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A NOTE ON MATRIX TRANSFORMATIONS OF HOLOMORPHIC DIRICHLET SERIES (*)

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Abstract: The aim of this note is to study matrix transformations of holomorphic Dirichlet series in bounded convex domains of $\mathbb{C}^n$. The problem considered here is motivated by the paper [1] of Borwein and Jakimovski for power series of one variable.

1 – Introduction

As is well-known the matrix transformation is one of the methods for summing series and sequences using an infinite matrix. Namely, having a matrix $[u_{jk}]_{j,k=1}^{\infty}$, a given series

$$\sum_{k=1}^{\infty} c_k \tag{1.1}$$

is transformed into the sequence $(\sigma_j)_{j=1}^{\infty}$ with

$$\sigma_j = \sum_{k=1}^{\infty} u_{jk}\, c_k \ . \tag{1.2}$$

The series (1.1) is said to be *summable to the sum* σ if, for all $j = 1, 2, ...$, the series on the right-hand side in (1.2) converges and

$$\lim_{j\to\infty} \sigma_j = \sigma \ .$$

The similar notion is also defined for functional series.

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Matrix transformations of power series of one complex variable has been studied previously by several authors. Most of papers dealed with Nörlund matrices, i.e; triangular matrices of a special form (see, e.g., [13, 14]). For the general case of the matrices there seem to be very few articles. Recently Borwein and Jakimovski [1] considered matrix transformations of power series in the complex plane $\mathbb{C}$ and obtained some results on this direction.

In our previous article [12] we studied matrix transformations of the class of multiple Dirichlet series with complex frequencies that define entire functions in $\mathbb{C}^n$.

The present paper continues [12]. We shall be concerned with matrix transformations of holomorphic Dirichlet series in a bounded convex domain of $\mathbb{C}^n$.

It should be noted that the techniques used in [1] do not work for Dirichlet series considered in our article [12] as well as in this paper, because they are essentially one-dimensional and moreover, of power series.

Also, since every entire function as well as every holomorphic function in a convex domain can be represented in the form of Dirichlet series with complex frequencies (see, e.g., [6, 9]) a study of Dirichlet series attracts a great attention. Some problems for these series have already been studied [3, 4, 5, 10, 11].

2 – Holomorphic Dirichlet series in a domain

We recall some basic notation which will be used in this paper.

$\mathcal{O}(\Omega)$ (Ω being a domain in $\mathbb{C}^n$) denotes the space of holomorphic functions in Ω, with the topology of uniform convergence on compact subsets of Ω.

If $z, \zeta \in \mathbb{C}^n$ then $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$; $\langle z, \zeta\rangle = z_1\zeta_1 + \cdots + z_n\zeta_n$.

Let Ω be a bounded convex domain in $\mathbb{C}^n$, with the supporting function defined as follows

$$H_\Omega(\zeta) = \sup_{z\in\Omega} \operatorname{Re}\langle z, \zeta\rangle , \qquad \zeta \in \mathbb{C}^n .$$

Let further $(\lambda^k)_{k=1}^\infty$ be a sequence of complex vectors in $\mathbb{C}^n$.

For a Dirichlet series

$$\sum_{k=1}^\infty c_k\, e^{\langle\lambda^k, z\rangle} , \qquad z \in \Omega , \tag{2.1}$$

there is the following characterization of the coefficients of this series when it converges for the topology of $\mathcal{O}(\Omega)$ [10] which is important and necessary for further study.

Theorem 2.1. *If the multiple Dirichlet series (2.1) converges for the topology of $\mathcal{O}(\Omega)$ and $|\lambda^k| \to \infty$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} \frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \le 0 \ . \tag{2.2}$$

Conversely, if the coefficients of (2.1) satisfy condition (2.2) and if

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0 \ , \tag{2.3}$$

then the series (2.1) converges absolutely for the topology of $\mathcal{O}(\Omega)$.

From this theorem it follows that if (2.3) holds, then the series (2.1) converges for the topology of $\mathcal{O}(\Omega)$ if and only if it converges absolutely for the topology of $\mathcal{O}(\Omega)$.

