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A NOTE ON THE THIRD-ORDER MOMENT STRUCTURE OF A BILINEAR MODEL WITH NON-INDEPENDENT SHOCKS

C.M. Martins

Abstract: Formulas for the third-order theoretical moments are obtained for the bilinear time series $X_t = \beta\, X_{t-k}\, \varepsilon_{t-l} + \varepsilon_t$, $k \geq l \geq 1$, assuming that $\{\varepsilon_t\}$ is a strictly stationary and ergodic sequence of random variables such that, for each $t \in \mathbb{Z}$, ε_t has some conditional moments that are finite. Thus, Gabr's results (1988), obtained with an independent and identically distributed Gaussian sequence $\{\varepsilon_t\}$, are generalized.

1 – Introduction

We consider the simple bilinear model $\{X_t\}_{t\in\mathbb{Z}}$:

$$X_t = \beta X_{t-k}\, \varepsilon_{t-l} + \varepsilon_t \ , \tag{1}$$

where β is a real constant and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of real random variables (r.v.). Model (1) is called *diagonal* if $k=l$, *superdiagonal* if $k>l$ and *subdiagonal* if $k<l$. It was firstly studied by Granger and Andersen (1978) considering $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ as a sequence of independent and identically distributed random variables (i.i.d. r.v.) with zero mean and variance σ^2, $\sigma > 0$. Assuming the normality of ε_t, $t \in \mathbb{Z}$, they proved that, in most cases, the autocorrelations of $\{X_t\}$ are equal to zero, which can lead it to be wrongly identified as a white noise (i.e. a sequence of centered and uncorrelated r.v.); so, they suggested the study of higher moments of $\{X_t\}$, namely the study of the autocorrelations of $\{X_t^2\}$, to obtain a characterization of $\{X_t\}$ different from a white noise. In the case of diagonal and superdiagonal models, Li (1984) deduced formulas for the first $k-1$ autocorrelations of $\{X_t^2\}$,

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supposing that $\{\varepsilon_t\}$ is i.i.d. with a Gaussian distribution and that $\{X_t\}$ is strictly stationary and has moments up to the fourth order. Assuming that $\{\varepsilon_t\}$ is a strictly stationary, ergodic sequence of r.v. whose conditional moments satisfy some particular hypotheses, Martins (1997a) proved that the autocorrelations of $\{X_t\}$ have the same behaviour as in the i.i.d. Gaussian case. Martins (1997b) also obtained the autocorrelation function of the process $\{X_t^2\}$ in the diagonal and superdiagonal cases, with $\{\varepsilon_t\}$ satisfying the above-mentioned conditions.

Gabr (1988) deduced formulas for the third-order theoretical moments for the bilinear time series model defined by (1), assuming that the error process satisfies Li's hypotheses and that $\{X_t\}$ is strictly stationary and has moments up to the third order. In this paper, we establish analogous properties for diagonal and superdiagonal models, supposing that $\{\varepsilon_t\}$ verifies the hypotheses considered by Martins (1997a). In this way, we generalize Gabr's results as we do require neither the normality nor the independence of the error process.

2 – Preliminary results

Let us then consider the simple bilinear model defined by (1), where the error process, $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, is now a strictly stationary, ergodic sequence of r.v.. Let us denote this general hypothesis by $\mathcal{H}$. Denoting the σ-field generated by $\{\varepsilon_t, \varepsilon_{t-1}, ...\}$ as $\underline{\varepsilon}_t$, and the conditional expectation given the past $\underline{\varepsilon}_t$ as $E(\cdot\,|\underline{\varepsilon}_t)$, it is also assumed that, for each $t\in\mathbb{Z}$, $E(\varepsilon_t^{2p}|\underline{\varepsilon}_{t-1}) = \mu_{2p} > 0$, $E(\varepsilon_t^{2p-1}|\underline{\varepsilon}_{t-1}) = 0$, $p=1,2,3$, in the diagonal case and $E(\varepsilon_t^2|\underline{\varepsilon}_{t-1}) = \mu_2 > 0$, $E(\varepsilon_t^{2p-1}|\underline{\varepsilon}_{t-1}) = 0$, $p=1,2$, in the superdiagonal case.

