PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 1 – 1999

ORTHOGONAL POLYNOMIALS AND QUADRATIC TRANSFORMATIONS

F. Marcellán and J. Petronilho

Abstract: Starting from a sequence $\{P_n\}_{n\geq 0}$ of monic polynomials orthogonal with respect to a linear functional **u**, we find a linear functional **v** such that $\{Q_n\}_{\geq 0}$, with either $Q_{2n}(x) = P_n(T(x))$ or $Q_{2n+1}(x) = (x-a)\,P_n(T(x))$ where T is a monic quadratic polynomial and $a \in \mathbb{C}$, is a sequence of monic orthogonal polynomials with respect to **v**. In particular, we discuss the case when **u** and **v** are both positive definite linear functionals. Thus, we obtain a solution for an inverse problem which is a converse, for quadratic mappings, of one analyzed in [11].

1 – Introduction and preliminaries

In this paper we analyze some problems related to quadratic transformations in the variable of a given system of monic orthogonal polynomials (MOPS). The first problem to be considered is the following:

P1. Let $\{P_n\}_{n\geq 0}$ be a MOPS and $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that

$$Q_{2n}(x) = P_n(T(x)) , \quad n \geq 0 , \tag{1}$$

where $T(x)$ is a (monic) polynomial of degree 2.

a) To find necessary and sufficient conditions in order to guarantee that $\{Q_n\}_{n\geq 0}$ be a MOPS.

b) In such conditions, to find the relation between the moment linear functionals corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

c) In particular, to characterize the positive definite case.

Received: May 3, 1997; *Revised*: October 24, 1997.
1991 Mathematics Subject Classification: Primary 42C05.
Keywords and Phrases: Orthogonal polynomials, Recurrence coefficients, Polynomial mappings, Stieltjes functions.

The motivations to study this problem appear in several works. Among others, we refer: T.S. Chihara [6] for the case $\{P_n\}_{n\geq 0}$ symmetric, $T(x) = x^2$ and requiring that $\{Q_n\}_{n\geq 0}$ were a symmetric MOPS; an important paper of Geronimo and Van Assche [11] — these authors have proved that given a sequence $\{P_n\}_{n\geq 0}$ of polynomials orthonormal with respect to some positive measure μ supported on the bounded interval $[-1,1]$ and a polynomial $T(x)$ of fixed degree $k \geq 2$ with distinct zeros and such that $|T(y_j)| \geq 1$, where y_j $(j=1,...,k-1)$ are the zeros of T', then there exists always a positive measure ν and a sequence of polynomials $\{Q_n\}_{n\geq 0}$ orthonormal with respect to ν such that $Q_{kn}(x) = P_n(T(x))$; M.H. Ismail [13], J. Charris, M.H. Ismail and S. Monsalve [4] in connection with sieved orthogonal polynomials; F. Peherstorfer [22],[23] related to orthogonality on several intervals; D. Bessis and P. Moussa [3],[21] for the analysis of orthogonality properties of iterated polynomial mappings; and Gover [12] related to the eigenproblem of a tridiagonal 2-Toeplitz matrix. Another kind of quadratic transformations were studied by P. Maroni [18],[20] and L.M. Chihara and T.S. Chihara [8].

Of course, to solve problem P1 we must give the expressions for the polynomials $Q_{2n+1}(x)$, in order to complete the set $\{Q_n\}_{n\geq 0}$. This suggests us the second problem that we will consider:

P2. The same assumptions and questions as in P1, but with (1) replaced by

$$Q_{2n+1}(x) = (x-a)\,P_n(T(x))\,, \qquad n \geq 0\,, \tag{2}$$

(a a fixed complex number).

In the next we will recall some basic definitions and results. The space of all polynomials with complex coefficients will be denoted by $\mathbb{P}$. Let $\mathbf{u}\colon \mathbb{P} \to \mathbb{C}$ be a linear functional. A sequence of polynomials $\{P_n\}_{n\geq 0}$ is called orthogonal with respect to $\mathbf{u}$ if each P_n has exact degree n and

$$\langle \mathbf{u}, P_n P_m \rangle = k_n\,\delta_{nm} \qquad (k_n \neq 0)$$

holds for all $n, m = 0, 1, 2, \ldots$. Given a linear functional $\mathbf{u}$, we say that $\mathbf{u}$ is regular or quasi-definite [6, p. 16] if there exists a sequence of polynomials orthogonal with respect to it. It is a basic fact that if $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are two polynomial sequences orthogonal with respect to the same linear functional then, for each n, $P_n(x) = c_n\,Q_n(x)$, where $\{c_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers. Therefore, in this paper we will consider monic orthogonal polynomial sequences

(MOPS). Every MOPS $\{P_n\}_{n\geq 0}$ satisfies a three-term recurrence relation

$$\begin{aligned} &P_{n+1}(x) = (x-\beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x)\,, \qquad n = 1,2,...,\\ &P_0(x) = 1\,, \qquad P_1(x) = x - \beta_0\ , \end{aligned} \tag{3}$$

with $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C}\backslash\{0\}$ for all n. Furthermore, according to a theorem of J. Favard, if $\{P_n\}_{n\geq 0}$ is a sequence of polynomials which satisfies the three term recurrence relation (3), with the conditions $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C}\backslash\{0\}$ for all n, then it is orthogonal with respect to some linear functional.

Of course, in the previous concepts we have considered "formal orthogonality", in the sense that the orthogonal polynomials $\{P_n\}_{n\geq 0}$ are only related to a numerical sequence $u_n := \langle \mathbf{u}, x^n\rangle$, $n = 0,1,2,...$, ignoring whether these numbers are actually moments of some weight or distribution function on some support or not. In order to answer question c) in problems P1 and P2 , we must analyze under what conditions a regular linear functional $\mathbf{u}$ is positive definite, i.e., $\langle \mathbf{u}, f\rangle > 0$ for all $f \in \mathbb{P}$ such that $f(x) \geq 0\,, \forall x \in \mathbb{R}$ and $f \not\equiv 0$. In fact, a sequence of polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to some linear functional $\mathbf{u}$ is said to be orthogonal in the positive-definite sense if $\mathbf{u}$ is positive-definite. By a representation theorem [6, Chapter II] a linear regular functional $\mathbf{u}$ is positive definite if and only if there exists an integral representation, in terms of a Stieltjes integral, of the form

$$\langle \mathbf{u}, f\rangle = \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}\sigma(x)\ ,$$

for every polynomial f, where σ is a distribution function, i.e., a function $\sigma : \mathbb{R} \to \mathbb{R}$ which is nondecreasing, it has infinitely many points of increase (those are the elements of the set $S := \{x\colon\ \sigma(x+\delta) - \sigma(x-\delta) > 0,\ \forall\,\delta > 0\}$, called the spectrum of σ) and all the moments

$$\int_{-\infty}^{+\infty} x^{2n}\, \mathrm{d}\sigma(x)\,, \qquad n = 0,1,2,\dots\ ,$$

are finite. Sometimes, we also say that $\mathrm{d}\sigma(x)$ is a distribution function or measure, and S is also called the support of $\mathrm{d}\sigma$, the notation $\mathrm{supp}(\mathrm{d}\sigma)$ being also used for S. A necessary and sufficient condition for $\{P_n(x)\}_{n\geq 0}$ to be orthogonal in the positive-definite sense (i.e., with respect to a positive-definite linear functional) is that $\{P_n(x)\}_{n\geq 0}$ satisfies a three-term recurrence relation as (3) with $\beta_n \in \mathbb{R}$ and $\gamma_n \in \mathbb{R}^+$ for all n. Notice that if a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ satisfies such a recurrence relation, then the corresponding distribution function,

σ, may not be uniquely determined. However, σ is uniquely determined, up to denumerable many points of discontinuity, if

$$\sum_{k=1}^{+\infty} p_k^2(x_0) = +\infty\,, \qquad p_k(x) := (u_0\,\gamma_1\,\gamma_2\cdots\gamma_k)^{-1/2}\,P_k(x) \tag{4}$$

(p_k is the orthonormal polynomial of degree k with positive leading coefficient) holds at a single real point x_0 (Freud [10, p. 66]). Furthermore, if σ is uniquely determined, up to the points of discontinuity of $\sigma(x)$, (4) holds for every real x_0 [10, p. 63].

Given a sequence of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ satisfying (3) with $\beta_n \in \mathbb{R}$ and $\gamma_n \in \mathbb{R}^+$ for all n, in order to obtain the corresponding distribution function σ — if it is unique — we introduce the associated polynomials of the first kind, $\{P_n^{(1)}(x)\}_{n\geq 0}$, which are defined by the shifted recurrence relation

$$\begin{aligned} &P_{n+1}^{(1)}(x) = (x-\beta_{n+1})\,P_n^{(1)}(x) - \gamma_{n+1}\,P_{n-1}^{(1)}(x)\,, \qquad n = 1,2,...,\\ &P_0^{(1)}(x) = 1\,, \qquad P_1^{(1)}(x) = x - \beta_1\ . \end{aligned}$$

They can also be described by

$$P_n^{(1)}(x) = \frac{1}{u_0}\left\langle \mathbf{u}_y, \frac{P_{n+1}(x) - P_{n+1}(y)}{x-y}\right\rangle, \qquad n = 0,1,\ldots\ .$$

This sequence of polynomials is important, because the asymptotic behavior of $P_n(x)$ and $P_{n-1}^{(1)}(x)$ gives us the Stieltjes transform of $\mathrm{d}\sigma(x)$. According to a well known result due to A. Markov (see W. Van Assche [27] and C. Berg [2])

$$\lim_{n\to\infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)} = \frac{1}{u_0}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(t)}{z-t}\,, \qquad z \in \mathbb{C}\backslash(X_1 \cup X_2)\ , \tag{5}$$

uniformly on compact subsets of $\mathbb{C}\backslash(X_1 \cup X_2)$, provided that σ is uniquely determined. Here, if we denote by x_{nj} ($j = 1,...,n$) the zeros of P_n, for each fixed number n, and put $Z_1 := \{x_{nj} \colon\ j = 1,...,n;\ n = 1,2,...\}$, then

$$\begin{aligned} &X_1 := Z_1' \quad \text{(set of accumulation points of } Z_1)\ ,\\ &X_2 := \Big\{x \in Z_1 \colon\ P_n(x) = 0 \ \text{ for infinitely many } \ n\Big\}\ . \end{aligned}$$

Notice that $\operatorname{supp}(\mathrm{d}\sigma) \subset X_1 \cup X_2 \subset \operatorname{co}(\operatorname{supp}(\mathrm{d}\sigma))$, where $\operatorname{co}(\operatorname{supp}(\mathrm{d}\sigma))$ is the convex hull of $\operatorname{supp}(\mathrm{d}\sigma)$. Now, the function $\sigma(x)$ can be recovered from (5) by applying the Stieltjes inversion formula. Putting

$$F(z;\sigma) := \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(t)}{t-z}\ ,$$

then, if $\operatorname{supp}(\mathrm{d}\sigma)$ is contained in an half-line,

$$\sigma(t_2)-\sigma(t_1)=\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{t_1}^{t_2}\Big[F(x+i\varepsilon;\sigma)-F(x-i\varepsilon;\sigma)\Big]\,\mathrm{d}x\ ,$$

where we assume that σ is normalized in the following way:

$$\sigma(t)=\frac{\sigma(t+0)+\sigma(t-0)}{2}\ .$$

The function $F(\,\cdot\,;\sigma)$ is called the Stieltjes function of the distribution function σ (or the Stieltjes transform of the corresponding measure).

Finally we recall some properties fulfilled by the zeros of the orthogonal polynomials in the positive definite case. Each $P_n(x)$, $n\ge 1$, has n real and simple zeros $x_{n,j}$ ($j=1,...,n$), which we will denote in increasing order by

$$x_{n,1}<x_{n,2}<...<x_{n,n}\ ,\qquad n=1,2,...\ .$$

The zeros of two consecutive polynomials $P_n(x)$ and $P_{n+1}(x)$, $n\ge 1$, interlace (separation theorem),

$$x_{n+1,j}<x_{n,j}<x_{n+1,j+1}\ ,\qquad 1\le j\le n,\quad n=1,2,...\ ,$$

so that there exist the limits

$$\xi:=\lim_{n\to\infty}x_{n,1}\ge-\infty\qquad\text{and}\qquad\eta:=\lim_{n\to\infty}x_{n,n}\le+\infty\ .$$

The interval $[\xi,\eta]$ is called the "true" interval of orthogonality of the sequence $\{P_n\}_{n\ge 0}$. $]\xi,\eta[$ is the smallest open interval containing the zeros of all the $P_n(x)$, $n\ge 1$, and $[\xi,\eta]$ is the smallest closed interval which is a supporting set for any distribution function σ with respect to which $\{P_n\}_{n\ge 0}$ is orthogonal (cf. [6, p. 29]). We also mention that the condition "$[\xi,\eta]$ compact" is sufficient in order that (4) holds [6, p. 110], hence if $[\xi,\eta]$ is compact then σ is uniquely determined.

2 – Problem P1

The "algebraic" properties of the solution for problem P1, i.e., the answer to the questions a) and b) in P1, have been presented in [16]. In this case, the completion of the system $\{Q_n\}_{n\ge 0}$ is given by using the sequence $\{P_n^*(c;\,\cdot\,)\}_{n\ge 0}$ of the monic kernel polynomials of K-parameter c corresponding to the sequence $\{P_n\}_{n\ge 0}$, defined only if $P_n(c)\ne 0$ for all $n=0,1,2,...$ by

$$P_n^*(c;x)=\frac{1}{x-c}\left[P_{n+1}(x)-\frac{P_{n+1}(c)}{P_n(c)}\,P_n(x)\right]\ ,$$

$\{P_n^*(c;\cdot)\}_{n\ge 0}$ being a MOPS with respect to $\mathbf{u}^* := (x-c)\mathbf{u}$ [6, p. 35]. The coefficients $\{\beta_n^*, \gamma_{n+1}^*\}_{n\ge 0}$ of the corresponding three-term recurrence are given by

$$(6)\qquad \beta_n^* = \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}\,, \qquad \gamma_{n+1}^* = \gamma_{n+1}\frac{P_{n+2}(c)\,P_n(c)}{P_{n+1}^2(c)}$$

for $n = 0, 1, 2, \ldots$.

