A NOTE ON MATRIX TRANSFORMATIONS OF HOLOMORPHIC DIRICHLET SERIES (*)

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Abstract: The aim of this note is to study matrix transformations of holomorphic Dirichlet series in bounded convex domains of $\mathbb{C}^n$. The problem considered here is motivated by the paper [1] of Borwein and Jakimovski for power series of one variable.

1 – Introduction

As is well-known the matrix transformation is one of the methods for summing series and sequences using an infinite matrix. Namely, having a matrix $[u_{jk}]_{j,k=1}^{\infty}$, a given series

$$\sum_{k=1}^{\infty} c_k \quad \text{(1.1)}$$

is transformed into the sequence $(\sigma_j)_{j=1}^{\infty}$ with

$$\sigma_j = \sum_{k=1}^{\infty} u_{jk} c_k \quad \text{(1.2)}$$

The series (1.1) is said to be *summable to the sum* $\sigma$ if, for all $j = 1, 2, \ldots$, the series on the right-hand side in (1.2) converges and

$$\lim_{j \to \infty} \sigma_j = \sigma \quad \text{.}$$

The similar notion is also defined for functional series.

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Matrix transformations of power series of one complex variable has been studied previously by several authors. Most of papers dealt with Nörlund matrices, i.e., triangular matrices of a special form (see, e.g., [13, 14]). For the general case of the matrices there seem to be very few articles. Recently Borwein and Jakimovski [1] considered matrix transformations of power series in the complex plane $\mathbb{C}$ and obtained some results on this direction.

In our previous article [12] we studied matrix transformations of the class of multiple Dirichlet series with complex frequencies that define entire functions in $\mathbb{C}^n$.

The present paper continues [12]. We shall be concerned with matrix transformations of holomorphic Dirichlet series in a bounded convex domain of $\mathbb{C}^n$.

It should be noted that the techniques used in [1] do not work for Dirichlet series considered in our article [12] as well as in this paper, because they are essentially one-dimensional and moreover, of power series.

Also, since every entire function as well as every holomorphic function in a convex domain can be represented in the form of Dirichlet series with complex frequencies (see, e.g., [6, 9]) a study of Dirichlet series attracts a great attention. Some problems for these series have already been studied [3, 4, 5, 10, 11].

2 – Holomorphic Dirichlet series in a domain

We recall some basic notation which will be used in this paper.

$O(\Omega)$ ($\Omega$ being a domain in $\mathbb{C}^n$) denotes the space of holomorphic functions in $\Omega$, with the topology of uniform convergence on compact subsets of $\Omega$.

If $z, \zeta \in \mathbb{C}^n$ then $|z| = (z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n)^{1/2}$; $\langle z, \zeta \rangle = z_1 \zeta_1 + \cdots + z_n \zeta_n$.

Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$, with the supporting function defined as follows

$$H_\Omega(\zeta) = \sup_{z \in \Omega} \Re \langle z, \zeta \rangle, \quad \zeta \in \mathbb{C}^n.$$

Let further $(\lambda_k)_{k=1}^\infty$ be a sequence of complex vectors in $\mathbb{C}^n$.

For a Dirichlet series

$$\sum_{k=1}^\infty c_k e^{\langle \lambda_k, z \rangle}, \quad z \in \Omega,$$

there is the following characterization of the coefficients of this series when it converges for the topology of $O(\Omega)$ [10] which is important and necessary for further study.
Theorem 2.1. If the multiple Dirichlet series (2.1) converges for the topology of $O(\Omega)$ and $|\lambda^k| \to \infty$ as $k \to \infty$, then

$$\limsup_{k \to \infty} \frac{\log |c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leq 0.$$  \hspace{1cm} (2.2)

Conversely, if the coefficients of (2.1) satisfy condition (2.2) and if

$$\lim_{k \to \infty} \frac{\log k}{|\lambda^k|} = 0,$$  \hspace{1cm} (2.3)

then the series (2.1) converges absolutely for the topology of $O(\Omega)$.

From this theorem it follows that if (2.3) holds, then the series (2.1) converges for the topology of $O(\Omega)$ if and only if it converges absolutely for the topology of $O(\Omega)$.

From now on a bounded convex domain $\Omega$ in $\mathbb{C}^n$ with the supporting function $H_\Omega(\zeta)$ and a sequence $(\lambda^k)_{k=1}^\infty$ of complex vectors in $\mathbb{C}^n$ satisfying condition (2.3) are considered to be given.

By virtue of Theorem 2.1, without loss of generality, we can assume that $0 \in \Omega$. Then it is clear that

$$0 < \alpha = \inf_{|\zeta|=1} H_\Omega(\zeta) \leq \beta = \sup_{|\zeta|=1} H_\Omega(\zeta) < \infty,$$

and, therefore

$$\alpha|\zeta| \leq H_\Omega(\zeta) \leq \beta|\zeta|, \quad \forall \zeta \in \mathbb{C}^n.$$  \hspace{1cm}

Also, to the sequence $(\lambda^k)_{k=1}^\infty$ we can associate the following sequence space

$$E_\Omega = \{ c = (c_k); \; (2.2) \; \text{satisfies} \}.$$  \hspace{1cm}

Note that for any $c = (c_k) \in E_\Omega$ and $t \in (0,1)$

$$\sum_{k=1}^\infty |c_k| e^{tH_\Omega(\lambda^k)} < +\infty.$$  \hspace{1cm} (2.4)

This inequality will be used very often in the sequel.