We also assume that the simple bilinear process $\{X_t\}$ is strictly stationary and that all its moments up to the third order exist. From Quinn (1982) and Azencott and Dacunha-Castelle (1984, pp. 30/32), it can be shown that a sufficient condition for the strict stationarity of the process $\{X_t\}$ is $\ln|\beta| + E(\ln|\varepsilon_t|) < 0$, provided that the error process $\{\varepsilon_t\}$ satisfies $\mathcal{H}$ and $E|\ln|\varepsilon_t|| < \infty$.

In this section we refer some results concerning the first and second order moments of the process $\{X_t\}$, obtained by Martins (1997a) from which we deduce necessary and sufficient conditions for the stationarity of the model.

For the diagonal model

$$X_t = \beta X_{t-k}\, \varepsilon_{t-k} + \varepsilon_t\,, \qquad k \geq 1\,, \tag{2}$$

we have $E(X_t) = \beta\mu_2$, $E(X_t X_{t-k}) = 2[E(X_t)]^2$ and $E(X_t X_{t-j}) = [E(X_t)]^2$,

$j \neq k$. The covariance of $\{X_t\}$ at lag j, $j \in \mathbb{N}$, is then given by

$$\operatorname{cov}(X_t, X_{t-j}) = \begin{cases} \beta^2 \mu_2^2 & \text{if } j = k, \\ 0 & \text{if } j \neq k \; . \end{cases}$$

After squaring (2), and using the hypotheses concerning conditional expectations and the strict stationarity of the process $\{X_t^2 \varepsilon_t^2\}$, we have

$$E(X_t^2) = \beta^2 E(X_t^2 \varepsilon_t^2) + \mu_2$$

and

$$\begin{aligned} E(X_t^2 \varepsilon_t^2) &= \beta^2 E\Big[X_{t-k}^2\, \varepsilon_{t-k}^2\, E(\varepsilon_t^2 | \underline{\varepsilon}_{t-1})\Big] + E\Big[E(\varepsilon_t^4 | \underline{\varepsilon}_{t-1})\Big] \\ &\quad + 2\,\beta E\Big[X_{t-k}\, \varepsilon_{t-k}\, E(\varepsilon_t^3 | \underline{\varepsilon}_{t-1})\Big] \\ &= \beta^2 \mu_2\, E(X_{t-k}^2\, \varepsilon_{t-k}^2) + \mu_4 \; . \end{aligned}$$

The fact that $E(X_t^2)$ exists and $\mu_2 > 0$ implies $\beta^2 \mu_2 < 1$ and

$$E(X_t^2 \varepsilon_t^2) = \frac{\mu_4}{1 - \beta^2 \mu_2} \; . \tag{3}$$

Finally, we obtain

$$E(X_t^2) = \frac{\beta^2 \mu_4}{1 - \beta^2 \mu_2} + \mu_2 \; . \tag{4}$$

It is easy to prove that $\beta^2 \mu_2 < 1$ implies $\ln|\beta| + E(\ln|\varepsilon_t|) < 0$, by Jensen's inequality, provided that $E|\ln|\varepsilon_t|| < +\infty$. Then we can establish the following necessary and sufficient condition concerning the stationarity of the process $\{X_t\}$.

