary sequences satisfying Davis' condition $D(u_n, v_n)$. For the proof of our main result it will be convenient to present the following lemma.

Lemma 1 (Davis (1979)). *Suppose $D(u_n, v_n)$ is satisfied by the stationary sequence $\{X_n\}$. Then, for every positive integer k,*

$$\lim_{n\to+\infty} \Big\{ P\Big(M_n \le u_n,\, W_n > v_n\Big) - P^k\Big(M_{n'} \le u_n,\, W_{n'} > v_n\Big) \Big\} = 0 \ ,$$

where $n' = [n/k]$.

In what follows the events $\{X_i \le u_n < X_{i+1}\}$ and $\{X_i > v_n \ge X_{i+1}\}$ are represented by A_i and B_i, respectively.

Definition 1. The sequence $\{X_n\}$ satisfies condition $\widetilde{D}(u_n, v_n)$ if $\lim\limits_{k\to+\infty} \limsup\limits_{n\to+\infty} k\,\widetilde{S}_{n,k} = 0$ where

$$\begin{aligned} \widetilde{S}_{n,k} = \sum_{1\le i<j<k\le n'-1} &\Big\{ P(A_i, A_j, A_k) + P(A_i, A_j, B_k) + P(A_i, B_j, B_k) \\ &+ P(B_i, B_j, B_k) + P(B_i, A_j, B_k) + P(A_i, B_j, A_k) \\ &+ P(B_i, B_j, A_k) + P(B_i, A_j, A_k) \Big\} \ . \end{aligned} \tag{2}$$

This condition restricts the occurence of three or more level crossings in a cluster.

The following theorem is the main result of this paper. We first present some assumptions of the theorem. Specifically, we will consider that $\{X_n\}$ satisfies

$$\lim_{n\to+\infty} nP(A_1) = \nu_1 \,, \qquad \lim_{n\to+\infty} nP(B_1) = \nu_2 \ , \tag{3}$$

$$\lim_{n\to+\infty} \sum_{1\le i<j\le n'-1} P(A_i, A_j) = \frac{\beta_1}{k} + o_k(1/k) \ , \tag{4}$$

$$\lim_{n\to+\infty} \sum_{1\le i<j\le n'-1} P(B_i, B_j) = \frac{\beta_2}{k} + o_k(1/k) \tag{5}$$

and

$$\lim_{n\to+\infty} \sum_{i=1}^{n'-1} \sum_{j=1}^{n'-1} P(A_i, B_j) = \frac{\beta_3}{k} + o_k(1/k) \ , \tag{6}$$

with ν_1, ν_2, β_1, β_2 and β_3 in $[0, +\infty[$. It should be remarked that, under stationarity, $\beta_1 \le \nu_1$, $\beta_2 \le \nu_2$, $\beta_3 \le \nu_1 - \beta_1$ and $\beta_3 \le \nu_2 - \beta_2$.

Theorem 1. *Suppose that the stationary sequence $\{X_n\}$ satisfies $D(u_n, v_n)$ and $\widetilde{D}(u_n, v_n)$ and that, for all positive integer k, (3), (4), (5) and (6) hold, where $\{u_n\}$ and $\{v_n\}$ are real sequences satisfying*

$$\lim_{n\to+\infty} P(X_1 > u_n) = P(X_1 \le v_n) = 0 \ . \tag{7}$$

Then,

$$\lim_{n\to+\infty} P\Big(M_n \le u_n,\, W_n > v_n\Big) = e^{-(\nu_1+\nu_2-\beta_1-\beta_2-\beta_3)} \ .$$

Proof: We start by observing that

$$\begin{aligned} &\{M_{n'} > u_n\} = \{X_1 > u_n\} \cup \Big\{\bigcup_{i=1}^{n'-1} A_i\Big\} \ , \\ &\{W_{n'} \le v_n\} = \{X_1 \le v_n\} \cup \Big\{\bigcup_{i=1}^{n'-1} B_i\Big\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} P\Big(M_{n'} \le u_n,\, W_{n'} > v_n\Big) &= 1 - P(M_{n'} > u_n) - P(W_{n'} \le v_n) \\ &\quad + P\Big(M_{n'} > u_n,\, W_{n'} \le v_n\Big) \ . \end{aligned} \tag{9}$$