Theorem 1 ([16]). *Let $\{P_n\}_{n\ge 0}$ be a MOPS and $\{Q_n\}_{n\ge 0}$ a simple set of monic polynomials such that*

$$Q_1(x) = x - b\,, \qquad Q_{2n}(x) = P_n(T(x)), \quad n \ge 0\;,$$

where $T(x)$ is a (monic) polynomial of degree 2 and $b \in \mathbb{C}$. Without loss of generality, write

$$T(x) = (x-a)\,(x-b) + c\;.$$

Then $\{Q_n\}_{n\ge 0}$ is a MOPS if and only if

$$P_n(c) \ne 0, \quad Q_{2n+1}(x) = (x-b)\,P_n^*(c;T(x))\,, \qquad n \ge 0\;.$$

In such conditions, if $\{P_n\}_{n\ge 0}$ satisfies the three-term recurrence relation (3) (with $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C}\backslash\{0\}$ for all n), then the coefficients $\tilde\beta_n$ and $\tilde\gamma_n$ for the corresponding three term recurrence relation satisfied by $\{Q_n\}_{n\ge 0}$ are given by

$$(7)\qquad \tilde\beta_{2n} = b, \quad \tilde\beta_{2n+1} = a\,, \qquad n \ge 0\;,$$

$$(8)\qquad \tilde\gamma_{2n-1} = -\frac{P_n(c)}{P_{n-1}(c)}, \quad \tilde\gamma_{2n} = -\frac{P_{n-1}(c)}{P_n(c)}\gamma_n\,, \qquad n \ge 1\;.$$

Moreover, if $\{P_n\}_{n\ge 0}$ is orthogonal with respect to the moment linear functional $\mathbf{u}$, then $\{Q_n\}_{n\ge 0}$ is orthogonal with respect to a moment linear functional $\mathbf{v}$ defined on the basis $\{T^n(x), (x-b)T^n(x)\}_{n\ge 0}$ of $\mathbb{P}$ by means of

$$(9)\qquad \langle \mathbf{v}, T^n(x)\rangle = \langle \mathbf{u}, x^n\rangle, \quad \langle \mathbf{v}, (x-b)\,T^n(x)\rangle = 0\,, \qquad n \ge 0\;.$$

Corollary 2. *Under the conditions of Theorem 1, the coefficients of the three-term recurrence relation verified by the MOPS's $\{P_n\}_{n\ge 0}$, $\{P_n^*(c;\cdot)\}_{n\ge 0}$ and $\{Q_n\}_{n\ge 0}$ are related by*

$$(10)\qquad \begin{aligned} &\beta_0 = \tilde\gamma_1 + c\,, \qquad \beta_n = \tilde\gamma_{2n+1} + \tilde\gamma_{2n} + c, \quad n \ge 1\;,\\ &\gamma_n = \tilde\gamma_{2n-1}\,\tilde\gamma_{2n}\,, \qquad n \ge 1\;,\\ &\beta_n^* = \tilde\gamma_{2n+1} + \tilde\gamma_{2n+2} + c\,, \qquad n \ge 0\;,\\ &\gamma_n^* = \tilde\gamma_{2n}\,\tilde\gamma_{2n+1}\,, \qquad n \ge 1\;. \end{aligned}$$

In order to answer question c), we must analyze under what conditions the linear functional $\mathbf{v}$ can be represented by some distribution function $\tilde{\sigma}$ provided that the given linear functional $\mathbf{u}$ is represented by some distribution function σ. In particular, we also must give the relation between the supports of $d\sigma$ and $d\tilde{\sigma}$. We will obtain an answer for these questions via the Markov theorem and the Stieltjes inversion formula, by using the technique described in the previous section. We begin by establishing some preliminary lemmas.

Lemma 3. *Under the conditions of Theorem 1,*

$$Q^{(1)}_{2n-1}(x) = (x-a)\, P^{(1)}_{n-1}(T(x))$$

holds for all $n = 1, 2, \dots$.

Proof: Put $P_n(x) \equiv \sum_{i=0}^{n} a_i^{(n)} x^i$, so that

$$P_n(x) - P_n(y) = (x-y) \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, x^{i-j}\, y^j \; . \tag{11}$$

Then, $P_n(T(x)) - P_n(T(y)) = [T(x) - T(y)] \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, T^{i-j}(x)\, T^j(y)$, and taking into account that $T(x) - T(y) = (x-y)\,[(x-a) + (y-b)]$, it follows that, for $n \geq 1$,

$$\begin{aligned}
Q^{(1)}_{2n-1}(x) &= \frac{1}{v_0} \left\langle \mathbf{v}_y,\, \frac{Q_{2n}(x) - Q_{2n}(y)}{x-y} \right\rangle = \frac{1}{u_0} \left\langle \mathbf{v}_y,\, \frac{P_n(T(x)) - P_n(T(y))}{x-y} \right\rangle \\
&= \frac{1}{u_0} \left\langle \mathbf{v}_y,\; [(x-a) + (y-b)] \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, T^{i-j}(x)\, T^j(y) \right\rangle \\
&= \frac{1}{u_0} \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, T^{i-j}(x) \Big[(x-a)\, \langle \mathbf{v}_y, T^j(y) \rangle + \langle \mathbf{v}_y, (y-b) T^j(y) \rangle \Big] \\
&= (x-a)\, \frac{1}{u_0} \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, T^{i-j}(x)\, \langle \mathbf{u}_y, y^j \rangle \\
&= (x-a)\, \frac{1}{u_0} \left\langle \mathbf{u}_y,\; \sum_{i=0}^{n-1} \sum_{j=0}^{i} a^{(n)}_{i+1}\, T^{i-j}(x)\, y^j \right\rangle \\
&= (x-a)\, \frac{1}{u_0} \left\langle \mathbf{u}_y,\, \frac{P_n(T(x)) - P_n(y)}{T(x) - y} \right\rangle, \quad \text{by (11)} \\
&= (x-a)\, P^{(1)}_{n-1}(T(x)) \; . \blacksquare
\end{aligned}$$

Lemma 4. *Let $a,b,c \in \mathbb{R}$, $T(x) \equiv (x-a)\,(x-b)+c$ and $\sigma(x)$ a distribution function such that* $\operatorname{supp}(\mathrm{d}\sigma) \subset [\xi,\eta]$, *with* $-\infty < \xi < \eta \leq +\infty$. *If* $c \leq \xi$, *then*

$$\int_{T^{-1}(]\xi,\eta[)} x^{2n}\,\frac{|x-a|}{T'(x)}\,\mathrm{d}\sigma(T(x)) < +\infty\,, \qquad n=0,1,2,\ldots\,. \tag{12}$$

Proof: Put $\sigma_T(x) := \sigma(T(x))$, $\Delta := (b-a)^2-4c$ and notice that

$$T^{-1}(]\xi,\eta[) \;=\; \left]\tfrac{a+b}{2}-s,\; \tfrac{a+b}{2}-r\right[\;\cup\; \left]\tfrac{a+b}{2}+r,\; \tfrac{a+b}{2}+s\right[\,,$$

with

$$r := \sqrt{\xi+\tfrac{\Delta}{4}}\,, \qquad s := \sqrt{\eta+\tfrac{\Delta}{4}}\;.$$

By expanding $x^{2n} = \sum_j [a_{nj}+b_{nj}(x-b)]\,T^j(x)$, one see that, in order to prove (12) it is sufficient to show that

$$\int_{T^{-1}(]\xi,\eta[)} |T(x)|^n\,\frac{|x-a|}{T'(x)}\,\mathrm{d}\sigma_T(x) < +\infty \tag{13}$$

and

$$\int_{T^{-1}(]\xi,\eta[)} |x-b|\,|T(x)|^n\,\frac{|x-a|}{T'(x)}\,\mathrm{d}\sigma_T(x) < +\infty \tag{14}$$

for all $n=0,1,2,\ldots$. For a fixed n, consider the functions f_n^+ and f_n^- defined by

$$f_n^{\pm}(y) := \begin{cases} |y|^n\left(1 \pm \dfrac{b-a}{2\sqrt{y+\Delta/4}}\right), & y > -\frac{\Delta}{4}\,, \\[2ex] \left|\dfrac{\Delta}{4}\right|^n, & y = -\frac{\Delta}{4}\;. \end{cases}$$

By hypothesis, we have $-\frac{\Delta}{4} \leq T(a) = c \leq \xi$. Hence, if $\xi = -\frac{\Delta}{4}$ then necessarily $c = -\frac{\Delta}{4}$, so that $a=b$ and $f_n^{\pm}(y) = |y^n|$ for $y \geq \xi \equiv -\frac{\Delta}{4}$; if $\xi > -\frac{\Delta}{4}$ we have $0 < r = \sqrt{\xi+\Delta/4} \leq \sqrt{y+\Delta/4}$ for $y \geq \xi$, so that $1/\sqrt{y+\Delta/4} \leq 1/r$ for $y \geq \xi$. In any case, we get

$$|f_n^{\pm}(y)| \leq \left(1+\frac{|b-a|}{2r}\right)|y|^n \quad \text{ for } \; y \geq \xi\;.$$

Therefore, since $y^n \in L_1(]\xi,\eta[;\sigma)$ — because σ is a distribution function —, we conclude that also $f_n \in L_1(]\xi,\eta[;\sigma)$, and then there exists

$$I_n^{\pm} := \int_\xi^\eta |f_n^{\pm}(y)|\,\mathrm{d}\sigma(y) = \int_\xi^\eta |y|^n\left|1 \pm \frac{b-a}{2\sqrt{y+\Delta/4}}\right|\mathrm{d}\sigma(y) \tag{15}$$

(notice that $f_n^{\pm}(y)$ is continuous for $y \in [\xi, +\infty[$). Now, $T(x)$ increases for $x > \frac{a+b}{2}$ and decreases for $x < \frac{a+b}{2}$, and

$$\frac{2(x-a)}{T'(x)} = 1 + \frac{b-a}{2\sqrt{T(x)+\Delta/4}} \quad \text{if } \ x > \tfrac{a+b}{2} \ ,$$
$$\frac{2(x-a)}{T'(x)} = 1 - \frac{b-a}{2\sqrt{T(x)+\Delta/4}} \quad \text{if } \ x < \tfrac{a+b}{2} \ .$$

Hence, if we make the substitution $x = \frac{a+b}{2} \pm \sqrt{y+\Delta/4}$ in the integral on the right-hand side of (15),

$$+\infty > I_n^{\pm} = \int_{\frac{a+b}{2}\pm r}^{\frac{a+b}{2}\pm s} |T(x)|^n \left| \frac{2(x-a)}{T'(x)} \right| \mathrm{d}\sigma_T(x) \ \geq 0 \ ,$$

i.e., (13) follows. To prove (14), define

$$g_n(y) := \begin{cases} \dfrac{(y-c)\,|y|^n}{2\sqrt{y+\Delta/4}}, & y > -\frac{\Delta}{4} \ , \\ 0, & y = -\frac{\Delta}{4} \ . \end{cases}$$

Since $c \leq \xi$, then for $y \geq \xi$ it holds $|y-c| = y - c \leq y + \frac{\Delta}{4}$. Hence

$$|g_n(y)| \leq \frac{|y|^n}{2}\sqrt{y+\frac{\Delta}{4}} \leq \frac{1}{4}\left(y^{2n} + y + \frac{\Delta}{4}\right) \qquad \text{for } \ y \geq \xi \ .$$

It follows that $g_n \in L_1(]\xi, \eta[; \sigma)$ and there exists

$$\text{(16)} \qquad J_n := \int_{\xi}^{\eta} |g_n(y)|\,\mathrm{d}\sigma(y) = \int_{\xi}^{\eta} \frac{(y-c)\,|y|^n}{2\sqrt{y+\Delta/4}}\,\mathrm{d}\sigma(y)$$

(notice also that $g_n(y)$ is continuous for $y \in [-\frac{\Delta}{4}, +\infty[$). Now, as before, making the substitutions $x = \frac{a+b}{2} \pm \sqrt{y+\Delta/4}$ we get

$$\begin{aligned} +\infty > J_n &= \int_{\frac{a+b}{2}\pm r}^{\frac{a+b}{2}\pm s} \frac{[T(x)-c]\,|T(x)|^n}{2\,|x-\frac{a+b}{2}|}\,\mathrm{d}\sigma(T(x)) \\ &= \int_{\frac{a+b}{2}\pm r}^{\frac{a+b}{2}\pm s} |x-b|\,|T(x)|^n \left|\frac{x-a}{T'(x)}\right| \mathrm{d}\sigma_T(x) \ , \end{aligned}$$

which completes the proof. ∎

Assume now that the moment sequence $\{u_n\}_{n\geq 0}$, corresponding to the linear functional $\mathbf{u}$ in Theorem 1, is uniquely determined by some distribution function σ. Then, up to the points of discontinuity of $\sigma(x)$, for every real x_0

$$\sum_{k=1}^{+\infty} p_k^2(x_0) = +\infty$$

holds. Now, using the relations in Corollary 2, for every real number t_0 we have

$$\sum_{k=1}^{+\infty} q_k^2(t_0) \geq \sum_{k=1}^{+\infty} q_{2k}^2(t_0) = \sum_{k=1}^{+\infty} p_k^2(T(t_0)) \ ,$$

where $q_k(x) := (v_0\, \tilde{\gamma}_1\, \tilde{\gamma}_2 \cdots \tilde{\gamma}_k)^{-1/2}\, Q_k(x)$. Hence if x_0 is a point of continuity of σ and it is known a priori that $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to some distribution function $\tilde{\sigma}$, then the points t_0 such that $x_0 = T(t_0)$ are points of continuity of $\tilde{\sigma}$. Therefore, we conclude that if $\sigma(t)$ is uniquely determined by the moment sequence $\{u_n\}_{n\geq 0}$ then $\tilde{\sigma}(t)$ is also uniquely determined by the moment sequence $\{v_n\}_{n\geq 0}$ corresponding to $\mathbf{v}$. In these conditions, by Markov Theorem and Lemma 3, we can write

$$\begin{aligned} F(z;\tilde{\sigma}) &= -v_0 \lim_{n\to\infty} \frac{Q_{2n-1}^{(1)}(z)}{Q_{2n}(z)} \\ &= -u_0 \lim_{n\to\infty} \frac{(z-a)\, P_{n-1}^{(1)}(T(z))}{P_n(T(z))} = (z-a)\, F(T(z);\sigma) \ , \end{aligned} \tag{17}$$

which gives the relation between $F(\,\cdot\,;\tilde{\sigma})$ and $F(\,\cdot\,;\sigma)$.