Theorem 2.1. *Let $\{X_t\}$ be the diagonal model defined by (2). Suppose that $\{\varepsilon_t\}$ satisfies $\mathcal{H}$ and $E(\varepsilon_t^{2p} | \underline{\varepsilon}_{t-1}) = \mu_{2p} > 0$, $E(\varepsilon_t^{2p-1} | \underline{\varepsilon}_{t-1}) = 0$, $p = 1, 2$. Suppose also that $E(X_t^2)$ exists and that $E|\ln|\varepsilon_t|| < +\infty$. Then the process $\{X_t\}$ is strictly and weakly stationary if and only if $\beta^2 \mu_2 < 1$.*

For the superdiagonal model

$$X_t = \beta X_{t-k}\, \varepsilon_{t-l} + \varepsilon_t \,, \qquad k > l \geq 1 \,, \tag{5}$$

we obtain $E(X_t) = 0$, $E(X_t X_{t-j}) = [E(X_t)]^2$, $\operatorname{cov}(X_t, X_{t-j}) = 0$, $j \in \mathbb{N}$, and $E(X_t^2) = \frac{\mu_2}{1 - \beta^2 \mu_2}$. We also can establish the following result.

Theorem 2.2. *Let $\{X_t\}$ be the superdiagonal model defined by (5). Suppose that $\{\varepsilon_t\}$ satisfies $\mathcal{H}$ and $E(\varepsilon_t^2|\underline{\varepsilon}_{t-1}) = \mu_2 > 0$, $E(\varepsilon_t|\underline{\varepsilon}_{t-1}) = 0$. Suppose also that $E(X_t^2)$ exists and that $E|\ln|\varepsilon_t|| < +\infty$. Then the process $\{X_t\}$ is strictly and weakly stationary if and only if $\beta^2\mu_2 < 1$.*

Taking into account the values obtained for the covariances of $\{X_t\}$, the superdiagonal model appears as a white noise and the diagonal model appears as a special MA(k) model.

In order to distinguish between these and bilinear models we need to investigate the behaviour of some moments of order greater than 2; in this sense, in the following sections we consider the analysis of the third-order moments of the process $\{X_t\}$.

3 – Third-order moments of $\{X_t\}$

The third-order moments of $\{X_t\}$ are defined by

$$\begin{aligned} R(s_1,s_2) &= E\Big[(X_t - E(X_t))\,(X_{t-s_1} - E(X_t))\,(X_{t-s_2} - E(X_t))\Big] \\ &= E(X_t\,X_{t-s_1}\,X_{t-s_2}) - E(X_t)\,\Big[\gamma(s_1) + \gamma(s_2) + \gamma(s_1 - s_2)\Big] \\ &\quad + 2[E(X_t)]^3 \ , \end{aligned} \tag{6}$$

where $s_1, s_2 \in \mathbb{Z}$ and $\gamma(s) = E(X_t X_{t-s})$, $s \in \mathbb{Z}$.

From Subba Rao and Gabr (1984), the following symmetry relations hold:

$$R(s_1,s_2) = R(s_2,s_1) = R(-s_1,\, s_2 - s_1) = R(s_1 - s_2,\, -s_2) \ ,$$

where $s_1, s_2 \in \mathbb{Z}$. So, it is sufficient to calculate $R(s_1,s_2)$ for $0 \le s_1 \le s_2$.

3.1. Diagonal model

Let us suppose that $\{X_t\}$ and $\{\varepsilon_t\}$ satisfy the general above-mentioned conditions for the diagonal model defined by (2). The next theorem gives the values of $R(s_1,s_2)$ for this model.

Theorem 3.1. *Let $\{X_t\}$ be the diagonal model defined by (2). Suppose that $\{\varepsilon_t\}$ satisfies $\mathcal{H}$ and $E(\varepsilon_t^{2p}|\underline{\varepsilon}_{t-1}) = \mu_{2p} > 0,\ E(\varepsilon_t^{2p-1}|\underline{\varepsilon}_{t-1}) = 0,\ p = 1, 2, 3.$ Suppose also that $\{X_t\}$ is strictly stationary and that $E(X_t^3)$ exists. Then*