From now on a bounded convex domain Ω in $\mathbb{C}^n$ with the supporting function $H_\Omega(\zeta)$ and a sequence $(\lambda^k)_{k=1}^\infty$ of complex vectors in $\mathbb{C}^n$ satisfying condition (2.3) are considered to be given.

By virtue of Theorem 2.1, without loss of generality, we can assume that $0 \in \Omega$. Then it is clear that

$$0 < \alpha = \inf_{|\zeta|=1} H_\Omega(\zeta) \le \beta = \sup_{|\zeta|=1} H_\Omega(\zeta) < \infty \ ,$$

and, therefore

$$\alpha|\zeta| \le H_\Omega(\zeta) \le \beta|\zeta| \,, \qquad \forall\, \zeta \in \mathbb{C}^n \ .$$

Also, to the sequence $(\lambda^k)_{k=1}^\infty$ we can associate the following sequence space

$$E_\Omega = \Big\{c = (c_k);\ (2.2) \text{ satisfies}\Big\} \ .$$

Note that for any $c = (c_k) \in E_\Omega$ and $t \in (0,1)$

$$\sum_{k=1}^\infty |c_k|\, e^{tH_\Omega(\lambda^k)} < +\infty \ . \tag{2.4}$$

This inequality will be used very often in the sequel.

Several properties of the space E_Ω were studied in [10], in particular, the characterization of its Köthe dual was obtained. We recall this result that is needed in the next section.

Denote by E_Ω^α the Köthe dual of E_Ω, i.e.

$$E_\Omega^\alpha = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k\, u_k \text{ converges absolutely for all } (c_k) \in E_\Omega \right\} .$$

Lemma 2.2 ([10, Corollary 3.2]). *The Köthe dual of the space E_Ω can be defined as follows*

$$E_\Omega^\alpha = \left\{ (d_k);\ \limsup_{k\to\infty} \frac{\log |d_k|}{H_\Omega(\lambda^k)} < 1 \right\} .$$

Also we have the following result.

Lemma 2.3. *Let (a_k) be a sequence of real numbers. Suppose that*

$$\limsup_{k\to\infty} \left\{ a_k + \frac{\mathrm{Re}\,\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \right\} < A < +\infty\,, \qquad \forall\, z \in \Omega\ . \tag{2.5}$$

Then

$$\limsup_{k\to\infty} a_k \le A - 1\ .$$

Proof: As the function $\mathrm{Re}\,\langle \lambda^k, z\rangle$ is plurisubharmonic in Ω and we already have condition (2.5), it is desirable to apply Hartogs' lemma for the sequence

$$\varphi_k(z) = a_k + \frac{\mathrm{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)}\,, \qquad z \in \Omega, \quad k = 1, 2, \ldots\ .$$

In this case we have only to prove the local boundedness of the sequence $(\varphi_k(z))$. Indeed, it is clear that $\varphi_k(z) \le a_k + 1$, $\forall\, z \in \Omega$. Moreover, from (2.5) it follows, in particular for $z = 0$, that $\limsup_{k\to\infty} a_k < A < +\infty$. These last two inequalities show that for each compact subset $K \subset \Omega$ there exists $M_K > 0$ such that

$$\varphi_k(z) \le a_k + 1 \le M_K\,, \qquad \forall\, z \in K, \quad \forall\, k \ge 1\ .$$

Now applying Hartogs' lemma (see, e.g. [7]) we obtain that if K is a compact in Ω and $\varepsilon > 0$ then for large k

$$\varphi_k(z) = a_k + \frac{\mathrm{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \le A + \frac{\varepsilon}{2}\,, \qquad \forall\, z \in K\ ,$$

which implies that for large k

$$\sup_{z\in K} \varphi_k(z) \le A + \frac{\varepsilon}{2} \;. \tag{2.6}$$

Furthermore, for such an $\varepsilon > 0$ we can choose K so large that

$$\begin{aligned} \sup_{z\in K} \varphi_k(z) &= a_k + \sup_{z\in K} \frac{\operatorname{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \\ &= a_k + \frac{H_K(\lambda^k)}{H_\Omega(\lambda^k)} \ge a_k + 1 - \frac{\varepsilon}{2} \;. \end{aligned} \tag{2.7}$$