We are now able to give an answer to the question c).

Theorem 5. *Let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to some uniquely determined distribution function $\sigma(x)$ and let $[\xi,\eta]$ (bounded or not) be the true interval of orthogonality of $\{P_n\}_{n\geq 0}$. Let b be a fixed real number, $T(x) \equiv (x-a)\,(x-b) + c$ a real polynomial of degree two and put $\Delta := (b-a)^2 - 4c$. Let $\{Q_n\}_{n\geq 0}$ be a sequence of polynomials such that*

$$Q_1(x) = x - b\,, \qquad Q_{2n}(x) = P_n(T(x))$$

for all $n = 0, 1, 2, \ldots$. Then, $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a positive definite linear functional if and only if

$$c \leq \xi\,, \qquad Q_{2n+1}(x) = (x-b)\, P_n^*(c;T(x)) \tag{18}$$

holds for all $n = 0, 1, 2, \ldots$.

In these conditions, $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the uniquely determined distribution function $d\tilde{\sigma}$

$$\mathrm{d}\tilde{\sigma}(x) = \frac{|x-a|}{T'(x)}\,\mathrm{d}\sigma(T(x))\,, \qquad r \leq |x - \tfrac{a+b}{2}| \leq s\,, \tag{19}$$

where

$$r := \sqrt{\xi + \tfrac{\Delta}{4}}\,, \qquad s := \sqrt{\eta + \tfrac{\Delta}{4}}\,.$$

Proof: First assume that conditions (18) hold. Since, for each positive integer number n, the zeros of P_n are in $]\xi, \eta[$, then the condition $c \leq \xi$ implies that $P_n(c) \neq 0$ for all $n = 0, 1, 2, \ldots$. From Theorem 1 it follows that $\{Q_n\}_{n\geq 0}$ is a MOPS. To conclude that it is a MOPS with respect to a positive measure, we only need to show that $\tilde{\beta}_n$ is real and $\tilde{\gamma}_{n+1}$ is positive for every $n = 0, 1, 2, \ldots$ (these notations are in accordance with Theorem 1). It is clear from (7) that $\tilde{\beta}_n$ is real and, since $\operatorname{sgn} P_n(x) = (-1)^n$ for $x \leq \xi$, so that $P_n(c)/P_{n-1}(c) < 0$, then from (8) we deduce $\tilde{\gamma}_n > 0$ for all $n = 1, 2, \ldots$.

Conversely, assume that $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a positive definite linear functional. From Theorem 1 it follows that $Q_{2n+1}(x)$ is given as in (18). Furthermore, Theorem 1 also gives $P_n(c) \neq 0$ for all $n = 0, 1, 2, \ldots$ and the relations in Corollary 2 hold. They will be used to show that $c \leq \xi$. In fact, we will prove [6, p. 108]

$$\begin{aligned} &(i) \quad c < \beta_n \quad \text{for} \;\; n = 0, 1, 2, \ldots\,, \\ &(ii) \quad \{\alpha_n(c)\}_{n\geq 1} \quad \text{is a chain sequence}\,, \end{aligned}$$

where

$$\alpha_n(x) := \frac{\gamma_n}{(\beta_{n-1} - x)\,(\beta_n - x)}\,, \qquad n = 1, 2, \ldots\,.$$

Since, by hypothesis, $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a positive definite linear functional, then $\tilde{\gamma}_n > 0$ $(n \geq 1)$, and (i) follows from Corollary 2. In order to prove (ii) define a sequence of parameters $\{m_n(c)\}_{n\geq 0}$ by

$$m_n(c) := 1 - \frac{P_{n+1}(c)}{(c-\beta_n)\,P_n(c)} \equiv \frac{\gamma_n\,P_{n-1}(c)}{(c-\beta_n)\,P_n(c)}\,, \qquad n = 0, 1, \ldots \;\; (P_{-1} \equiv 0)$$

(which is well defined according to (i) and the conditions $P_n(c) \neq 0$ for all $n \geq 0$). Now, we get

$$\alpha_n(c) = m_n(c)\,[1 - m_{n-1}(c)]\,, \qquad n = 1, 2, \ldots\,, \tag{20}$$

and also, by (8) and (i), for $n \geq 1$ it holds $m_n(c) = 1 - P_{n+1}(c) \,/\, [(c-\beta_n) P_n(c)] = 1 - \tilde{\gamma}_{2n+1}/(\beta_n - c) < 1$ and $m_n(c) = \gamma_n P_{n-1}(c) \,/\, [(c-\beta_n)\, P_n(c)] = \tilde{\gamma}_{2n}/(\beta_n - c) > 0$, so that

(21) $$m_0(c) = 0\,, \quad 0 < m_n(c) < 1, \quad n = 1, 2, \dots\,.$$

It follows from (20) and (21) that $\{\alpha_n(c)\}_{n\geq 1}$ is a chain sequence, $\{m_n(c)\}_{n\geq 0}$ being the corresponding minimal parameter sequence (cf. [6, p. 110]). Thus $c \leq \xi$.

Now, under such conditions, let $\mathrm{d}\tilde{\sigma}$ be the distribution function with respect to which $\{Q_n\}_{n\geq 0}$ is orthogonal. According to (17), for fixed $\varepsilon > 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} F(x+i\varepsilon;\tilde{\sigma}) - F(x-i\varepsilon;\tilde{\sigma}) &= \int_\xi^\eta \left(\frac{x-a+i\varepsilon}{t - T(x+i\varepsilon)} - \frac{x-a-i\varepsilon}{t-T(x-i\varepsilon)} \right) \mathrm{d}\sigma(t) \\ &= i \int_\xi^\eta f_\varepsilon(t,x)\; \mathrm{d}\sigma(t)\;, \end{aligned}$$

where

$$f_\varepsilon(t,x) := \frac{2\varepsilon[(x-a)^2 - c + \varepsilon^2 + t]}{[T(x) + \varepsilon^2 - t]^2 + 4\,\varepsilon^2 (t + \Delta/4)}\;.$$

Thus

$$\begin{aligned} f_\varepsilon(t,x) = {} & \left(1 + \frac{b-a}{2\sqrt{t+\Delta/4}}\right) \frac{\varepsilon}{\left(x - \frac{a+b}{2} - \sqrt{t+\Delta/4}\right)^2 + \varepsilon^2} \\ & + \left(1 - \frac{b-a}{2\sqrt{t+\Delta/4}}\right) \frac{\varepsilon}{\left(x - \frac{a+b}{2} + \sqrt{t+\Delta/4}\right)^2 + \varepsilon^2} \qquad \text{for } \; t > -\tfrac{\Delta}{4}\;, \end{aligned}$$

and

$$f_\varepsilon(-\tfrac{\Delta}{4},\, x) = 2\,\varepsilon \left[\frac{(b-a)\,(x - \frac{a+b}{2})}{\left[\left(x - \frac{a+b}{2}\right)^2 + \varepsilon^2\right]^2} + \frac{1}{\left(x - \frac{a+b}{2}\right)^2 + \varepsilon^2} \right]\;.$$

Hence, since $\xi \geq c \geq -\frac{\Delta}{4}$, we have

$$\begin{aligned} \int_\xi^\eta f_\varepsilon(t,x)\, \mathrm{d}\sigma(t) = {} & \int_\xi^\eta \left(1 + \frac{b-a}{2\sqrt{t+\Delta/4}}\right) \frac{\varepsilon}{\left(x - \frac{a+b}{2} - \sqrt{t+\Delta/4}\right)^2 + \varepsilon^2}\, \mathrm{d}\sigma(t) \\ & + \int_\xi^\eta \left(1 - \frac{b-a}{2\sqrt{t+\Delta/4}}\right) \frac{\varepsilon}{\left(x - \frac{a+b}{2} + \sqrt{t+\Delta/4}\right)^2 + \varepsilon^2}\, \mathrm{d}\sigma(t)\;, \end{aligned}$$

where it must be understood that the terms $\frac{b-a}{2\sqrt{t+\Delta/4}}$ do not appear in this expression if $\xi = -\frac{\Delta}{4}$ (remark that the condition $c \leq \xi = -\frac{\Delta}{4}$ also implies $a = b$).

In each of these integrals, we make the change of variables $u = \sqrt{t+\Delta/4}$, so that $t = T(\frac{a+b}{2} \pm u)$. Thus

$$\int_\xi^\eta f_\varepsilon(t,x)\,\mathrm{d}\sigma(t) = \int_r^s \left(1+\frac{b-a}{2\,u}\right) \frac{\varepsilon}{\left(x-\left(\frac{a+b}{2}+u\right)\right)^2+\varepsilon^2}\,\mathrm{d}\sigma\,\left(T(\tfrac{a+b}{2}+u)\right)$$
$$+\int_r^s \left(1-\frac{b-a}{2\,u}\right) \frac{\varepsilon}{\left(x-\left(\frac{a+b}{2}-u\right)\right)^2+\varepsilon^2}\,\mathrm{d}\sigma\,\left(T(\tfrac{a+b}{2}-u)\right)\,.$$

Now, in the first of these integrals we make the substitution $v = \frac{a+b}{2}+u$ and in the second one $v = \frac{a+b}{2}-u$, which leads to

$$\text{(22)}\quad \int_\xi^\eta f_\varepsilon(t,x)\,\mathrm{d}\sigma(t) = 2\int_{\frac{a+b}{2}+r}^{\frac{a+b}{2}+s} g_\varepsilon(x,v)\,\mathrm{d}\sigma_T(v) - 2\int_{\frac{a+b}{2}-s}^{\frac{a+b}{2}-r} g_\varepsilon(x,v)\,\mathrm{d}\sigma_T(v)$$

where

$$\sigma_T(v) := \sigma(T(v))\,, \qquad g_\varepsilon(x,v) := \frac{\varepsilon}{(x-v)^2+\varepsilon^2}\,\frac{v-a}{T'(v)}$$

(notice that if $\xi = -\frac{\Delta}{4}$, i.e., $r=0$, then the factor $(v-a)/T'(v)$ does not appear in the definition of g_ε). Denote

$$S_+ := \left]\tfrac{a+b}{2}+r,\ \tfrac{a+b}{2}+s\right[, \qquad S_- := \left]\tfrac{a+b}{2}-s,\ \tfrac{a+b}{2}-r\right[$$

and let $]t_1,t_2[$ (with $t_1 < t_2$) be an open interval (bounded or not) such that either

$$\tfrac{a+b}{2}+r \le t_1 < t_2 \le \tfrac{a+b}{2}+s \quad \text{or} \quad \tfrac{a+b}{2}-s \le t_1 < t_2 \le \tfrac{a+b}{2}-r\,.$$

Then the following holds:

(i) for almost all values of v in $S_\pm$, the function $g_\varepsilon(x,v)$ is continuous with respect to x in the open interval $]t_1,t_2[$;

(ii) $|g_\varepsilon(x,v)| \le G_\varepsilon(v) := (v-a)/\varepsilon\,T'(v)$ for all $x \in\,]t_1,t_2[$ and $G_\varepsilon(v)$ is an integrable function over $S_\pm$ with respect to $\sigma_T(v)$ (by Lemma 4); and

(iii) if t_1 or t_2 is infinite (which can occur only if $s=+\infty$, i.e., $\eta=+\infty$), so that $]t_1,t_2[\, = \,]t_1,+\infty[$ if $\frac{a+b}{2}+r \le t_1 < t_2$, or $]t_1,t_2[\, = \,]-\infty,t_2[$ if $t_1 < t_2 \le \frac{a+b}{2}-r$, we have, in the first case,

$$\int_{t_1}^{+\infty} |g_\varepsilon(x,v)|\,\mathrm{d}x = \lim_{t_2\to+\infty}\int_{t_1}^{t_2} \frac{\varepsilon}{(x-v)^2+\varepsilon^2}\,\mathrm{d}x\,\frac{v-a}{T'(v)}$$
$$= \left(\frac{\pi}{2} - \arctan\frac{t_1-v}{\varepsilon}\right)\frac{v-a}{T'(v)} \le \pi\,\frac{v-a}{T'(v)} =: G_1(v)\,,$$

with $G_1(v)$ an integrable function over S_+ with respect to $\sigma_T(v)$, and, in the same way, for the second case

$$\int_{-\infty}^{t_2} |g_\varepsilon(x,v)|\,\mathrm{d}x \le G_1(v) \ ,$$

$G_1(v)$ being also an integrable function over S_- with respect to $\sigma_T(v)$.