$$R(s_1, s_2) = \begin{cases} \dfrac{3\,\beta^3\mu_4}{1-\beta^2\mu_2}\,(\beta^2\mu_4-\mu_2) + \beta^3(\mu_6+2\,\mu_2^3), & s_1=s_2=0, \\ \dfrac{\beta}{1-\beta^2\mu_2}\Big[\mu_4-\mu_2^2+\beta^2\mu_2(\mu_4-\mu_2^2+2\,\beta^2\mu_2^3)\Big], & s_1=s_2=k, \\ \dfrac{\beta^3\mu_2}{1-\beta^2\mu_2}\,\lambda - 2\,\beta^3\mu_2^3, & s_1=0,\ s_2=k, \\ \dfrac{\beta^{2n+1}\mu_2^{2n}}{1-\beta^2\mu_2}\,\lambda, & s_1=0,\ s_2=nk,\ n=2,3,..., \\ \beta^3\mu_2^3, & s_1=k,\ s_2=2\,k, \\ 0, & \text{otherwise}\ , \end{cases}$$

where $\lambda = \beta^4(3\,\mu_4^2 - \mu_2\,\mu_6) + \beta^2(\mu_6 - 3\,\mu_2\,\mu_4) + 2\,\mu_4.$

Proof: The values of $\gamma(s)$ were already indicated in section 2. These values are given by

$$\gamma(s) = \begin{cases} 2[E(X_t)]^2 & \text{if } s=k, \\ [E(X_t)]^2 & \text{if } s\neq k,\ s>0\ . \end{cases}$$

Consider the case $s_1 = s_2 = 0$. From (6) we have

$$R(0,0) = E(X_t^3) - 3\,E(X_t)\,E(X_t^2) + 2[E(X_t)]^3\ . \tag{7}$$

If we raise both sides of (2) to the third order, denote the quantity $n!/[p!(n-p)!]$ as C_p^n and take expectations, we have

$$\begin{aligned} E(X_t^3) &= \sum_{i=0}^{3} C_i^3\,\beta^i\,E\Big[X_{t-k}^i\,\varepsilon_{t-k}^i\,E(\varepsilon_t^{3-i}|\underline{\varepsilon}_{t-1})\Big] \\ &= 3\,\beta\,\mu_2^2 + \beta^3 E(X_t^3\,\varepsilon_t^3) \end{aligned} \tag{8}$$

and

$$\begin{aligned} E(X_t^3\,\varepsilon_t^3) &= \sum_{i=0}^{3} C_i^3\,\beta^i\,E\Big[X_{t-k}^i\,\varepsilon_{t-k}^i\,E(\varepsilon_t^{6-i}|\underline{\varepsilon}_{t-1})\Big] \\ &= \mu_6 + \frac{3\,\beta^2\mu_4^2}{1-\beta^2\mu_2}\ . \end{aligned}$$

Inserting this result into (8), we obtain

$$R(0,0) = \frac{3\,\beta^3\mu_4}{1-\beta^2\mu_2}\,(\beta^2\mu_4-\mu_2) + \beta^3(\mu_6+2\,\mu_2^3)\ .$$

For $s_1 = s_2 = s > 0$ we have, from (6),

$$R(s,s) = E(X_t\,X_{t-s}^2) - 2\,E(X_t)\,\gamma(s) - E(X_t)\,E(X_t^2) + 2[E(X_t)]^3\ . \tag{9}$$

Using (2), we can write

$$E(X_t\,X_{t-s}^2) = \beta\,E(X_{t-k}\,\varepsilon_{t-k}\,X_{t-s}^2) + E(\varepsilon_t\,X_{t-s}^2)\ .$$