From Bonferroni's inequality we get

$$\begin{aligned} &\sum_{i=1}^{n'-1} P(A_i) - \sum_{1\le i<j\le n'-1} P(A_i, A_j) \le \\ &\qquad \le P(M_{n'} > u_n) \\ &\qquad \le P(X_1 > u_n) + \sum_{i=1}^{n'-1} P(A_i) - \sum_{1\le i<j\le n'-1} P(A_i, A_j) \\ &\qquad\quad + \sum_{1\le i<j<k\le n'-1} P(A_i, A_j, A_k) \ . \end{aligned} \tag{10}$$

Using now stationarity it results

$$\lim_{n\to+\infty} \sum_{i=1}^{n'-1} P(A_i) = \lim_{n\to+\infty} (n'-1)\, P(A_1) = \frac{\nu_1}{k}$$

and

$$\limsup_{n\to+\infty} \sum_{1\le i<j<k\le n'-1} P(A_i, A_j, A_k) \;\le\; \limsup_{n\to+\infty} \widetilde{S}_{n,k} \;=\; o_k(1/k)\ .$$

Hence, attending to (4), (7) and (10), we have

$$\begin{aligned}\frac{\beta_1-\nu_1}{k} + o_k(1/k) &\le \liminf_{n\to+\infty} \Big\{-P(M_{n'} > u_n)\Big\}\\ &\le \limsup_{n\to+\infty} \Big\{-P(M_{n'} > u_n)\Big\}\\ &\le \frac{\beta_1-\nu_1}{k} + o_k(1/k)\ .\end{aligned} \tag{11}$$

Analogously we prove

$$\begin{aligned}\frac{\beta_2-\nu_2}{k} + o_k(1/k) &\le \liminf_{n\to+\infty} \Big\{-P(W_{n'} \le v_n)\Big\}\\ &\le \limsup_{n\to+\infty} \Big\{-P(W_{n'} \le v_n)\Big\}\\ &\le \frac{\beta_2-\nu_2}{k} + o_k(1/k)\ .\end{aligned} \tag{12}$$

Furthermore, using (8) and Boole's inequality, we obtain

$$\begin{aligned}&P\Big(M_{n'} > u_n,\, W_{n'} \le v_n,\, v_n < X_1 \le u_n\Big) =\\ &\qquad = P\Big(\bigcup_{i=1}^{n'-1} A_i,\ \bigcup_{i=1}^{n'-1} B_i,\ \ v_n < X_1 \le u_n\Big)\\ &\qquad \le \sum_{i=1}^{n'-1}\sum_{j=1}^{n'-1} P(A_i, B_j)\end{aligned} \tag{13}$$

and thus

$$\begin{aligned}&\limsup_{n\to+\infty} P\Big(M_{n'} > u_n,\, W_{n'} \le v_n\Big) =\\ &\qquad = \limsup_{n\to+\infty} P\Big(M_{n'} > u_n,\, W_{n'} \le v_n,\, v_n < X_1 \le u_n\Big)\\ &\qquad \le \frac{\beta_3}{k} + o_k(1/k)\ .\end{aligned} \tag{14}$$

On the other hand, applying Bonferroni's inequality, we have, with $B = \bigcup_{i=1}^{n'-1} B_i$,

$$
\begin{aligned}
P\Big(M_{n'} > u_n,\, W_{n'} \le v_n\Big) &\ge P\Big(\bigcup_{i=1}^{n'-1} A_i,\ \bigcup_{i=1}^{n'-1} B_i\Big) \\
&\ge \sum_{i=1}^{n'-1} P(A_i, B) - \sum_{1\le i<j\le n'-1} P(A_i, A_j, B)
\end{aligned}
\tag{15}
$$

and, using again the same inequality, we get

$$
\begin{aligned}
&\liminf_{n\to+\infty} \sum_{i=1}^{n'-1} P(A_i, B) \ge \\
&\ge \liminf_{n\to+\infty} \sum_{i=1}^{n'-1} \sum_{j=1}^{n'-1} P(A_i, B_j) - \limsup_{n\to+\infty} \sum_{i=1}^{n'-1} \sum_{1\le j<k\le n'-1} P(A_i, B_j, B_k)\ .
\end{aligned}
\tag{16}
$$