Therefore, integrating from t_1 to t_2 both sides of (22) with respect to x, (i)–(iii) can be used to justify a change in the order of integration (cf. Cramér [9, pp. 68,69]), and in this way we get

$$\int_{t_1}^{t_2} \Big[\int_{\xi}^{\eta} f_\varepsilon(t,x)\,\mathrm{d}\sigma(t)\Big]\,\mathrm{d}x = 2\int_{S_+} h_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) - 2\int_{S_-} h_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) \ ,$$

where

$$h_{t_1,t_2}(\varepsilon,v) := \int_{t_1}^{t_2} g_\varepsilon(x,v)\,\mathrm{d}x = \left(\arctan\frac{t_2-v}{\varepsilon} - \arctan\frac{t_1-v}{\varepsilon}\right)\frac{v-a}{T'(v)} \ .$$

Notice that

$$\lim_{\varepsilon\to 0^+} h_{t_1,t_2}(\varepsilon,v) = \left(\pi\,\chi_{]t_1,t_2[}(v) + \frac{\pi}{2}\,\chi_{\{t_1,t_2\}}(v)\right)\frac{v-a}{T'(v)} \ , \tag{23}$$

so that the functions $h^+_{t_1,t_2}(\varepsilon,v)$ and $h^-_{t_1,t_2}(\varepsilon,v)$ defined by

$$h^{\pm}_{t_1,t_2}(\varepsilon,v) := \begin{cases} h_{t_1,t_2}(\varepsilon,v), & v\in S_\pm \ \ \varepsilon>0, \\ \lim\limits_{\varepsilon\to 0^+} h_{t_1,t_2}(\varepsilon,v), & v\in S_\pm \ \ \varepsilon = 0 \ . \end{cases}$$

satisfy:

(i) for almost all values of v in $S_\pm$, $h^{\pm}_{t_1,t_2}(\varepsilon,v)$ is right-continuous with respect to ε in the point $\varepsilon=0$; and

(ii) $|h^{\pm}_{t_1,t_2}(\varepsilon,v)| \le G_2(v) := \pi(v-a)/T'(v)$ for all $\varepsilon\ge 0$ and $v\in S_\pm$, $G_2(v)$ being an integrable function over $S_\pm$ with respect to $\sigma_T(v)$.

Therefore, it holds [9, p. 67]

$$\lim_{\varepsilon\to 0^+}\int_{S_\pm} h_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) = \lim_{\varepsilon\to 0^+}\int_{S_\pm} h^{\pm}_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) = \int_{S_\pm} h^{\pm}_{t_1,t_2}(0,v)\,\mathrm{d}\sigma_T(v) \ .$$

Now, if t_1 and t_2 are points of continuity of the distribution function defined by $\frac{|v-a|}{T'(v)}\,\mathrm{d}\sigma_T(v)$ (which is a distribution function by Lemma 4), then, according to

(23), we have

$$\int_{S_+} h^+_{t_1,t_2}(0,v)\,\mathrm{d}\sigma_T(v) = \int_{S_+} \pi\,\chi_{]t_1,t_2[}(v)\,\frac{v-a}{T'(v)}\,\mathrm{d}\sigma_T(v)$$
$$= \begin{cases} 0 & \text{if } \frac{a+b}{2}-s \le t_1 < t_2 \le \frac{a+b}{2}-r, \\ \pi\displaystyle\int_{t_1}^{t_2} \frac{v-a}{T'(v)}\,\mathrm{d}\sigma_T(v) & \text{if } \frac{a+b}{2}+r \le t_1 < t_2 \le \frac{a+b}{2}+s\ , \end{cases}$$

and

$$\int_{S_-} h^-_{t_1,t_2}(0,v)\,\mathrm{d}\sigma_T(v) = \begin{cases} \pi\displaystyle\int_{t_1}^{t_2} \frac{v-a}{T'(v)}\,\mathrm{d}\sigma_T(v) & \text{if } \frac{a+b}{2}-s \le t_1 < t_2 \le \frac{a+b}{2}-r, \\ 0 & \text{if } \frac{a+b}{2}+r \le t_1 < t_2 \le \frac{a+b}{2}+s\ . \end{cases}$$

Thus, from the Stieltjes inversion formula and using the previous conclusions, at the points t_1 and t_2 of continuity of $\tilde\sigma$ and $\frac{|v-a|}{T'(v)}\,\mathrm{d}\sigma_T(v)$, we get

$$\begin{aligned} \tilde\sigma(t_2)-\tilde\sigma(t_1) &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{t_1}^{t_2} \Big[F(x+i\varepsilon;\tilde\sigma) - F(x-i\varepsilon;\tilde\sigma)\Big]\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{t_1}^{t_2} \Big[\int_\xi^\eta f_\varepsilon(t,x)\,\mathrm{d}\sigma(t)\Big]\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{S_+} h_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) - \frac{1}{\pi}\int_{S_-} h_{t_1,t_2}(\varepsilon,v)\,\mathrm{d}\sigma_T(v) \\ &= \frac{1}{\pi}\int_{S_+} h^+_{t_1,t_2}(0,v)\,\mathrm{d}\sigma_T(v) - \frac{1}{\pi}\int_{S_-} h^-_{t_1,t_2}(0,v)\,\mathrm{d}\sigma_T(v) \\ &= \begin{cases} \displaystyle\int_{t_1}^{t_2} \frac{v-a}{T'(v)}\,\mathrm{d}\sigma_T(v), & \text{if } \frac{a+b}{2}+r \le t_1 < t_2 \le \frac{a+b}{2}+s, \\ -\displaystyle\int_{t_1}^{t_2} \frac{v-a}{T'(v)}\,\mathrm{d}\sigma_T(v) & \text{if } \frac{a+b}{2}-s \le t_1 < t_2 \le \frac{a+b}{2}-r\ . \end{cases} \end{aligned}$$

and formula (19) follows.

Remark 1. The support of $\mathrm{d}\tilde\sigma$ is contained in the union of two intervals (eventually a unique interval if $\xi = -\frac{\Delta}{4}$):

$$\operatorname{supp}(\mathrm{d}\tilde\sigma) \subset \Big[\tfrac{a+b}{2}-s,\ \tfrac{a+b}{2}-r\Big] \cup \Big[\tfrac{a+b}{2}+r,\ \tfrac{a+b}{2}+s\Big] = T^{-1}([\xi,\eta])\ .$$

Corollary 6. *Under the conditions of Theorem 5, if σ is absolutely continuous, so that $\mathrm{d}\sigma(x) = w(x)\,\mathrm{d}x$, then $\tilde\sigma$ is absolutely continuous and $\mathrm{d}\tilde\sigma(x) = \tilde w(x)\,\mathrm{d}x$ where*

$$\tilde w(x) := |x-a|\,w(T(x))\,, \qquad r \le |x - \tfrac{a+b}{2}| \le s\ . \tag{24}$$

Proof: By (19) we get

$$\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}x}(x) = \begin{cases} (x-a)\, w(T(x)) & \text{if } \frac{a+b}{2}+r < x < \frac{a+b}{2}+s, \\ -(x-a)\, w(T(x)) & \text{if } \frac{a+b}{2}-s < x < \frac{a+b}{2}-r\ . \end{cases}$$

Taking into account that $c \le \xi$, we deduce (24). ∎

Remark 2. Theorem 5 agrees with the results of Geronimo and Van Assche. In fact, consider the Borel measures μ_0 and μ induced, respectively, by σ and $\tilde{\sigma}$, and let A be a Borel set in $S \equiv \mathrm{supp}(\mathrm{d}\sigma)$. Then, if $T_1^{-1}(x)$ and $T_2^{-1}(x)$ stand for the two possible inverse functions for appropriate restrictions of $T(x)$, so that $T_1(x) = T(x)$ for $x \in\,]-\infty, \frac{a+b}{2}]$ and $T_2(x) = T(x)$ for $x \in [\frac{a+b}{2}, +\infty[$, then we have

$$\mu(T_i^{-1}(A)) = \int_{T_i^{-1}(A)} \mathrm{d}\tilde{\sigma}(x) = \int_{T_i^{-1}(A)} (-1)^i\, \frac{x-a}{T'(x)}\, \mathrm{d}\sigma(T(x))\,, \qquad i = 1, 2\ ,$$

which leads, by means of the change of variable $t = T_i(x)$ (notice that $T_1(x)$ is decreasing and $T_2(x)$ is increasing), to

$$\text{(25)} \qquad \mu(T_i^{-1}(A)) = \int_A w_i(t)\, \mathrm{d}\mu_0(t), \quad w_i(t) := \frac{T_i^{-1}(t) - a}{T'(T_i^{-1}(t))}\,, \qquad i = 1, 2\ .$$

This was the formula (corresponding to the quadratic case) used by Geronimo and Van Assche to start with the approach presented in [11, p. 561].

Remark 3. One see that, at least for the quadratic case, it is not need to impose, a priori, the restrictions "T with distinct zeros" and "supp(dσ) compact", considered in [11]. Furthermore, in [11] it is assumed the condition $|T(y_i)| \ge 1$ on the zeros y_i of T', which in our case corresponds to the condition $T(\frac{a+b}{2}) \equiv -\frac{\Delta}{4} \notin\,]\xi, \eta[$. We have shown that $c \le \xi$ is a necessary condition for the orthogonality of $\{Q_n\}_{n\ge0}$, which implies $-\frac{\Delta}{4} \le \xi$ (because $c = T(a) \ge -\frac{\Delta}{4}$). Hence $-\frac{\Delta}{4} \notin\,]\xi, \eta[$ must hold necessarily for the orthogonality of $\{Q_n\}_{n\ge0}$.

3 – Problem P2

While for the solution of problem P1 it plays a remarkable role the sequence $\{P_n^*(c;\cdot)\}_{n\ge0}$ of the monic kernel polynomials of K-parameter c corresponding to the sequence $\{P_n\}_{n\ge0}$, for the solution of P2 it is the sequence $\{P_n(\,\cdot\,;\lambda)\}_{n\ge0}$ of the so called co-recursive polynomials that plays the key role. These polynomials, which are defined by the relation

$$P_n(x;\lambda) := P_n(x) - \lambda\, P_{n-1}^{(1)}(x)$$

($\lambda \in \mathbb{C}$), were introduced and studied by T.S. Chihara in [7] (we notice that some generalizations of these co-recursive polynomials were provided by H.A. Slim [24] and F. Marcellán, J.S. Dehesa, A. Ronveaux [15]). The polynomials of the sequence $\{P_n(\,\cdot\,;\lambda)\}_{n\ge 0}$ satisfy the same recurrence relation (3) as $\{P_n\}_{n\ge 0}$, but with different initial conditions, namely,

$$\begin{aligned} &P_{n+1}(x;\lambda) = (x-\beta_n)\, P_n(x;\lambda) - \gamma_n\, P_{n-1}(x;\lambda)\,, \qquad n = 1,2,...,\\ &P_0(x;\lambda) = 1\,, \qquad P_1(x;\lambda) = x - (\beta_0+\lambda)\;. \end{aligned} \tag{26}$$

Therefore, $\{P_n(\cdot;\lambda)\}_{n\ge 0}$ is a MOPS with respect to some linear functional, which we will denote by $\mathbf{u}(\lambda)$.

If λ is real and $\mathbf{u}$ is a positive definite linear functional (and then so is $\mathbf{u}(\lambda)$ as well as the linear functional corresponding to the associated polynomials $\{P_n^{(1)}\}_{n\ge 0}$), then denoting by x_{nj}, $x_{nj}^{(1)}$ and $x_{nj}(\lambda)$ $(j=1,...,n)$ the zeros of P_n, $P_n^{(1)}$ and $P_n(\,\cdot\,;\lambda)$, respectively, ordered in such a way that $x_{n,j} < x_{n,j+1}$, it was stated in [7] that

$$\lambda<0 \;\;\Rightarrow\;\; x_{n,j}(\lambda) < x_{n,j} < x_{n,j}^{(1)} < x_{n,j+1}(\lambda) < x_{n,j+1}, \;\; j=1,...,n-1\;. \tag{27}$$

Therefore, denoting by $[\xi(\lambda),\eta(\lambda)]$ the true interval of orthogonality of $\{P_n(\,\cdot\,;\lambda)\}_{n\ge 0}$, it follows that, for $\lambda<0$, $\xi(\lambda) \le \xi < \eta(\lambda) \le \eta$. Let us prove that $\xi(\lambda) \ge \xi + \lambda$. We recall that, in general, the coefficient of x^{n-1} of the polynomial P_n of an MOPS $\{P_n\}_{n\ge 0}$ satisfying a three-term recurrence relation such as (3) is equal to $-\sum_{j=0}^{n-1}\beta_j$ (cf. [6, p. 19]) and then

$$\sum_{j=1}^{n} x_{nj} = \sum_{j=0}^{n-1}\beta_j\,, \qquad n = 1,2,...\;. \tag{28}$$

Hence, using (28) and the corresponding property for $P_n(\,\cdot\,;\lambda)$, we can write $\sum_{j=1}^{n} x_{nj}(\lambda) = (\beta_0+\lambda) + \sum_{j=1}^{n-1}\beta_j = \lambda + \sum_{j=0}^{n-1}\beta_j = \lambda + \sum_{j=1}^{n} x_{nj}$, so that, taking into account (27),

$$x_{11}(\lambda) = \lambda + x_{11}\,, \qquad x_{n1}(\lambda) = \lambda + x_{n1} + \sum_{j=2}^{n}(x_{nj} - x_{nj}(\lambda)) > \lambda + x_{n1}, \;\; n\ge 2\;.$$

Therefore, $\xi(\lambda) = \lim_{n\to\infty} x_{n1}(\lambda) \geq \lambda + \lim_{n\to\infty} x_{n1} = \lambda + \xi$. We conclude that, in general, the true interval of orthogonality of $\{P_n(\,\cdot\,;\lambda)\}_{n\geq 0}$ is contained in $[\xi+\lambda, \eta]$, if $\lambda < 0$. However, for any λ, it was proved in [7] that the zeros of $P_n(\,\cdot\,;\lambda)$ are all in $]\xi, \eta[$ for all n if and only if

$$\lim_{n\to+\infty} \frac{P_n(\xi)}{P^{(1)}_{n-1}(\xi)} \equiv A \leq \lambda \leq B \equiv \lim_{n\to+\infty} \frac{P_n(\eta)}{P^{(1)}_{n-1}(\eta)} \ ,$$

where A (B) must be replaced by $-\infty$ $(+\infty)$ in case $\xi = -\infty$ $(\eta = +\infty)$.