Taking now the cases $s<k$, $s>k$ and $s=k$ separately and using the strict stationarity of the processes involved and the hypotheses about conditional moments of ε_t, we obtain

$$E(X_t\,X_{t-s}^2) = \begin{cases} \beta\,\mu_4\Big(1+\dfrac{3\,\beta^2\mu_2}{1-\beta^2\mu_2}\Big), & s=k,\\[2ex] \beta\,\mu_2\Big(\mu_2+\dfrac{\beta^2\mu_4}{1-\beta^2\mu_2}\Big), & s\neq k\ , \end{cases}$$

which implies

$$R(s,s) = \begin{cases} 0, & s\neq k\\[2ex] \dfrac{\beta}{1-\beta^2\mu_2}\Big(\mu_4+\beta^2\mu_2\,\mu_4-\beta^2\mu_2^3+2\,\beta^4\mu_2^4-\mu_2^2\Big), & s=k\ . \end{cases}$$

Let us now consider the case $s_1 = 0$, $s_2 = s > 0$. In this case we have

$$R(0,s) = E(X_t^2\,X_{t-s}) - E(X_t)\,E(X_t^2) - 2\,E(X_t)\,\gamma(s) + 2[E(X_t)]^3\ . \tag{10}$$

If we square (2), multiply by X_{t-s}, take expectations and apply the hypotheses concerning conditional moments of ε_t, we obtain

$$E(X_t^2\,X_{t-s}) = \beta^2 E(X_{t-k}^2\,\varepsilon_{t-k}^2\,X_{t-s}) + \beta\mu_2^2\ . \tag{11}$$

If $s<k$, it can be shown that

$$E(X_t^2 X_{t-s}) = \beta\mu_2\Big(\frac{\beta^2\mu_4}{1-\beta^2\mu_2}+\mu_2\Big) = E(X_t)\,E(X_t^2)$$

and $R(0,s) = 0$.

If $s \geq k$, let us put $s = nk+m$, $n\in\mathbb{N}$, $m = 0,1,...,k-1$.

Denoting the expectation $E(X_{t-k}^2\,\varepsilon_{t-k}^2\,X_{t-s})$ as V_{s-k}, we can show that

$$V_s = \beta^2\mu_2\,V_{s-k} + \beta\,\mu_2\,\mu_4\,, \qquad s \geq k\;,$$

which is a difference equation in the quantity V_s.

Considering separately the cases $m = 0$ and $1 \leq m \leq k-1$, we obtain the solution for this difference equation:

$$V_{nk+m} = \begin{cases} \dfrac{\beta\mu_2}{1-\beta^2\mu_2}\left[\mu_4 + (\beta^2\mu_2)^n\lambda\right], & m = 0, \\ \dfrac{\beta\mu_2}{1-\beta^2\mu_2}\,\mu_4, & m = 1,\dots,k-1\;, \end{cases}$$

where $\lambda = 3\,\beta^2\mu_4(\beta^2\mu_4 - \mu_2) + \beta^2\mu_6(1-\beta^2\mu_2) + 2\,\mu_4$.

Inserting this formulas into (11) and incorporating the results obtained into (10) we obtain

$$R(0,s) = \begin{cases} \dfrac{\beta^3\mu_2}{1-\beta^2\mu_2}\,\lambda - 2\,\beta^3\mu_2^3, & s = k, \\ \dfrac{\beta^{2n+1}\mu_2^n}{1-\beta^2\mu_2}\,\lambda, & s = nk,\;\; n = 2,3,\dots, \\ 0, & \text{otherwise}\;. \end{cases}$$

Finally, we have to consider $s_1 = s,\; s_2 = s+r,\; s \geq 1,\; r \geq 1$.

In this case it can be shown that

$$R(s,s+r) = \begin{cases} \beta^3\mu_2^3, & s = r = k, \\ 0, & \text{otherwise}\;, \end{cases}$$

which ends the proof. ∎

3.2. Superdiagonal model

Taking $l = k-m$, $1 \leq m \leq k-1$ in (5), the superdiagonal model can be written as

(12) $$X_t = \beta X_{t-k}\,\varepsilon_{t-k+m} + \varepsilon_t\;,$$

where $1 \leq m \leq k-1,\;\; k \geq 2$.