Moreover, since

$$
\begin{aligned}
&\sum_{i=1}^{n'-1} \sum_{1\le j<k\le n'-1} P(A_i, B_j, B_k) = \\
&\qquad = \sum_{1\le i<j<k\le n'-1} P(A_i, B_j, B_k) + P(B_i, A_j, B_k) + P(B_i, B_j, A_k) \\
&\qquad \le \widetilde{S}_{n,k}
\end{aligned}
$$

and $\widetilde{D}(u_n, v_n)$ holds, from (16) it results

$$
\liminf_{n\to+\infty} \sum_{i=1}^{n'-1} P(A_i, B) \ge \frac{\beta_3}{k} + o_k(1/k)\ .
$$

Let's recall (15). Considering again Boole's inequality we get

$$
\begin{aligned}
\limsup_{n\to+\infty} \sum_{1\le i<j\le n'-1} P(A_i, A_j, B) &\le \limsup_{n\to+\infty} \sum_{1\le i<j\le n'-1} \sum_{k=1}^{n'-1} P(A_i, A_j, B_k) \\
&\le \limsup_{n\to+\infty} \widetilde{S}_{n,k} = o_k(1/k)
\end{aligned}
\tag{17}
$$

and thus

$$
\liminf_{n\to+\infty} P\Big(M_{n'} > u_n,\, W_{n'} \le v_n\Big) \ge \frac{\beta_3}{k} + o_k(1/k)\ .
\tag{18}
$$

From (14) and (18), we have

$$\begin{aligned}\frac{\beta_3}{k}+o_k(1/k) &\leq \liminf_{n\to+\infty} P\Big(M_{n'}>u_n,\, W_{n'}\leq v_n\Big)\\ &\leq \limsup_{n\to+\infty} P\Big(M_{n'}>u_n,\, W_{n'}\leq v_n\Big)\\ &\leq \frac{\beta_3}{k}+o_k(1/k)\ .\end{aligned}\tag{19}$$

Finally, putting $\alpha=\nu_1+\nu_2-\beta_1-\beta_2-\beta_3$ we conclude from (9), (11), (12) and (19), that

$$\begin{aligned}1-\frac{\alpha}{k}+o_k(1/k) &\leq \liminf_{n\to+\infty} P\Big(M_{n'}\leq u_n,\, W_{n'}>v_n\Big)\\ &\leq \limsup_{n\to+\infty} P\Big(M_{n'}\leq u_n, W_{n'}>v_n\Big)\\ &\leq 1-\frac{\alpha}{k}+o_k(1/k)\end{aligned}\tag{20}$$

which implies

$$\limsup_{n\to+\infty}\left|P\Big(M_{n'}\leq u_n,\, W_{n'}>v_n\Big)-1+\frac{\alpha}{k}\right| = o_k(1/k)\ .\tag{21}$$

Observe now that

$$\begin{aligned}&\limsup_{n\to+\infty}\left|P\Big(M_n\leq u_n,\, W_n>v_n\Big)-e^{-\alpha}\right|\leq\\ &\quad\leq \limsup_{n\to+\infty}\left|P\Big(M_n\leq u_n,\, W_n>v_n\Big)-P^k\Big(M_{n'}\leq u_n,\, W_{n'}>v_n\Big)\right|\\ &\qquad+\limsup_{n\to+\infty}\left|P^k\Big(M_{n'}\leq u_n,\, W_{n'}>v_n\Big)-\Big(1-\frac{\alpha}{k}\Big)^k\right|\\ &\qquad+\left|e^{-\alpha}-\Big(1-\frac{\alpha}{k}\Big)^k\right|\ .\end{aligned}\tag{22}$$

Using Lemma 1, the first term of the right hand side of (22) is zero. Moreover, using the well known inequality

$$\left|\prod_{i=1}^k a_i-\prod_{i=1}^k b_i\right| \leq \sum_{i=1}^k |a_i-b_i|$$

with $a_1,...,a_k,\ b_1,...,b_k$ in $[0,1]$, we conclude that the second term of the right hand side of (22) is bounded by $\limsup\limits_{n\to+\infty} k|P(M_{n'}\leq u_n, W_{n'}>v_n)-(1-\frac{\alpha}{k})|$.