The co-recursive polynomials are important in order to establish the regularity conditions for a linear functional associated with an inverse polynomial modification of a regular functional (Maroni, [17]). In fact, if $\mathbf{u}$ is regular, for fixed $\lambda, c \in \mathbb{C}$ and being $\mathbf{u}(\lambda,c)$ defined by the distributional equation $(x-c)\,\mathbf{u}(\lambda,c) = -\lambda\,\mathbf{u}$, i.e.,

$$\mathbf{u}(\lambda,c) = u_0\,\delta_c - \lambda(x-c)^{-1}\,\mathbf{u} \ , \tag{29}$$

where δ_c stands for the Dirac measure at the point c and $(x-c)^{-1}\mathbf{u}$ is the linear functional defined by

$$\Big\langle (x-c)^{-1}\,\mathbf{u},\, f \Big\rangle := \Big\langle \mathbf{u},\, \frac{f(x)-f(c)}{x-c} \Big\rangle \ ,$$

it was shown in [17] that $\mathbf{u}(\lambda,c)$ is regular if and only if $\lambda \neq 0$ and $P_n(c;\lambda) \neq 0$ for all $n = 0,1,2,\ldots$. In such conditions, the corresponding MOPS, $\{P_n(\,\cdot\,;\lambda,c)\}_{n\geq 0}$, is given by

$$P_n(x;\lambda,c) := P_n(x) - \frac{P_n(c;\lambda)}{P_{n-1}(c;\lambda)}\,P_{n-1}(x)\,, \qquad n \geq 0 \ .$$

For the set of the coefficients $\{\beta_n(\lambda,c),\ \gamma_{n+1}(\lambda,c)\}_{n\geq 0}$ of the corresponding three-term recurrence relation we have the relations

$$\beta_0(\lambda,c) = \beta_0 + P_1(c;\lambda)\,, \qquad \beta_n(\lambda,c) = \beta_n + \frac{P_{n+1}(c;\lambda)}{P_n(c;\lambda)} - \frac{P_n(c;\lambda)}{P_{n-1}(c;\lambda)} \ , \tag{30}$$

$$\gamma_1(\lambda,c) = \lambda\,P_1(c;\lambda)\,, \qquad \gamma_{n+1}(\lambda,c) = \gamma_n\,\frac{P_{n+1}(c;\lambda)\,P_{n-1}(c;\lambda)}{P^2_n(c;\lambda)} \ , \tag{31}$$

for $n = 1,2,\ldots$.

Theorem 7. *Let $\{P_n\}_{n\geq 0}$ be an MOPS and $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that*

$$Q_2(a) = \lambda\,, \qquad Q_{2n+1}(x) = (x-a)\,P_n(T(x)), \quad n \geq 0 \ ,$$

where $T(x)$ is a monic polynomial of degree 2 and $a, \lambda \in \mathbb{C}$. Without loss of generality, write

$$T(x) = (x-a)\,(x-b) + c\ .$$

Then, $\{Q_n\}_{n\geq 0}$ is a MOPS if and only if

$$\lambda \neq 0\,, \quad P_n(c;\lambda) \neq 0, \quad Q_{2n}(x) = P_n(T(x);\lambda, c)\,, \quad n \geq 0\ . \tag{32}$$

In such conditions, if $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (3), then the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ for the corresponding three-term recurrence relation for $\{Q_n\}_{n\geq 0}$ can be determined according to the relations

$$\tilde{\beta}_{2n} = a, \quad \tilde{\beta}_{2n+1} = b\,, \quad n \geq 0\ , \tag{33}$$

$$\tilde{\gamma}_1 = -\lambda, \quad \tilde{\gamma}_{2n} = -\frac{P_n(c;\lambda)}{P_{n-1}(c;\lambda)}, \quad \tilde{\gamma}_{2n+1} = -\gamma_n \frac{P_{n-1}(c;\lambda)}{P_n(c;\lambda)}\,, \quad n \geq 1\ . \tag{34}$$

Moreover, if $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the linear functional $\mathbf{u}$, then $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a linear functional $\mathbf{v}$ defined on the basis $\{T^n(x), (x-a)T^n(x)\}_{n\geq 0}$ of $\mathbb{P}$ by the relations

$$\langle \mathbf{v}, T^n(x)\rangle = \langle \mathbf{u}(\lambda, c), x^n\rangle, \quad \langle \mathbf{v}, (x-a)\,T^n(x)\rangle = 0\,, \quad n \geq 0\ . \tag{35}$$

Proof: Expand the polynomial T as

$$T(x) = x^2 + p\,x + q\,, \quad p := -(a+b), \quad q := ab + c\ .$$

Assume that $\{Q_n\}_{n\geq 0}$ is a MOPS. Thus, it satisfies a three-term recurrence relation

$$\begin{aligned} &x\,Q_n(x) = Q_{n+1}(x) + \tilde{\beta}_n\,Q_n(x) + \tilde{\gamma}_n\,Q_{n-1}(x)\,, \quad n = 1, 2, \ldots\ , \\ &Q_0(x) = 1\,, \quad Q_1(x) = x - \tilde{\beta}_0\ , \end{aligned} \tag{36}$$

with $\tilde{\gamma}_n \neq 0$ for $n \geq 1$. It is clear that $\tilde{\beta}_0 = a$ and $\tilde{\gamma}_1 = -\lambda$. Then $\lambda \neq 0$. In the three-term recurrence relation (3) for $\{P_n\}_{n\geq 0}$ replace x by $x^2 + px + q$ and then multiply by $x - a$, so that

$$(x^2 + px + q)\,Q_{2n+1}(x) = Q_{2n+3}(x) + \beta_n Q_{2n+1}(x) + \gamma_n Q_{2n-1}(x)\,, \quad n \geq 0\ . \tag{37}$$

Now, use successively (36) to expand $x\,Q_{2n+1}(x)$ and $x^2\,Q_{2n+1}(x)$ as a linear combination of the $Q_i(x)$ and then substitute the obtained expressions in the left hand side of (37). This yields a linear combination of elements of the sequence

$\{Q_n\}_{n\geq 0}$ which vanishes identically. Therefore, since this sequence is a basis for $\mathbb{P}$, we find the following relations:

$$\tilde{\beta}_{2n+2} + \tilde{\beta}_{2n+1} + p = 0\,, \quad n \geq 0\;, \tag{38}$$

$$\tilde{\gamma}_{2n+2} + \tilde{\beta}^2_{2n+1} + \tilde{\gamma}_{2n+1} + p\,\tilde{\beta}_{2n+1} + q = \beta_n\,, \quad n \geq 0\;, \tag{39}$$

$$\tilde{\beta}_{2n+1} + \tilde{\beta}_{2n} + p = 0\,, \quad n \geq 0\;, \tag{40}$$

$$\tilde{\gamma}_{2n}\,\tilde{\gamma}_{2n+1} = \gamma_n\,, \quad n \geq 1\;. \tag{41}$$

Combining (40) (after the change of indices $n \to n+1$) with (38) it leads to $\tilde{\beta}_{2n+3} = \tilde{\beta}_{2n+1}$ for $n \geq 0$, so that $\tilde{\beta}_{2n+1} = \tilde{\beta}_1 = b$ $(n \geq 0)$ and, consequently, from (38), $\tilde{\beta}_{2n} = -b - p = a$ $(n \geq 0)$. Hence, we can rewrite (39) as $\tilde{\gamma}_{2n+2} + \tilde{\gamma}_{2n+1} = \beta_n - c$ $(n \geq 0)$, or, according to (41),

$$\tilde{\gamma}_{2n+2} + \frac{\gamma_n}{\tilde{\gamma}_{2n}} = \beta_n - c\,, \quad n \geq 1\;. \tag{42}$$

Now, define recurrently a sequence $\{y_n\}_{n\geq 0}$ by

$$y_0 = 1\,, \quad y_{n+1} = -\tilde{\gamma}_{2n+2}\,y_n, \quad n \geq 0\;.$$

Remark that $y_n \neq 0$ for all $n \geq 0$, hence $\tilde{\gamma}_{2n+2} = -y_{n+1}/y_n$ for $n \geq 0$; if we substitute in (42), we can deduce

$$y_{n+1} = (c - \beta_n)\,y_n - \gamma_n\,y_{n-1}\,, \quad n \geq 1\;. \tag{43}$$

For $n = 0$, we have $y_1 = -\tilde{\gamma}_2 = -(\beta_0 - c - \tilde{\gamma}_1) = -(\beta_0 - c + \lambda)$, so that

$$y_0 = 1\,, \quad y_1 = c - (\beta_0 + \lambda)\;. \tag{44}$$

It follows from (43) and (44) that $y_n = P_n(c;\lambda)$ for $n \geq 0$ and, therefore,

$$P_n(c;\lambda) \neq 0\,, \quad n \geq 0\;.$$

Furthermore, we have

$$\tilde{\gamma}_{2n+2} = -\frac{y_{n+1}}{y_n} = -\frac{P_{n+1}(c;\lambda)}{P_n(c;\lambda)}, \quad \tilde{\gamma}_{2n+3} = \frac{\gamma_{n+1}}{\tilde{\gamma}_{2n+2}} = -\gamma_{n+1}\frac{P_n(c;\lambda)}{P_{n+1}(c;\lambda)}\,, \quad n \geq 0\;.$$

Now, using these relations, the change of indices $n \to 2n$ in the recurrence relation (36) yields

$$(x-a)\,Q_{2n}(x) = (x-a)\,P_n(T(x)) - \frac{P_n(c;\lambda)}{P_{n-1}(c;\lambda)}\,(x-a)\,P_{n-1}(T(x))\,, \quad n \geq 0\,,$$

and then the expression for $Q_{2n}(x)$ as in (32) follows. Thus, conditions (32) are necessary for the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$. We also have proved that relations (33) and (34) hold.

Conversely, it is straightforward to verify that conditions (32) are sufficient in order to guarantee that $\{Q_n\}_{n\geq 0}$ be a MOPS. For that, define complex numbers $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ by formulas (33) and (34). These parameters are well defined, according to the hypothesis that $P_n(c;\lambda)\neq 0$ for all $n\geq 0$, and, in addition, $\tilde{\gamma}_n\neq 0$ for all $n\geq 1$. Then $\{Q_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (36), so that, by Favard's theorem, it is a MOPS.

Finally, if conditions (32) are satisfied, we prove that $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the linear functional $\mathbf{v}$ defined by (35). For this, since $\{Q_n\}_{n\geq 0}$ is a MOPS, it is sufficient to show that

$$\langle \mathbf{v},1\rangle \neq 0\,, \qquad \langle \mathbf{v},Q_n\rangle = 0, \quad n\geq 1\,.$$

If we set $n=0$ in the first relation of (35), then $\langle \mathbf{v},1\rangle = \langle \mathbf{u}(\lambda,c),1\rangle \neq 0$ follows because $\mathbf{u}(\lambda,c)$ is regular. Next, notice that

$$\Big\langle \mathbf{v},\,(x-a)\,f(T(x))\Big\rangle = 0\,, \qquad \Big\langle \mathbf{v},f(T(x))\Big\rangle = \Big\langle \mathbf{u}(\lambda,c),\,f(x)\Big\rangle\,,$$

for all $f\in\mathbb{P}$. Hence $\langle \mathbf{v},Q_{2n+1}(x)\rangle = \langle \mathbf{v},(x-a)P_n(T(x))\rangle = 0$ and $\langle \mathbf{v},Q_{2n+2}(x)\rangle = \langle \mathbf{v},P_{n+1}(T(x);\lambda,c)\rangle = \langle \mathbf{u}(\lambda,c),P_{n+1}(x;\lambda,c)\rangle = 0$ holds for $n=0,1,2,\ldots$, which completes the proof. ∎

Corollary 8. *Under the conditions of Theorem 7, the coefficients of the three-term recurrence relation verified by the MOPS's $\{P_n\}_{n\geq 0}$, $\{P_n(\,\cdot\,;\lambda,c)\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are related by*

$$\begin{aligned}
&\beta_0(\lambda,c) = \tilde{\gamma}_1 + c\,, \qquad \beta_n(\lambda,c) = \tilde{\gamma}_{2n+1}+\tilde{\gamma}_{2n}+c, \quad n\geq 1\,,\\
&\gamma_n(\lambda,c) = \tilde{\gamma}_{2n-1}\,\tilde{\gamma}_{2n}\,, \qquad n\geq 1\,,\\
&\beta_n = \tilde{\gamma}_{2n+1}+\tilde{\gamma}_{2n+2}+c\,, \qquad n\geq 0\,,\\
&\gamma_n = \tilde{\gamma}_{2n}\,\tilde{\gamma}_{2n+1}\,, \qquad n\geq 1\,.
\end{aligned} \tag{45}$$

In order to provide an answer for question c) in P2, we establish some preliminar lemmas.