For $\{X_t\}$ defined by (12) we obtained in section 2

$$E(X_t) = 0 \; ,$$

$$E(X_t^2) = \frac{\mu_2}{1-\beta^2\mu_2} \; ,$$

$$\gamma(s) = E(X_t\, X_{t-s}) = 0 \, , \qquad s > 0 \; .$$

The fact that $E(X_t) = 0$, implies

$$R(s_1, s_2) = E(X_t\, X_{t-s_1}\, X_{t-s_2}) \, , \qquad s_1, s_2 \in \mathbb{Z} \; .$$

Using an analogous methodology we can prove the next result.

Theorem 3.2. *Let $\{X_t\}$ be the superdiagonal model defined by (12). Suppose that $\{\varepsilon_t\}$ satisfies $\mathcal{H}$ and $E(\varepsilon_t^2|\underline{\varepsilon}_{t-1}) = \mu_2 > 0$, $E(\varepsilon_t^{2p-1}|\underline{\varepsilon}_{t-1}) = 0$, $p=1,2$. Suppose also that $\{X_t\}$ is strictly stationary and that $E(X_t^3)$ exists. Then*

$$R(s_1, s_2) = \begin{cases} \dfrac{\beta\mu_2^2}{1-\beta^2\mu_2}, & s_1 = k-m, \;\; s_2 = k, \\[2ex] 0, & \textit{otherwise} \; . \end{cases}$$

4 – Simulation studies

The results obtained can be useful in bilinear time series modelling, particularly in the choice of the orders k and l of some simple bilinear models for which the error process is not Gaussian. It is well known that some real time series are well described by models with a non Gaussian error process (e.g. Engle (1982) and Weiss (1984) proposed the modelling of some financial time series by ARMA processes with ARCH errors). Thus, with the results obtained here it is possible to consider, as an alternative for the study of these series, nonlinear models with such a kind of error process.

In order to illustrate the practical interest of these results, some simulation studies were performed, considering $\{\varepsilon_t\}$ as a sequence of i.i.d. symmetrically distributed r.v. with zero mean. The distributions considered here are the the Student distribution with 7 d.f. ($\varepsilon_t \sim t(7)$) and the uniform distribution in the interval $[-1,1]$ ($\varepsilon_t \sim U[-1,1]$). In each case, the values of β were chosen in order to satisfy the condition $\beta^2\mu_2 < 1$. The values considered for (k,l) are $(3,1)$ and $(2,2)$. We construct realizations of $\{X_t\}$, of length 200, and the model is replicated 200 times. The sample third-order moments are calculated for each replication and, for each (s_1, s_2), the mean $\bar{R}_{s_1 s_2}$ of the sample third-order moments in the set of the replications, is recorded.

Table I gives $\bar{R}_{s_1 s_2}$, $s_1, s_2 = 1, \ldots, 5$, for the superdiagonal model with $(k, l) = (3, 1)$ as well as the corresponding theoretical values (in the parenthesis) of $R(s_1, s_2)$, $s_1, s_2 = 1, \ldots, 5$. The distribution considered for ε_t is $t(7)$ and the value of β is 0.5. It can be seen that simulation results agree well with theoretical results presented in Theorem 3.2, namely, the simulated values in the cells $(1, 3)$ (or $(3, 1)$) are much larger than any other values.

Table I

$X_t = 0.5\, X_{t-3}\, \varepsilon_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim t(7)$

s_1 \ s_2	0	1	2	3	4	5
0	−.064	.060	.117	.090	−.011	.140
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
1	.060	.140	.019	1.553	−.011	.007
	(0.0)	(0.0)	(0.0)	(1.508)	(0.0)	(0.0)
2	.116	.019	−.036	−.078	−.041	−.060
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
3	.089	1.538	−.078	−.238	−.138	−.006
	(0.0)	(1.508)	(0.0)	(0.0)	(0.0)	(0.0)
4	−.011	−.011	−.041	−.137	.037	.016
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
5	.137	.007	−.059	−.006	.016	−.062
	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)	(0.0)