Combining (2.6)–(2.7) gives that for each $\varepsilon > 0$ there exists k_0 such that

$$a_k \le A - 1 + \varepsilon \,, \qquad \forall\, k \ge k_0 \;,$$

which means that $\limsup_{k\to\infty} a_k \le A - 1$. The proof is complete. ∎

3 – Matrix transformations of holomorphic Dirichlet series

Denote by $E_\Omega(\mathcal{U})$ the class of all matrices $[u_{jk}]_{j,k=1}^\infty$ having the property that whenever the sequence $c = (c_k) \in E_\Omega$ the sequence of functions $(f_j(z))_{j=1}^\infty$ given by

$$f_j(z) := \sum_{k=1}^\infty u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle} \,, \qquad j = 1, 2, \dots \;, \tag{3.1}$$

converges locally uniformly in Ω, each Dirichlet series $\sum_{k=1}^\infty u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}$ being convergent in Ω, $j = 1, 2, \dots$.

We shall study conditions for a given matrix $[u_{jk}]_{j,k=1}^\infty$ to belong to the class $E_\Omega(\mathcal{U})$.

Theorem 3.1. *If the following conditions hold:*

$$\exists \lim_{j\to\infty} u_{jk} = u_k \,, \qquad k = 1, 2, \dots \;, \tag{3.2}$$

and

$$\limsup_{k\to\infty} \left(\sup_{j\ge 1} \frac{\log |u_{jk}|}{H_\Omega(\lambda^k)} \right) \le 0 \;, \tag{3.3}$$

then the matrix $[u_{jk}]$ belongs to $E_\Omega(\mathcal{U})$.

Proof: Assume that conditions (3.2) and (3.3) hold. Let $c = (c_k) \in E_\Omega$. Take an arbitrary compact subset K of Ω. Then $K \subset s\,\Omega$ for some $s \in (0,1)$.

Due to condition (3.2), for every $k \in \mathbb{N}$ the sequence $(u_{jk})_{j=1}^{\infty}$ is bounded and therefore,

$$Q_k := \sup_{j\geq 1} \log |u_{jk}| < +\infty\,, \qquad \forall\, k \geq 1\ .$$

Hence,

$$|u_{jk}| \leq e^{Q_k}\,, \qquad \forall\, k \geq 1,\ \ \forall\, j \geq 1\ .$$

Furthermore, by condition (3.3), for $s_1 = (1-s)/2$, there exists $N(s)$ such that

$$\frac{\log |u_{jk}|}{H_\Omega(\lambda^k)} \leq s_1\,, \qquad \forall\, k > N(s),\ \ \forall\, j \geq 1\ ,$$

or equivalently,

$$|u_{jk}| \leq e^{s_1 H_\Omega(\lambda^k)}\,, \qquad \forall\, k > N(s),\ \ \forall\, j \geq 1\ . \tag{3.4}$$

Then we have for all $j \geq 1$

$$\begin{aligned} \sup_{z\in K}\Big|\sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle\lambda^k,z\rangle}\Big| &\leq \sum_{k=1}^{\infty} |u_{jk}\, c_k| \sup_{z\in s\Omega} |e^{\langle\lambda^k,z\rangle}| \\ &= \sum_{k=1}^{N(s)} |u_{jk}\, c_k|\, e^{sH_\Omega(\lambda^k)} + \sum_{k=N(s)+1}^{\infty} |u_{jk}\, c_k|\, e^{sH_\Omega(\lambda^k)} \\ &\leq \sum_{k=1}^{N(s)} |c_k|\, e^{Q_k+sH_\Omega(\lambda^k)} + \sum_{k=N(s)+1}^{\infty} |c_k|\, e^{(s_1+s)H_\Omega(\lambda^k)} < +\infty\,, \end{aligned}$$

due to (2.4).

Thus, each series $\sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle\lambda^k,z\rangle}$, $j = 1,2,...$, converges absolutely for the topology of the space $\mathcal{O}(\Omega)$ and therefore, represents a holomorphic function $f_j(z)$ in Ω.

We now prove that the sequence (f_j) converges uniformly on K.