Lemma 9. *Under the conditions of Theorem 7,*

$$Q^{(1)}_{2n-1}(x) = (x-b)\,P^{(1)}_{n-1}(T(x);\lambda,c)\,, \qquad n=1,2,\ldots\,.$$

Proof: We begin by putting $P_n(x;\lambda,c) \equiv \sum_{i=0}^n a_i^{(n)} x^i$ and then, mutatis mutandis, we follow the same steps as in the proof of Lemma 3. ∎

Lemma 10. *Under the conditions of Lemma 4, if $c \le \xi$ and $\int_\xi^\eta \frac{\mathrm{d}\sigma(x)}{|x-c|} < +\infty$, then*

$$\int_{T^{-1}(]\xi,\eta[)} x^{2n} \frac{\mathrm{d}\sigma(T(x))}{|x-a|\,T'(x)} < +\infty\,, \qquad n = 0,1,2,\dots\,. \tag{46}$$

Proof: First, assume $c < \xi$. Then, we can write

$$\begin{aligned}\int_{T^{-1}(]\xi,\eta[)} x^{2n} \frac{\mathrm{d}\sigma(T(x))}{|x-a|\,T'(x)} &= \int_{T^{-1}(]\xi,\eta[)} \frac{x^{2n}}{|T(x)-c|} \frac{|x-b|}{T'(x)}\,\mathrm{d}\sigma(T(x))\\ &\le \frac{1}{\xi-c} \int_{T^{-1}(]\xi,\eta[)} x^{2n} \frac{|x-b|}{T'(x)}\,\mathrm{d}\sigma(T(x)) \;<\; +\infty\end{aligned}$$

where the last inequality follows from Lemma 4, interchanging the roles of a and b. Suppose now that $c = \xi$. Since we can write $x^{2n} = \sum_j [a_{nj} + b_{nj}(x-a)]\,T^j(x)$, to prove (46) it is sufficient to show that

$$\int_{T^{-1}(]\xi,\eta[)} |T(x)|^n \frac{\mathrm{d}\sigma_T(x)}{|x-a|\,T'(x)} < +\infty \tag{47}$$

and

$$\int_{T^{-1}(]\xi,\eta[)} |T(x)|^n \frac{\mathrm{d}\sigma_T(x)}{T'(x)} < +\infty \tag{48}$$

for all $n=0,1,2,\dots$. In fact, using

$$\frac{1}{|T(x)-c|}\left(1 \pm \frac{a-b}{2\sqrt{T(x)+\Delta/4}}\right) = \frac{2}{(x-a)\,T'(x)}\,, \qquad x \in T^{-1}(]\xi,\eta[)\;,$$

where we take the sign $+$ if $x > \frac{a+b}{2}$ and the sign $-$ if $x < \frac{a+b}{2}$, we can verify that

$$\int_{T^{-1}(]\xi,\eta[)} |T(x)|^n \frac{\mathrm{d}\sigma(T(x))}{|x-a|\,T'(x)} \;\le\; 2\left(1+\frac{|b-a|}{2r}\right) \int_\xi^\eta \frac{|y|^n}{|y-c|}\,\mathrm{d}\sigma(y) \;<+\infty\;.$$

The last integral is finite since, for each fixed n, we can find nonnegative numbers k_j ($j=0,\dots,n+1$) such that $|y|^n/|y-c| \le k_0/|y-c| + \sum_{j=0}^n k_{j+1}|y|^j$, and because $\int_\xi^\eta \frac{\mathrm{d}\sigma(x)}{|x-c|} < +\infty$. Hence (47) is proved. To prove (48), observe first that if $\xi > -\frac{\Delta}{4}$ then $a \ne b$, hence $1/|T'(x)| = 1/2\sqrt{T(x)+\Delta/4} \le 1/2r$, so that

$$\int_{T^{-1}(]\xi,\eta[)} |T(x)|^n \frac{\mathrm{d}\sigma(T(x))}{T'(x)} \;\le\; \frac{1}{r}\int_\xi^\eta |y|^n\,\mathrm{d}\sigma(y) \;<+\infty\;;$$

and, if $\xi=-\frac{\Delta}{4}$ then $c=-\frac{\Delta}{4}$ and $a=b$, hence $(x-a)\,T'(x)=2[T(x)-c]$ and $|T(x)|^n|x-a|\leq(|T(x)|^{2n}+|x-a|^2)/2=(|T(x)|^{2n}+|T(x)-c|)/2$, so that

$$\int_{T^{-1}(]\xi,\eta[)}|T(x)|^n\,\frac{\mathrm{d}\sigma(T(x))}{T'(x)}=\int_{T^{-1}(]\xi,\eta[)}|T(x)|^n\,|x-a|\,\frac{\mathrm{d}\sigma(T(x))}{|x-a|\,T'(x)}$$
$$\leq\frac{1}{2}\int_{\xi}^{\eta}\left(1+\frac{|y|^{2n}}{|y-c|}\right)\mathrm{d}\sigma(y)\;<+\infty\;.\;\blacksquare$$

Remark 4. If $c<\xi$, then the condition $\int_\xi^\eta\frac{\mathrm{d}\sigma(x)}{|x-c|}<+\infty$ in (46) holds.

Theorem 11. *Let $\{P_n\}_{n\geq0}$ be a MOPS with respect to some uniquely determined distribution function $\mathrm{d}\sigma(x)$ and let $[\xi,\eta]$ be the true interval of orthogonality of $\{P_n\}_{n\geq0}$, with $-\infty<\xi\leq+\infty$. Let a and λ be fixed real numbers, $T(x)\equiv(x-a)(x-b)+c$ a real polynomial of degree two and put $\Delta:=(b-a)^2-4c$. Let $\{Q_n\}_{n\geq0}$ be a sequence of polynomials such that*

$$Q_2(a)=\lambda\,,\qquad Q_{2n+1}(x)=(x-a)\,P_n(T(x))$$

for all $n=0,1,2,\ldots$. Assume that one of the following two conditions hold:

(**i**) $c\leq\xi+\lambda$;

(**ii**) $c\leq\xi,\;\;-\infty<\lim\limits_{n\to+\infty}\dfrac{P_n(\xi)}{P^{(1)}_{n-1}(\xi)}\equiv A\leq\lambda\leq B\equiv\lim\limits_{n\to+\infty}\dfrac{P_n(\eta)}{P^{(1)}_{n-1}(\eta)}$

(with $B\equiv+\infty$ if $\eta=+\infty$).

Then, $\{Q_n\}_{n\geq0}$ is a MOPS with respect to a positive definite linear functional if and only if

$$\lambda<0\,,\qquad Q_{2n}(x)=P_n(T(x);\lambda,c)\;,\tag{49}$$

hold for all $n=0,1,2,\ldots$.

In these conditions, $\{Q_n\}_{n\geq0}$ is orthogonal with respect to the uniquely determined distribution function $\tilde\sigma$ defined as

$$\mathrm{d}\tilde\sigma(x)=M\,\delta_a(x)-\frac{\lambda}{|x-a|}\,\frac{\mathrm{d}\sigma(T(x))}{T'(x)}\,,\qquad r<|x-\tfrac{a+b}{2}|<s\;,\tag{50}$$

where

$$M:=u_0+\lambda\,F(c;\sigma)\geq0\,,\qquad r:=\sqrt{\xi+\tfrac{\Delta}{4}}\,,\qquad s:=\sqrt{\eta+\tfrac{\Delta}{4}}\;.$$

Proof: Suppose that $\{Q_n\}_{n\geq0}$ is a MOPS with respect to a positive definite linear functional. It follows immediately from Theorem 7 that $\lambda=-\tilde\gamma_1<0$ and $Q_{2n+1}(x)$ is given as in (49).

Conversely, assume that conditions (49) hold. We will show that $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a positive definite linear functional if (i) or (ii) hold. Assume that (i) holds. Since, for $\lambda<0$, the zeros of $P_n(\,\cdot\,;\lambda)$ belong to $]\xi+\lambda,\eta[$, then the condition $c\leq\xi+\lambda$ implies that $P_n(c;\lambda)\neq 0$ for all $n=0,1,2,\ldots$. Hence, from Theorem 7, $\{Q_n\}_{n\geq 0}$ is a MOPS. Now, it is obvious from (33) that $\tilde{\beta}_n$ is real for every $n=0,1,2,\ldots$ and, using again the fact that all the zeros of $P_n(\,\cdot\,;\lambda)$ are in $]\xi+\lambda,\eta[$, so that $\operatorname{sgn}P_n(x;\lambda)=(-1)^n$ for $x\leq\xi+\lambda$, then $P_n(c;\lambda)/P_{n-1}(c;\lambda)<0$ holds and (34) gives $\tilde{\gamma}_n>0$ for all $n=1,2,\ldots$. Thus, $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a positive definite linear functional in the case (i). Assume now that (ii) holds. Then, the zeros of $P_n(\,\cdot\,;\lambda)$ are in $]\xi,\eta[$, and so the condition $c\leq\xi$ implies that $P_n(c;\lambda)\neq 0$ for all $n=0,1,2,\ldots$, hence $\{Q_n\}_{n\geq 0}$ is a MOPS. As before, we can show that $\tilde{\beta}_n$ is real and $\tilde{\gamma}_n>0$ for all n, so that $\{Q_n\}_{n\geq 0}$ is a MOPS in the positive definite sense, also in situation (ii).

It remains to show that the distribution function, $\tilde{\sigma}$, with respect to which $\{Q_n\}_{n\geq 0}$ is orthogonal is given by (50). As for problem P1, one can easily show that $\tilde{\sigma}$ is uniquely determined. Now, observe that, since $\lambda<0$, condition (i) implies that $c<\xi$. On the other hand, conditions (ii) imply $\int_\xi^\eta \frac{\mathrm{d}\sigma(t)}{|t-c|}<+\infty$. In fact, by hypothesis $-\infty<\lim_{n\to+\infty}\frac{P_n(\xi)}{P_{n-1}^{(1)}(\xi)}\equiv A\leq\lambda<0$, then $\lim_{n\to+\infty}\frac{P_{n-1}^{(1)}(\xi)}{P_n(\xi)}\equiv\frac{1}{A}$, hence there exists the integral $\int_{-\infty}^{+\infty}\frac{\mathrm{d}\sigma(t)}{t-\xi}\equiv F(\xi;\sigma)=-\frac{u_0}{A}$, so that $\int_\xi^\eta\frac{\mathrm{d}\sigma(t)}{|t-c|}=\int_\xi^\eta\frac{\mathrm{d}\sigma(t)}{t-c}\leq\int_\xi^\eta\frac{\mathrm{d}\sigma(t)}{t-\xi}<+\infty$. We have shown that the hypotheses of Lemma 10 are verified, in both situations (i) and (ii). Thus, we can define a distribution function, σ_1, such that

$$\mathrm{d}\sigma_1(x):=\frac{\mathrm{d}\sigma(T(x))}{|x-a|\,T'(x)}\ ,$$

with $\operatorname{supp}(\mathrm{d}\sigma_1)\subset T^{-1}([\xi,\eta])$. Let's show that

$$(51)\qquad F(z;\sigma_1)=\frac{1}{z-a}\Big[F(T(z);\sigma)-F(c;\sigma)\Big],\qquad z\in\mathbb{C}\backslash T^{-1}(]\xi,\eta[)\ .$$

In fact, with the notations of the proof of Lemma 4, for $z\in\mathbb{C}\backslash T^{-1}(]\xi,\eta[)$ we can write

$$(52)\quad F(z;\sigma_1)=\int_{-\infty}^{+\infty}\frac{1}{t-z}\,d\sigma_1(t)=\int_{-\infty}^{+\infty}\frac{1}{t-z}\,\frac{\mathrm{d}\sigma(T(t))}{|t-a|\,T'(t)}=I_1(z)+I_2(z)$$

where

$$I_1(z):=\int_{\frac{a+b}{2}+r}^{\frac{a+b}{2}+s}\frac{\mathrm{d}\sigma(T(t))}{(t-z)\,(t-a)\,T'(t)}\ ,\qquad I_2(z):=-\int_{\frac{a+b}{2}-s}^{\frac{a+b}{2}-r}\frac{\mathrm{d}\sigma(T(t))}{(t-z)\,(t-a)\,T'(t)}\ .$$

Now, by hypothesis, we have $\xi \geq c \geq -\Delta/4$. Hence, if $\xi > -\Delta/4$ then also $y > -\Delta/4$ for $y \in [\xi, \eta]$, and since for $y > -\Delta/4$ it holds

$$\begin{aligned}\frac{1}{(z-a)\,(y-T(z))} &= \frac{1}{z-a}\,\frac{1}{y-c} \\ &\quad - \frac{1}{2\left(z-\frac{a+b}{2}-\sqrt{y+\Delta/4}\right)\left(\frac{b-a}{2}+\sqrt{y+\Delta/4}\right)\sqrt{y+\Delta/4}} \\ &\quad + \frac{1}{2\left(z-\frac{a+b}{2}+\sqrt{y+\Delta/4}\right)\left(\frac{b-a}{2}-\sqrt{y+\Delta/4}\right)\sqrt{y+\Delta/4}}\ ,\end{aligned}$$

integrating both sides of this equality with respect to $\sigma(y)$ over $[\xi, \eta]$ we get

$$\frac{1}{z-a}\int_\xi^\eta \frac{d\sigma(y)}{y-T(z)} = \frac{1}{z-a}\int_\xi^\eta \frac{d\sigma(y)}{y-c} + I_1(z) + I_2(z)\ . \tag{53}$$

We have used the relations

$$T'(t) = 2\,\mathrm{sign}(T'(t))\,\sqrt{T(t)+\Delta/4}\,, \qquad t-a = \tfrac{b-a}{2} + \tfrac{1}{2}\,T'(t)\ ,$$

$$t - z = -(z - \tfrac{a+b}{2} - \tfrac{1}{2}\,T'(t))\,, \qquad T(\tfrac{a+b}{2} \pm r) = \xi\,, \qquad T(\tfrac{a+b}{2} \pm s) = \eta\ ,$$

as well as the substitution $y = T(t)$ in the above integrals I_1 and I_2 in order to show that

$$I_1(z) = -\frac{1}{2}\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{\left(z-\frac{a+b}{2}-\sqrt{y+\Delta/4}\right)\left(\frac{b-a}{2}+\sqrt{y+\Delta/4}\right)\sqrt{y+\Delta/4}}\ ,$$

$$I_2(z) = \frac{1}{2}\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{\left(z-\frac{a+b}{2}+\sqrt{y+\Delta/4}\right)\left(\frac{b-a}{2}-\sqrt{y+\Delta/4}\right)\sqrt{y+\Delta/4}}\ .$$