Hence, by (21) and (22), we deduce that

$$\begin{aligned}\lim_{k\to+\infty}\limsup_{n\to+\infty}\left|P\Big(M_n\le u_n,\,W_n>v_n\Big)-e^{-\alpha}\right| &\le \\ &\le \lim_{k\to+\infty}\left|e^{-\alpha}-\Big(1-\frac{\alpha}{k}\Big)^k\right|=0\end{aligned} \tag{23}$$

which enables us to conclude that $\lim_{n\to+\infty} P(M_n\le u_n, W_n>v_n)=e^{-\alpha}$. ∎

The following two results are important tools on the establishment of the asymptotic independence of the maxima and minima.

Corollary 1. *Suppose that $\{X_n\}$ is a stationary sequence under the assumptions of Theorem 1. Then, $\{M_n\le u_n\}$ and $\{W_n>v_n\}$ are asymptotically independent if and only if $\beta_3=0$.*

Proof: Since $D(u_n,v_n)$ holds, we obtain $\lim_{n\to+\infty}\{P(M_n\le u_n)-P^k(M_{n'}\le u_n)\}$ $=0$ and $\lim_{n\to+\infty}\{P(W_n>v_n)-P^k(W_{n'}>v_n)\}=0$.

On the other hand, it results from (11) that

$$\limsup_{n\to+\infty}\left|P(M_{n'}\le u_n)-\Big(1-\frac{\nu_1-\beta_1}{k}\Big)\right|=o_k(1/k)\ .$$

Therefore, with the arguments used in (22) and (23), we deduce that

$$\lim_{n\to+\infty} P(M_n\le u_n)=e^{-\nu_1+\beta_1}\ . \tag{24}$$

Similarly we prove that

$$\lim_{n\to+\infty} P(W_n>v_n)=e^{-\nu_2+\beta_2}\ . \tag{25}$$

So $\{M_n\le u_n\}$ and $\{W_n>v_n\}$ are asymptotically independent if and only if $\beta_3=0$. ∎

The proofs of Theorem 1 and Corollary 1 enables us to establish the following theorem. Firstly we must define another local dependence condition, weaker than $\widetilde{D}(u_n,v_n)$.

Definition 2. The sequence $\{X_n\}$ satisfies condition $\widetilde{C}(u_n,v_n)$ if $\lim_{k\to+\infty}\limsup_{n\to+\infty} k\,\widetilde{C}_{n,k}=0$ where

$$\widetilde{C}_{n,k}\;=\sum_{1\le i<j<k\le n'-1}\Big\{P(A_i,A_j,A_k)+P(B_i,B_j,B_k)\Big\}\ .$$

Indeed, we will prove that, if $\beta_3 = 0$, it is enough to consider $\widetilde{C}(u_n, v_n)$ instead of $\widetilde{D}(u_n, v_n)$.

Theorem 2. *Suppose that the stationary sequence $\{X_n\}$ satisfies $D(u_n, v_n)$ and $\widetilde{C}(u_n, v_n)$ where $\{u_n\}$ and $\{v_n\}$ are real sequences satisfying, for all positive integer k, (3), (4), (5), (7) and (6) with $\beta_3 = 0$. Then, $\{M_n \leq u_n\}$ and $\{W_n > v_n\}$ are asymptotically independent with*

$$\lim_{n \to +\infty} P\Big(M_n \leq u_n,\, W_n > v_n\Big) = e^{-(\nu_1+\nu_2-\beta_1-\beta_2)} \ .$$

Proof: Observe that we established (24) and (25) only using the first and the fourth terms of $\widetilde{S}_{n,k}$. Moreover, with $\beta_3 = 0$, from (14) we deduce

$$\limsup_{n \to +\infty} P\Big(M_{n'} > u_n,\, W_{n'} \leq v_n\Big) = o_k(1/k) \ .$$

Then, (20) is similarly obtained (with $\beta_3 = 0$), and the result follows immediately. ∎

3 – Example

Let $\{Y_n\}$ and $\{Z_n\}$ be independent sequences of i.i.d. random variables, with marginal distribution functions H and G respectively. Suppose that $G(0) = H(0) = 0$ and assume that there exists a real sequence $\{u_n\}$ satisfying

$$\lim_{n \to +\infty} n(1-H(u_n)) = \tau_Y \quad \text{and} \quad \lim_{n \to +\infty} n(1-G(u_n)) = \tau_Z \ ,$$

with τ_Y and τ_Z in $[0, +\infty[$.