Let ε be any positive number. We choose $N_1 \geq N(s)$ so that

$$\sum_{k=N_1+1}^{\infty} |c_k|\, e^{(s_1+s)H_\Omega(\lambda^k)} < \frac{\varepsilon}{4}\ . \tag{3.5}$$

Denote

$$C(N_1) := \sum_{k=1}^{N_1+1} |c_k|\, e^{sH_\Omega(\lambda^k)}\ . \tag{3.6}$$

Consider the first N_1 columns of the matrix $[u_{jk}]$. From condition (3.2) it follows that there exists N_2 such that

$$(3.7)\qquad |u_{pk} - u_{qk}| < \frac{\varepsilon}{2\,C(N_1)}\,, \qquad \forall\, k = 1, 2, ..., N_1, \quad \forall\, p, q > N_2\ .$$

Then, for all $p, q > N_2$, we get

$$\begin{aligned}\sup_{z\in K}\Big|f_p(z) - f_q(z)\Big| &\le \sum_{k=1}^{\infty}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)}\\ &= \sum_{k=1}^{N_1}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)}\\ &\le \frac{\varepsilon}{2\,C(N_1)}\sum_{k=1}^{N_1}|c_k|\,e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)}\\ &= \frac{\varepsilon}{2} + \sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)}\ ,\end{aligned}$$

due to (3.6)–(3.7).

Concerning the last series, by virtue of (3.4)–(3.5) we have

$$\sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)} \;\le\; 2\sum_{k=N_1+1}^{\infty}|c_k|\,e^{(s_1+s)H_\Omega(\lambda^k)} \;<\; \frac{\varepsilon}{2}\ .$$

The theorem is proved. ∎

Theorem 3.2. *If the matrix $[u_{jk}]$ belongs to $E_\Omega(\mathcal{U})$, then the condition (3.2) and the following condition*

$$(3.8)\qquad \limsup_{k\to\infty}\left(\frac{\log|u_{jk}|}{H_\Omega(\lambda^k)}\right) \le 0\,, \qquad \forall\, j = 1, 2, \dots\ ,$$

must necessarily hold.

This theorem is a consequence of two results given below. Namely, the first part of the theorem follows from Proposition 3.3, while the second one is a consequence of Proposition 3.4 applying for $x_k = u_{jk},\ j = 1, 2, \dots$.

Proposition 3.3. *Suppose that for all "unit vectors" $a^{(m)}$, $m = 1, 2, ...,$ in E_Ω with*

$$a_k^{(m)} = \begin{cases} 1, & \textit{if } k = m,\\ 0, & \textit{otherwise}\ ,\end{cases}$$

the sequence $(f_j^{(m)}(z))_{j=1}^{\infty}$ *defined by*

$$f_j^{(m)}(z) := \sum_{k=1}^{\infty} u_{jk}\, a_k^{(m)}\, e^{\langle \lambda^k, z\rangle}\,, \qquad j = 1, 2, \dots\,, \tag{3.9}$$

converges at the point $z = 0$. *Then condition (3.2) is satisfied.*

Proposition 3.4. *Let* (x_k) *be a given sequence of complex numbers. Suppose that whenever* $(c_k) \in E_\Omega$ *the series* $\sum_{k=1}^{\infty} x_k\, c_k\, e^{\langle \lambda^k, z\rangle}$ *converges in* Ω. *Then*

$$\limsup_{k\to\infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \le 0\ .$$

Proof of Proposition 3.3: Obviously, for each "unit vector" $a^{(m)}$ of the space E_Ω the sequence (3.9) is well defined. Furthermore, from a convergence of the sequence $(f_j^{(m)}(0))_{j=1}^{\infty}$, which in this case has a form $(u_{jm})_{j=1}^{\infty}$, it follows that $u_m = \lim_{j\to\infty} u_{jm}$, $m \in \mathbb{N}$, exists. Thus condition (3.2) is satisfied. ∎

Proof of Proposition 3.4: From the assumption in the Proposition it follows that $(x_k\, e^{\langle \lambda^k, z\rangle})_{k=1}^{\infty} \in E_\Omega^\alpha$, $\forall\, z \in \Omega$, $\forall\, j \ge 1$. By Lemma 2.2 we have

$$\limsup_{k\to\infty} \frac{\log |x_k| + \mathrm{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} < 1\,, \qquad \forall\, z \in \Omega\ .$$

Applying Lemma 2.3 gives

$$\limsup_{k\to\infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \le 0\ .$$

The proof is completed. ∎

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