Therefore, (51) follows from (52) and (53), in case $\xi > -\Delta/4$. Consider now the other possible case, $\xi = -\Delta/4$. Then, also $c = -\Delta/4$, so that $a = b$, $(t-a)\,T'(t) = 2[T(t)-c]$, $t - z = a - z + \sqrt{T(t)-c}$ if $t > a$ and $t - z = a - z - \sqrt{T(t)-c}$ if $t < a$. Hence, it follows directly from the definition of I_1 and I_2, again after the change of variable $y = T(t)$,

$$I_1(z) = -\frac{1}{2}\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{(z-a-\sqrt{y-c})\,(y-c)}\,, \qquad I_2(z) = -\frac{1}{2}\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{(z-a+\sqrt{y-c})\,(y-c)}$$

and, consequently,

$$\begin{aligned}F(z;\sigma_1) = I_1(z) + I_2(z) &= -\frac{1}{2}\int_\xi^\eta \left(\frac{1}{z-a-\sqrt{y-c}} + \frac{1}{z-a+\sqrt{y-c}}\right)\frac{\mathrm{d}\sigma(y)}{y-c} = \\ &= (z-a)\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{(y-T(z))\,(y-c)} = \frac{1}{z-a}\left[\int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{y-T(z)} - \int_\xi^\eta \frac{\mathrm{d}\sigma(y)}{y-c}\right]\ ,\end{aligned}$$

which completes the proof of (51). Now, we proceed by finding the relationship between the Stieltjes functions $F(\cdot\,;\sigma)$ and $F(\cdot\,;\tilde\sigma)$. By Markov Theorem and Lemma 9 we can write

$$F(z;\tilde\sigma) = (z-b)\,F(T(z);\sigma_{\lambda,c})\ ,$$

where $\sigma_{\lambda,c}$ is the distribution function with respect to which the sequence $\{P_n(\cdot\,;\lambda,c)\}_{n\geq 0}$ is orthogonal, and since (Marcellán [14, p. 116])

$$F(z;\sigma_{\lambda,c}) = -\frac{1}{z-c}\,[u_0+\lambda\,F(z;\sigma)]\ ,$$

we obtain

$$F(z;\tilde\sigma) = -\frac{u_0}{z-a} - \frac{\lambda}{z-a}\,F(T(z);\sigma) = -\frac{u_0+\lambda\,F(c;\sigma)}{z-a} - \lambda\,F(z;\sigma_1)\ . \tag{54}$$

One see that $\tilde\sigma$ has, possibly, a mass point at $x=a$, with jump $M:=u_0+\lambda F(c;\sigma)$. Combining (54) with (51), we obtain (50). Finally, we show that the jump M is nonnegative. In fact, in situation (i), we have $0<1/(t-c)\leq 1/(\xi-c)$, for $t\in[\xi,\eta]$, so that

$$0\leq\int_\xi^\eta \frac{1}{t-c}\,\mathrm{d}\sigma(t)\leq\int_\xi^\eta\frac{1}{\xi-c}\,\mathrm{d}\sigma(t)\ \Longrightarrow\ 0\leq F(c;\sigma)\leq\frac{u_0}{\xi-c}=-\frac{u_0}{\lambda}\,\frac{-\lambda}{\xi-c}\leq-\frac{u_0}{\lambda}$$

(notice that $0<\frac{-\lambda}{\xi-c}\leq 1$ because it holds $0<-\lambda\leq\xi-c$), hence $u_0+\lambda F(c;\sigma)\geq 0$; and, in situation (ii), we deduce $F(c;\sigma)\leq F(\xi;\sigma)=-u_0/A\leq -u_0/\lambda$ (remark that $-\infty<A\leq\lambda<0$) and so, again, $u_0+\lambda F(c;\sigma)\geq 0$.

Remark 5. One can deduce that $\mathrm{d}\tilde\sigma$ has a mass point at $x=a$, with jump $M\equiv u_0+\lambda F(c;\sigma)$, by showing that

$$\Big[\sum_{n=0}^{+\infty} q_n^2(a)\Big]^{-1} = M\ ,\qquad q_n(x):=\Big(v_0\prod_{i=1}^{n}\tilde\gamma_i\Big)^{-1/2}Q_n(x)\ .$$

In fact, using the well known identity $P_n(c)\,P^{(1)}_{n-2}(c)-P_{n-1}(c)\,P^{(1)}_{n-1}(c)=-\prod_{i=1}^{n-1}\gamma_i$ $(n\geq 1)$, it is straightforward to verify that $P_n(c)\,P_{n-1}(c;\lambda)-P_n(c;\lambda)\,P_{n-1}(c)=\lambda\prod_{i=1}^{n-1}\gamma_i$ $(n\geq 1)$, and so we easily get $Q_{2n}(a)=P_n(c;\lambda,c)=\lambda\prod_{i=1}^{n-1}\gamma_i/P_{n-1}(c;\lambda)$, $Q_{2n+1}(a)=0$. Therefore, since, by Theorem 7, $\prod_{i=1}^{2n}\tilde\gamma_i=$

$\lambda \prod_{i=1}^{n-1} \gamma_i \, P_n(c;\lambda)/P_{n-1}(c;\lambda)$, we can write

$$\begin{aligned}
\sum_{n=0}^{+\infty} q_n^2(a) &= \frac{1}{v_0} + \sum_{n=1}^{+\infty} \frac{Q_{2n}^2(a)}{v_0 \prod_{i=1}^{2n} \tilde{\gamma}_i} = \frac{1}{u_0} + \frac{1}{u_0} \sum_{n=1}^{+\infty} \frac{\lambda \prod_{i=1}^{n-1} \gamma_i}{P_{n-1}(c;\lambda)\, P_n(c;\lambda)} = \\
&= \frac{1}{u_0} + \frac{1}{u_0} \sum_{n=1}^{+\infty} \left(\frac{P_n(c)}{P_n(c;\lambda)} - \frac{P_{n-1}(c)}{P_{n-1}(c;\lambda)} \right) = \frac{1}{u_0} \lim_{n\to+\infty} \frac{P_n(c)}{P_n(c;\lambda)} = \\
&= \frac{1}{u_0} \lim_{n\to+\infty} \frac{P_n(c)}{P_n(c) - \lambda\, P_{n-1}^{(1)}(c)} = \left[u_0 \left(1 - \lambda \lim_{n\to+\infty} \frac{P_{n-1}^{(1)}(c)}{P_n(c)} \right) \right]^{-1} ,
\end{aligned}$$

hence, if $1 \neq \lambda \lim P_{n-1}^{(1)}(c)/P_n(c)$, then there exists a mass point at $x = a$, with jump $u_0[1 - \lambda \lim P_{n-1}^{(1)}(c)/P_n(c)] = u_0 + \lambda\, F(c;\sigma)$.

Remark 6. In Theorem 3 in [11] is described how the Stieltjes transforms associated with $\sigma^{(1)}$ and $\tilde{\sigma}^{(1)}$, the orthogonality measures for $\{P_n^{(1)}\}_{n\geq 0}$ and $\{Q_n^{(1)}\}_{n\geq 0}$, respectively, are related. Our result gives a direct proof of the relation between σ and $\tilde{\sigma}$.

Corollary 12. *Under the conditions of Theorem 11, if σ is absolutely continuous, i.e., $\mathrm{d}\sigma(x) = w(x)\,\mathrm{d}x$ and w is supported on the interval $[\xi,\eta]$, then*

$$\mathrm{d}\tilde{\sigma}(x) = M\delta_a(x)\,\mathrm{d}x - \frac{\lambda}{|x-a|}\, w(T(x))\,\mathrm{d}x \tag{55}$$

with

$$\mathrm{supp}(\mathrm{d}\tilde{\sigma}) \subset \left[\tfrac{a+b}{2} - s,\ \tfrac{a+b}{2} - r\right] \cup \{a\} \cup \left[\tfrac{a+b}{2} + r,\ \tfrac{a+b}{2} + s\right] .$$

4 – Examples

As an application of Theorem 11, we can find examples of families of orthogonal polynomials $\{Q_n\}_{n\geq 0}$ such that there exist mass points located at the interior of the support of the absolutely continuous part of the corresponding measure. This fact was observed by Geronimo and Van Assche [11, p. 580] by starting with some families of Jacobi polynomials. We also refer that, in a very interesting work [5], J. Charris, G. Salas and V. Silva studied another family of orthogonal polynomials, $\{R_n\}_{n\geq 0}$, for which this fact appears. Consider

$$x\, R_{2n} = R_{2n+1} + \frac{n+2}{4(n+1)}\, R_{2n-1}\,, \qquad x\, R_{2n+1} = R_{2n+2} + \frac{n+1}{4(n+2)}\, R_{2n}$$

$(n \geq 0)$, with initial conditions $R_{-1} = 0$ and $R_0 = 1$. They found that $\{R_n\}_{n\geq 0}$ is orthogonal with respect to $\mathrm{d}\mu := \frac{1}{\pi}\sqrt{1-x^2}\,\mathrm{d}x + \frac{1}{2}\,\delta_0(x)\,\mathrm{d}x,\ -1 \leq x \leq 1$. We will find this sequence $\{R_n\}_{n\geq 0}$ in our next example 1, for the particular choice $\alpha = 1/2,\ \lambda = -1/8$.

Example 1. Consider the sequence $\{U_n\}_{n\geq 0}$ of the Chebyshev polynomials of second kind, $U_n(\cos\theta) := \sin(n+1)\theta/\sin\theta$ $(n \geq 0)$, which is orthogonal (in the positive definite sense) with respect to $2\pi^{-1}\sqrt{1-x^2}\,\mathrm{d}x$, $-1 \leq x \leq 1$. The monic polynomials $\{\widehat{U}_n\}_{n\geq 0}$ corresponding to $\{U_n\}_{n\geq 0}$ are given by $\widehat{U}_n = 2^{-n}\,U_n(x)$ and satisfy the three-term recurrence relation $x\,\widehat{U}_n = \widehat{U}_{n+1} + \frac{1}{4}\,\widehat{U}_{n-1}$ $(n \geq 1)$, with initial conditions $\widehat{U}_0 = 1,\ \widehat{U}_1 = x$. Let $\{P_n\}_{n\geq 0}$ be a MOPS defined by $P_n(x) \equiv \alpha^n\,\widehat{U}_n(\frac{x}{\alpha})$, with $\alpha > 0$. Therefore, we easily check that $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (3), with $\beta_n := 0,\ \gamma_{n+1} := \alpha^2/4$ $(n = 0, 1, \ldots)$. Let $\lambda \in \mathbb{R}$ and $\{Q_n\}_{n\geq 0}$ a sequence of monic polynomials such that

$$(56) \quad Q_2(0) := \lambda, \quad Q_{2n+1}(x) := x\,P_n(x^2-\alpha) \equiv \alpha^n\,x\,\widehat{U}_n\Big(\frac{x^2-\alpha}{\alpha}\Big), \quad n = 0, 1, 2, \ldots .$$

Hence, we have $T(x) = x^2 - \alpha,\ a = b = 0,\ c = -\alpha$ (with the notations of Theorem 7). Using $\widehat{U}_n(1) = (n+1)/2^n = (-1)^n\,\widehat{U}_n(-1)$ and $U_n^{(1)} \equiv U_n$, we see that $P_n(-\alpha;\lambda) = \alpha^n\,\widehat{U}_n(-1) - \lambda\,\alpha^{n-1}\,\widehat{U}_{n-1}(-1) = (-\alpha/2)^n\,[1 + n(1+2\lambda/\alpha)]$. Then, by Theorem 7, $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a quasi-definite linear functional if and only if

$$\lambda \neq 0\,, \qquad 1 + n(1+2\lambda/\alpha) \neq 0\,, \qquad Q_{2n}(x) = \alpha^n\,\widehat{U}_n\Big(\frac{x^2-\alpha}{\alpha};\lambda,-\alpha\Big)\,,$$

hold for all $n = 0, 1, 2, \ldots$. In this case, the coefficients of the corresponding three-term recurrence relation are explicitly given by

$$\tilde{\beta}_n = 0\,, \quad \tilde{\gamma}_{2n+1} = \frac{\alpha}{2}\,\frac{1+(n-1)\,(1+2\lambda/\alpha)}{1+n(1+2\lambda/\alpha)}\,, \quad \tilde{\gamma}_{2n+2} = \frac{\alpha}{2}\,\frac{1+(n+1)\,(1+2\lambda/\alpha)}{1+n(1+2\lambda/\alpha)}\,,$$

for all $n = 0, 1, \ldots$.

In order to determine conditions for $\{Q_n\}_{n\geq 0}$ to be orthogonal in the positive-definite sense, notice first that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $\mathrm{d}\sigma(x) := w(x)\,\mathrm{d}x$, where w is the weight function

$$w(x) := \frac{2}{\pi\alpha^2}\,\sqrt{\alpha^2-x^2}\,, \qquad -\alpha \leq x \leq \alpha\ .$$

The corresponding Stieltjes function is given by $F(z;\sigma) = -2\,\alpha^{-2}(z - \sqrt{z^2-\alpha^2})$, where the square root is such that $|z + \sqrt{z^2-\alpha^2}| > \alpha$ whenever $z \in \mathbb{C}\backslash[-\alpha,\alpha]$

(cf. [26, p. 176]). Thus, $u_0 \equiv \int_{-\alpha}^{\alpha} \mathrm{d}\sigma(t) = 1$, $F(-\alpha;\sigma) = 2/\alpha$ and (with the notations of Theorem 11) $A = -\alpha/2$, $B = \alpha/2$, $\xi = -\eta = -\alpha$. Consequently, by Theorem 11, $\{Q_n\}_{n\geq 0}$ satisfying (56) is an MOPS in the positive definite sense if

$$-\frac{\alpha}{2} \leq \lambda < 0\,, \qquad Q_{2n}(x) = \alpha^n\, \widehat{U}_n\Big(\frac{x^2-\alpha}{\alpha};\lambda,-\alpha\Big), \quad n = 0,1,2,\ldots\,.$$

In such conditions, according to Corollary 12, $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the distribution function

$$\mathrm{d}\tilde{\sigma}(x) := \Big(1+\frac{2\lambda}{\alpha}\Big)\, \delta_0(x)\, \mathrm{d}x - \frac{2\lambda}{\pi\alpha^2}\sqrt{2\alpha - x^2}\, \mathrm{d}x\,, \qquad -\sqrt{2\alpha} \leq x \leq \sqrt{2\alpha}\,.$$

Notice that choosing $\lambda = -\alpha/2$, then $\mathrm{d}\tilde{\sigma}(x) = (\pi\alpha)^{-1}\sqrt{2\alpha - x^2}\,\mathrm{d}x$, $-\sqrt{2\alpha} \leq x \leq \sqrt{2\alpha}$, so that $\{Q_n\}_{n\geq 0}$ is also defined by a linear transformation in the variable of the Chebyshev polynomials: $Q_n(x) \equiv (2\alpha)^{n/2}\, \widehat{U}_n(x/\sqrt{2\alpha})$. In this case, there is no mass point at $x = 0$. However, if we choose λ such that $-\alpha/2 < \lambda < 0$, then there is always a mass point, located at $x = 0$, which is an interior point of the support of the absolutely continuous part of the measure $\mathrm{d}\tilde{\sigma}(x)$.