Let $\{T_n\}$ be an i.i.d. sequence, independent of $\{Y_n\}$ and $\{Z_n\}$, with support $\{1, 2, 3\}$ and $P(T_1 = i) = p_i$, $i = 1, 2, 3$.

Define

$$X_n = \begin{cases} Y_n, & T_n = 1, \\ \max\{Y_{n-2}, Z_n\}, & T_n = 2, \\ -Y_{n-1}, & T_n = 3 \ . \end{cases}$$

We easily prove that $\{X_n\}$ is stationary and 2-dependent with marginal distribution function

$$F(x) = H(x)\, p_1 + H(x)\, G(x)\, p_2 + (1 - H(-x))\, p_3 \,, \qquad x \in \mathbb{R} \ ,$$

and satisfies $D(u_n, -u_n)$.

Moreover, $\{X_n\}$ does not satisfy either $D^*(u_n, -u_n)$ or $D''(u_n)$ once

$$\limsup_{n\to+\infty} n \sum_{j=2}^{[n/k]} P\Big(X_1 > u_n,\, X_j \le u_n < X_{j+1}\Big) \;\to\; \tau_Y\, p_1\, p_2\,, \qquad k\to+\infty\ .$$

We will prove now that $\widetilde{D}(u_n, -u_n)$ holds. Observe first that $\lim_{n\to+\infty} nF(-u_n) = \tau_Y p_3$ and

$$\lim_{n\to+\infty} n(1-F(u_n)) = \tau_Y\, p_1 + (\tau_Y+\tau_Z)\, p_2\ . \tag{26}$$

Indeed, since $\sum_{1\le i<j<k\le n'-1} P(A_i, A_j, A_k)$ is bounded by

$$\frac{n}{k} \sum_{3\le i<j\le n'-1} P(A_1, A_i, A_j) \le$$

$$\le \frac{n}{k} \sum_{2\le i<j\le n'-1} P\Big(X_1>u_n,\, A_i,\, A_j\Big)$$

$$\le \frac{n}{k} \sum_{i=2}^{n'-3} \Big\{ P\Big(X_1>u_n,\, X_{i+1}>u_n,\, X_{i+3}>u_n\Big) + \sum_{j=i+3}^{n'-1} P\Big(X_1>u_n,\, X_{i+1}>u_n,\, X_{j+1}>u_n\Big)\Big\}$$

$$\le \frac{n}{k} \sum_{i=2}^{n'-3} P(X_1>u_n)\, P(X_{i+3}>u_n) + \frac{n}{k} \sum_{i=2}^{n'-3} \sum_{j=i+4}^{n'} P\Big(X_1>u_n,\, X_{i+1}>u_n\Big)\, P(X_j>u_n)$$

$$\le \frac{n^2}{k^2}\Big(P(X_1>u_n)\Big)^2 + \frac{n^2}{k^2}\, P(X_1>u_n) \sum_{i=2}^{n'-3} P\Big(X_1>u_n,\, X_{i+1}>u_n\Big)$$

$$= \frac{n^2}{k^2}\Big(P(X_1>u_n)\Big)^2 + \frac{n^2}{k^2}\, P(X_1>u_n) \Big\{ P\Big(X_1>u_n,\, X_3>u_n\Big) + \sum_{i=4}^{n'-2} P(X_1>u_n)\, P(X_i>u_n)\Big\}$$

$$\le \frac{2n^2}{k^2}\Big(P(X_1>u_n)\Big)^2 + \frac{n^3}{k^3}\Big(P(X_1>u_n)\Big)^3$$

using (26), we conclude that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} k \sum_{1\le i<j<k\le n'-1} P(A_i, A_j, A_k) \;=\; 0 \;.$$

Analogously we prove the same for the other terms of $\widetilde{S}_{n,k}$. Then $\widetilde{D}(u_n, -u_n)$ holds.

The 2-dependence and the stationarity shall help us again on the computation of the parameters.