Table II records $\bar{R}_{s_1 s_2}$, $s_1, s_2 = 1, \ldots, 5$, for the diagonal model with $k = 2$ as well as the corresponding theoretical values (in the parenthesis) of $R(s_1, s_2)$, $s_1, s_2 = 1, \ldots, 5$. In this case $\varepsilon_t \sim U[-1, 1]$ and $\beta = 1.0$. We can see that there are various cells that are apparently significant, namely the ones corresponding to the following pairs (s_1, s_2): $(0, 0)$, $(2, 2)$, $(0, 2)$, $(0, 4)$ and $(2, 4)$ (as well as the corresponding cells (s_2, s_1)). This fact leads us to think our time series could be well described by a diagonal model with $k = 2$, according to the results of Theorem 3.1.

Table II

$X_t = X_{t-2}\,\varepsilon_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim U[-1,1]$

s_1 \ s_2	0	1	2	3	4	5
0	.087 (.097)	−.005 (0.0)	.128 (.134)	−.003 (0.0)	.064 (.008)	−.004 (0.0)
1	−.005 (0.0)	−.004 (0.0)	−.003 (0.0)	−.002 (0.0)	−.002 (0.0)	.001 (0.0)
2	.126 (.134)	−.003 (0.0)	.209 (.215)	−.001 (0.0)	.033 (.037)	−.005 (0.0)
3	−.004 (0.0)	−.002 (0.0)	−.001 (0.0)	.001 (0.0)	.000 (0.0)	−.002 (0.0)
4	.061 (.008)	−.002 (0.0)	.033 (.037)	.000 (0.0)	−.004 (0.0)	−.001 (0.0)
5	−.006 (0.0)	.001 (0.0)	−.004 (0.0)	−.002 (0.0)	−.001 (0.0)	−.002 (0.0)

Finally, we notice that examples of discrete distributions for the error process can also be considered, as no assumptions about densities are imposed in this study.

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REFERENCES

[1] Azencott, R. and Dacunha-Castelle, D. – *Séries d'Observations Irrégulières, Modélisation et Prévision*, Masson, Paris, 1984.

[2] Engle, R.F. – Autoregressive conditional heteroscedasticity with estimates of the variance of the UK inflation, *Econometrica,* 50 (1982), 987–1008.

[3] Gabr, M.M. – On the third-order moment structure and bispectral analysis of some bilinear time series, *Journal of Time Series Analysis,* 9 (1988), 11–20.

[4] Granger, C.W.J. and Andersen, A. – *An Introduction to Bilinear Time Series Models*, Vandenhoeck and Ruprecht, Göttingen, 1978.

[5] Li, W.K. – On the autocorrelation structure and identification of some bilinear time series, *Journal of Time Series Analysis,* 5 (1984), 173–181.

[6] Martins, C.M. – A note on the autocorrelations related to a bilinear model with non-independent shocks, *Statistics & Probability Letters,* 36 (1997a), 245–250.

[7] Martins, C.M. – *On the autocorrelation structure of a bilinear model with non-independent shocks*, Tech. Rep., 97-04, Dep. Mat. Univ. Coimbra, 1997b.

[8] Quinn, B.G. – Stationarity and invertibility of simple bilinear models, *Stoch. Processes Appl.,* 12 (1982), 225–230.

[9] Subba Rao, T. and Gabr, M.M. – An introduction to bispectral analysis and bilinear time series models, *Lecture Notes in Statistics,* 24 (1984), Berlin: Springer-Verlag.

[10] Weiss, A.A. – ARMA models with ARCH errors, *Journal of Time Series Analysis,* 5 (1984), 129–143.

C.M. Martins,
Dep. de Matemática, Faculdade de Ciências e Tecnologia da Universidade de Coimbra,
Universidade de Coimbra – PORTUGAL