Example 2. We recover a result by Van Assche [25]. We omit the details (see [1]). Consider the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ defined by a three-term recurrence relation with recurrence coefficients

$$\tilde{\beta}_{2n} := a\,, \quad \tilde{\beta}_{2n+1} := b\,, \quad \tilde{\gamma}_{2n+1} := g\,, \quad \tilde{\gamma}_{2n+2} := h \qquad (n = 0,1,\ldots)\,,$$

where $a, b \in \mathbb{R}$ and $g, h > 0$. Let $T(x) := (x-a)\,(x-b)$ and $\lambda := -g$. Starting with $P_n(x) \equiv \alpha^n\, \widehat{U}_n(\frac{x-\beta}{\alpha})$, $\alpha := 2\sqrt{gh}$, $\beta := g+h$, one can use Theorem 11 to derive the representations

$$Q_{2n+1}(x) = (gh)^{n/2}\, (x-a)\, U_n(z)\,, \qquad Q_{2n}(x) = (gh)^{n/2}\Big[U_n(z) + \sqrt{\tfrac{g}{h}}\, U_{n-1}(z)\Big]\,,$$

where $z := [(x-a)\,(x-b) - g - h]\,(4\,gh)^{-1/2}$, $\{Q_n\}_{n\geq 0}$ being orthogonal with respect to the distribution function

$$\mathrm{d}\tilde{\sigma}(x) := \pi\sqrt{\tfrac{h}{g}}\Big(1-\frac{\min(g,h)}{h}\Big)\,\delta_a(x) + \frac{\chi_E(x)}{|x-a|}\left\{1 - \frac{[(x-a)\,(x-b)-g-h]^2}{4\,gh}\right\}^{1/2}\mathrm{d}x$$

(in fact, this is the expression of $\mathrm{d}\tilde{\sigma}(x)$ given by (55) up to the factor $\sqrt{g/h}\,/\pi$), where χ_E denotes the characteristic function of the set

$$E := \Big[\tfrac{a+b}{2} - s,\ \tfrac{a+b}{2} - r\Big] \cup \Big[\tfrac{a+b}{2} + r,\ \tfrac{a+b}{2} + s\Big]\,,$$

r and s being defined by

$$r := \left(|\sqrt{g} - \sqrt{h}|^2 + |\tfrac{a-b}{2}|^2\right)^{1/2}, \qquad s := \left(|\sqrt{g} + \sqrt{h}|^2 + |\tfrac{a-b}{2}|^2\right)^{1/2} .$$

In this example, notice that there is a mass point located at $x=0$ if $g<h$, and there is no mass point if $g \geq h$. For $a = b = 0$, we recover a result by Chihara [6, p. 91].

Example 3. Now, let $\{P_n\}_{n\geq 0}$ be the sequence of the monic Laguerre polynomials, $P_n \equiv L_n^{(\nu)}$. They can be defined by a three-term recurrence relation as (3), with $\beta_n := 2n + \nu + 1$ and $\gamma_{n+1} := (n+1)(n+\nu+1)$ for $n = 0, 1, ...,$ provided that $\nu \neq -1, -2, ...$ (orthogonality conditions [19]). Consider $\lambda \in \mathbb{R}$ and a sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ such that

$$Q_2(0) := \lambda\,, \qquad Q_{2n+1}(x) := x\, L_n^{(\nu)}(x^2), \quad n = 0, 1, 2, ... \, . \tag{57}$$

With the notations of Theorem 7, we have $T(x) = x^2$, $a=b=c=0$. Moreover, using the explicit representation $L_n^{(\nu)}(x) = (-1)^n\, n!\, \sum_{k=0}^{n} \binom{n+\nu}{n-k} (-x)^k/k!$, we can deduce $L_n^{(\nu)}(0) = (-1)^n\, (\nu+1)_n$ and $(L_{n-1}^{(\nu)})^{(1)}(0) = (-1)^n\, n!\, \sum_{k=1}^{n} \binom{n+\nu}{n-k} (-1)^k \cdot$ $\cdot\, (\nu+1)_k/k!$, where $(\alpha)_0 := 1$, $(\alpha)_n := \alpha\,(\alpha+1)\cdots(\alpha+n-1) = \Gamma(n+\alpha)/\Gamma(\alpha)$ for $\alpha \neq 0, -1, -2, ...$ (Γ denotes the Gamma function). Hence, by Theorem 7, we find that $\{Q_n\}_{n\geq 0}$ is a MOPS with respect to a quasi-definite linear functional if and only if

$$\lambda \neq 0\,, \qquad \sum_{k=1}^{n} \binom{n+\nu}{n-k} (-1)^k\, \frac{n!}{k!}\, \frac{(\nu+1)_k}{(\nu+1)_n} \neq \frac{1}{\lambda}\,, \qquad Q_{2n}(x) = L_n^{(\nu)}(x^2; \lambda, 0) \,,$$

hold for all $n = 0, 1, 2, ...$.

In order to have orthogonality in the positive definite sense, we must impose $\nu > -1$, and in this case $\{L_n^{(\nu)}\}_{n\geq 0}$ is orthogonal with respect to $\mathrm{d}\sigma(x) := w(x)\,\mathrm{d}x$, where

$$w(x) := x^\nu\, \mathrm{e}^{-x}\,, \qquad 0 \leq x < +\infty$$

(then $\xi = 0$, $\eta = +\infty$). If $\nu > 0$, we have $F(0;\sigma) = \int_0^{+\infty} t^{\nu-1}\, \mathrm{e}^{-t}\, \mathrm{d}t = \Gamma(\nu) = \Gamma(\nu+1)/\nu = u_0/\nu$ and $A = -u_0/F(0;\sigma) = -\nu$, $B \equiv +\infty$. Therefore, by Theorem 11, $\{Q_n\}_{n\geq 0}$ satisfying (57) is an MOPS in the positive definite sense if

$$-\nu \leq \lambda < 0\,, \qquad Q_{2n}(x) = L_n^{(\nu)}(x^2; \lambda, 0), \quad n = 0, 1, 2, ... \, .$$

In such conditions, according to Corollary 12, $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the distribution function

$$\mathrm{d}\tilde{\sigma}(x) := \Gamma(\nu+1) \left(1 + \frac{\lambda}{\nu}\right) \delta_0(x)\, \mathrm{d}x - \lambda\, |x|^{2\nu-1}\, \mathrm{e}^{-x^2}\, \mathrm{d}x\,, \qquad -\infty < x < +\infty\,.$$

Choosing $\lambda = -\nu$, then $\mathrm{d}\tilde{\sigma}(x) = \nu\,|x|^{2\mu}\,\mathrm{e}^{-x^2}\,\mathrm{d}x$, with $\mu := \nu - \frac{1}{2}$, hence $\{Q_n\}_{n\geq 0}$ is the sequence of the generalized Hermite polynomials: $Q_n \equiv H_n^{(\mu)}$, $\mu > -\frac{1}{2}$ (cf. [6, p. 157]). However, if we choose λ such that $-\nu < \lambda < 0$, then there is always a mass point, located at $x = 0$.

Remark 7. The quadratic transformations studied here give us new examples of families of semiclassical orthogonal polynomials (for details about this class of orthogonal polynomials, see, e.g., [19] and [14]). In fact, it can be proved that if the starting MOPS $\{P_n\}_{n\geq 0}$ is semiclassical, then $\{Q_n\}_{n\geq 0}$ as defined in problems P1 or P2 is also semiclassical. In particular, related to problem P1, in [16] the classification of all possible sequences $\{Q_n\}_{n\geq 0}$ in case that $\{P_n\}_{n\geq 0}$ be a classical MOPS has been given.

ACKNOWLEDGEMENTS – We thank the referee for their valuable comments and remarks.

This paper was finished during a stay of the second author in Departamento de Matemáticas, Universidad Carlos III de Madrid, with financial support of a Grant from *Junta Nacional de Investigação Científica e Tecnológica* (JNITC) — BD976 — and *Centro de Matemática da Universidade de Coimbra* (CMUC) of Portugal. The work of the first author was supported by the *Dirección General de Enseñanza Superior* (DGES) of Spain — PB96-0120-C03-01.

REFERENCES

[1] Alvarez-Nodarse, R., Marcellán, F. and Petronilho, J. – *On Some Polynomial Mappings for Measures. Applications*, In "Proceedings of the International Workshop on Orthogonal Polynomials in Mathematical Physics 96" (M. Alfaro, R. Álvarez-Nodarse, G.L. Lagomasino, F. Marcellán, Eds.), Leganés, Madrid, 1997, 1–22.

[2] Berg, C. – Markov's theorem revisited, *J. Approx. Theory.,* 78 (1994), 260–275.

[3] Bessis, D. and Moussa, P. – Orthogonality properties of iterated polynomial mappings, *Comm. Math. Phys.,* 88 (1983), 503–529.

[4] Charris, J., Ismail, M.H. and Monsalve, S. – On sieved orthogonal polynomials, X: general blocks of recurrence relations, *Pacific J. of Math.,* 163(2) (1994), 237–267.

[5] Charris, J., Salas, G. and Silva, V. – Polinomios ortogonales relacionados con problemas espectrales, *Rev. Col. de Mat.,* 27 (1991), 35–80.

[6] Chihara, T.S. – On co-recursive orthogonal polynomials, *Proc. Amer. Math. Soc.,* 8 (1957), 899–905.

[7] Chihara, T.S. – *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[8] Chihara, L.M. and Chihara, T.S. – A class of nonsymmetric orthogonal polynomials, *J. Math. Anal. and Applic.,* 126 (1987), 275–291.

[9] Cramér, H. – *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1946.

[10] Freud, G. – *Orthogonal Polynomials*, Pergamon Press, Oxford, 1971.

[11] Geronimo, J. and Van Assche, W. – Orthogonal polynomials on several intervals via a polynomial mapping, *Trans. Amer. Math. Soc.,* 308(2) (1988), 559–581.

[12] Gover, M.J.C. – The eigenproblem of a tridiagonal 2-Toeplitz matrix, *Linear Alg. and Its Applic.,* 197-198 (1994), 63–78.

[13] Ismail, M.H. – On sieved orthogonal polynomials, III: orthogonality on several intervals, *Trans. Amer. Math. Soc.,* 249(1) (1986), 89–111.

[14] Marcellán, F. – *Polinomios ortogonales semiclásicos. Una aproximación constructiva*, In "Actas V Simposium Polinomios Ortogonales" (A. Cachafeiro, E. Godoy, Eds.), Vigo (Spain), 1988, 100–123.

[15] Marcellán, F., Dehesa, J.S. and Ronveaux, A. – On orthogonal polynomials with perturbed recurrence relations, *J. Comp. Appl. Math.,* 30 (1990), 203–212.

[16] Marcellán, F. and Petronilho, J. – Eigenproblems for tridiagonal 2-Toeplitz matrices and quadratic polynomial mappings, *Linear Alg. and Its Applic.,* 260 (1997), 169–208.

[17] Maroni, P. – Sur la suite de polynômes orthogonaux associée à la forme $\mathbf{u} = \delta_c + \lambda(x-c)^{-1}\mathcal{L}$, *Periodica Mathematica Hungarica,* 21(3) (1990), 223–248.

[18] Maroni, P. – Sur la décomposition quadratique d'une suite de polynômes orthogonaux I, *Riv. di Mat. Pura ed Appl.,* 6 (1990), 19–53.

[19] Maroni, P. – *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, In "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, A. Ronveaux, Eds.), Proc., Erice, 1990, IMACS. *Ann. Comp. Appl. Math.,* 9 (1991), 95–130.

[20] Maroni, P. – Sur la décomposition quadratique d'une suite de polynômes orthogonaux II, *Port. Math.,* 50(3) (1993), 305–329.

[21] Moussa, P. – Iteration des polynômes et proprietés d' orthogonalité, *Ann. Inst. Henri Poincaré,* 44 (1986), 315–325.

[22] Peherstorfer, F. – On Bernstein–Szegö orthogonal polynomials on several intervals, *SIAM J. Math. Anal.,* 21 (1990), 461–482.

[23] Peherstorfer, F. – On Bernstein–Szegö orthogonal polynomials on several intervals II, *J. Approx. Theory,* 64 (1991), 123–161.

[24] Slim, H.A. – On co-recursive orthogonal polynomials and their application to potential scattering, *J. Math. Anal. and Applic.,* 136 (1988), 1–19.

[25] Van Assche, W. – Asymptotics properties of orthogonal polynomials from their recurrence formula, I, *J. Approx. Theory,* 44 (1985), 258–276.

[26] Van Assche, W. – Asymptotics for Orthogonal Polynomials, *Lecture Notes in Mathematics,* 1265, Springer-Verlag, Berlin, 1987.

[27] Van Assche, W. – Orthogonal polynomials, associated polynomials and functions of the second kind, *J. Comp. Appl. Math.*, 37 (1991), 237–249.

Francisco Marcellán,
Depto. de Matemáticas, Escuela Politécnica Superior, Univ. Carlos III de Madrid,
Butarque 15, 28911 Leganés, Madrid – SPAIN
E-mail: pacomarc@ing.uc3m.es

and

José Petronilho,
Depto. de Matemática, Faculdade de Ciências e Tecnologia, Univ. Coimbra,
Apartado 3008, 3000 Coimbra – PORTUGAL
E-mail: josep@mat.uc.pt