Let us start by calculating ν_1. In fact observing that $\lim_{n\to+\infty} P(X_1 \le u_n < X_2, T_2 = 3) = 0$ and using the Total Probability Rule, we have

$$\begin{aligned} n\,P(X_1 \le u_n < X_2) &= n\,P(Y_1 \le u_n < Y_2)\,p_1\,p_1 \\ &\quad + n\,P\Big(Y_1 \le u_n,\, \max\{Y_0, Z_2\} > u_n\Big)\,p_1\,p_2 \\ &\quad + n\,P\Big(\max\{Y_{-1}, Z_1\} \le u_n,\, Y_2 > u_n\Big)\,p_1\,p_2 \\ &\quad + n\,P\Big(\max\{Y_{-1}, Z_1\} \le u_n,\, \max\{Y_0, Z_2\} > u_n\Big)\,p_2\,p_2 \\ &\quad + n\,P(Y_1 > u_n)\,p_1\,p_3 + n\,P(\max\{Y_0, Z_2\} > u_n)\,p_2\,p_3 \;. \end{aligned} \tag{27}$$

Therefore $\nu_1 = \lim_{n\to+\infty} nP(X_1 \le u_n < X_2) = (\tau_Y + \tau_Z)\,p_2 + \tau_Y\,p_1$.

Using similar arguments and observing that

$$P\Big(X_1 > -u_n \ge X_2,\, T_2 = 1\Big) = P\Big(X_1 > -u_n \ge X_2,\, T_2 = 2\Big) \to 0\,, \qquad n \to +\infty \;,$$

it results $\nu_2 = \tau_Y p_3$.

In what concerns the evaluation of β_1, we have

$$\begin{aligned} \sum_{1\le i<j\le n'-1} P(A_i, A_j) &= \sum_{j=3}^{n'-1} (n'-j)\,P(A_1, A_j) \\ &= (n'-3)\,P(A_1, A_3) + \sum_{j=4}^{n'-1} (n'-j)\,P(A_1, A_j) \;. \end{aligned}$$

Since

$$\begin{aligned} \sum_{j=4}^{n'-1} (n'-j)\,P(A_1, A_j) &\le n' \sum_{j=4}^{n'-1} P(X_2 > u_n)\,P(X_{j+1} > u_n) \\ &\le \frac{n^2}{k^2} \Big(P(X_2 > u_n)\Big)^2 \;, \end{aligned}$$

it follows that

$$\lim_{n\to+\infty} \sum_{1\le i<j\le n'-1} P(A_i, A_j) = \lim_{n\to+\infty} \frac{n}{k} P(A_1, A_3) + o_k(1/k) .$$

For the computation of $P(A_1, A_3)$ we must use again the arguments used in (27). We first observe that $nP(A_1, A_3, C)$ is asymptotically zero if C is one of the events:

$$\{T_2{=}1,\, T_4{=}1\}, \quad \{T_2{=}2,\, T_4{=}1\}, \quad \{T_2{=}2,\, T_4{=}2\}, \quad \{T_2{=}3\} \ \text{ or } \ \{T_4{=}3\} .$$

Thus, with straightforward calculus, we deduce that $\beta_1 = \tau_Y\, p_1\, p_2$.

Moreover it is very easy to obtain $\beta_2 = 0$.

On the other hand, the computation of β_3 follows the steps used above. In fact, as

$$\begin{aligned}\lim_{n\to+\infty} \sum_{i=1}^{n'-1} \sum_{j=1}^{n'-1} P(A_i, B_j) &= \sum_{j=2}^{n'-1} (n'-j)\, P(A_1, B_j) + \sum_{j=2}^{n'-1} (n'-j)\, P(B_1, A_j) \\ &= (n'-2)\, P(A_1, B_2) + (n'-3)\, P(A_1, B_3) \\ &\quad + (n'-2)\, P(B_1, A_2) + (n'-3)\, P(B_1, A_3) + o_k(1/k)\end{aligned}$$

and $\lim_{n\to+\infty} nP(A_1, B_3) = \lim_{n\to+\infty} nP(B_1, A_3) = 0$, it results $\beta_3 = \tau_Y(p_1p_3 + p_2p_3)$.

Finally, we conclude that

$$\lim_{n\to+\infty} P\Big(M_n \le u_n,\, W_n > v_n\Big) = e^{-\alpha}$$

where $\alpha = \tau_Y + \tau_Z p_2 - \tau_Y(p_1p_2 + p_1p_3 + p_2p_3)$.

It should be noticed that $u_n = u_n(x)$ and $v_n = v_n(y)$. Hence, the parameters τ_Y, τ_Z, ν_1, ν_2, β_1, β_2 and β_3 depend on the real x and y. Then, clearly $\alpha = \alpha(x, y)$.

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