Several properties of the space $E_\Omega$ were studied in [10], in particular, the characterization of its Köthe dual was obtained. We recall this result that is needed in the next section.
Denote by $E^\alpha_\Omega$ the Kőthe dual of $E_\Omega$, i.e.

$$E^\alpha_\Omega = \left\{ (u_k); \sum_{k=1}^{\infty} c_k u_k \text{ converges absolutely for all } (c_k) \in E_\Omega \right\}.$$

**Lemma 2.2** ([10, Corollary 3.2]). The Kőthe dual of the space $E_\Omega$ can be defined as follows

$$E^\alpha_\Omega = \left\{ (d_k); \limsup_{k \to \infty} \frac{\log |d_k|}{H_\Omega(\lambda^k)} < 1 \right\}.$$

Also we have the following result.

**Lemma 2.3.** Let $(a_k)$ be a sequence of real numbers. Suppose that

$$\limsup_{k \to \infty} \left\{ a_k + \frac{\text{Re} \langle \lambda^k, z \rangle}{H_\Omega(\lambda^k)} \right\} < A < +\infty, \quad \forall z \in \Omega.$$

Then

$$\limsup_{k \to \infty} a_k \leq A - 1.$$

**Proof:** As the function $\text{Re} \langle \lambda^k, z \rangle$ is plurisubharmonic in $\Omega$ and we already have condition (2.5), it is desirable to apply Hartogs’ lemma for the sequence

$$\varphi_k(z) = a_k + \frac{\text{Re} \langle \lambda^k, z \rangle}{H_\Omega(\lambda^k)}, \quad z \in \Omega, \quad k = 1, 2, \ldots.$$

In this case we have only to prove the local boundedness of the sequence $(\varphi_k(z))$. Indeed, it is clear that $\varphi_k(z) \leq a_k + 1, \forall z \in \Omega$. Moreover, from (2.5) it follows, in particular for $z = 0$, that $\limsup_{k \to \infty} a_k < A < +\infty$. These last two inequalities show that for each compact subset $K \subset \Omega$ there exists $M_K > 0$ such that

$$\varphi_k(z) \leq a_k + 1 \leq M_K, \quad \forall z \in K, \quad \forall k \geq 1.$$

Now applying Hartogs’ lemma (see, e.g. [7]) we obtain that if $K$ is a compact in $\Omega$ and $\varepsilon > 0$ then for large $k$

$$\varphi_k(z) = a_k + \frac{\text{Re} \langle \lambda^k, z \rangle}{H_\Omega(\lambda^k)} \leq A + \frac{\varepsilon}{2}, \quad \forall z \in K,$$
which implies that for large $k$

$$
\sup_{z \in K} \varphi_k(z) \leq A + \frac{\varepsilon}{2}.
$$

Furthermore, for such an $\varepsilon > 0$ we can choose $K$ so large that

$$
\sup_{z \in K} \varphi_k(z) = a_k + \sup_{z \in K} \frac{\text{Re}(\lambda^k, z)}{H_\Omega(\lambda^k)} = a_k + \frac{H_K(\lambda^k)}{H_\Omega(\lambda^k)} \geq a_k + \frac{\varepsilon}{2}.
$$

Combining (2.6)–(2.7) gives that for each $\varepsilon > 0$ there exists $k_0$ such that

$$
a_k \leq A - 1 + \varepsilon, \quad \forall k \geq k_0,
$$

which means that $\limsup_{k \to \infty} a_k \leq A - 1$. The proof is complete.

3 – Matrix transformations of holomorphic Dirichlet series

Denote by $E_\Omega(\mathcal{U})$ the class of all matrices $[u_{jk}]_{j,k=1}^{\infty}$ having the property that whenever the sequence $c = (c_k) \in E_\Omega$ the sequence of functions $(f_j(z))_{j=1}^{\infty}$ given by

$$
f_j(z) := \sum_{k=1}^{\infty} u_{jk} c_k e^{(\lambda^k, z)}, \quad j = 1, 2, \ldots,
$$

converges locally uniformly in $\Omega$, each Dirichlet series $\sum_{k=1}^{\infty} u_{jk} c_k e^{(\lambda^k, z)}$ being convergent in $\Omega$, $j = 1, 2, \ldots$.

We shall study conditions for a given matrix $[u_{jk}]_{j,k=1}^{\infty}$ to belong to the class $E_\Omega(\mathcal{U})$.

**Theorem 3.1.** If the following conditions hold:

$$
\exists \lim_{j \to \infty} u_{jk} = u_k, \quad k = 1, 2, \ldots,
$$

and

$$
\limsup_{k \to \infty} \left( \sup_{j \geq 1} \frac{\log|u_{jk}|}{H_\Omega(\lambda^k)} \right) \leq 0,
$$

then the matrix $[u_{jk}]$ belongs to $E_\Omega(\mathcal{U})$. 
Proof: Assume that conditions (3.2) and (3.3) hold. Let $c = (c_k) \in E_\Omega$. Take an arbitrary compact subset $K$ of $\Omega$. Then $K \subset s\Omega$ for some $s \in (0,1)$.

Due to condition (3.2), for every $k \in \mathbb{N}$ the sequence $(u_{jk})_{j=1}^\infty$ is bounded and therefore,

$$Q_k := \sup_{j \geq 1} \log |u_{jk}| < +\infty, \quad \forall k \geq 1.$$

Hence,

$$|u_{jk}| \leq e^{Q_k}, \quad \forall k \geq 1, \quad \forall j \geq 1.$$

Furthermore, by condition (3.3), for $s_1 = (1-s)/2$, there exists $N(s)$ such that

$$\log \frac{|u_{jk}|}{H_\Omega(\lambda^k)} \leq s_1, \quad \forall k > N(s), \quad \forall j \geq 1,$$

or equivalently,

$$(3.4) \quad |u_{jk}| \leq e^{s_1 H_\Omega(\lambda^k)}, \quad \forall k > N(s), \quad \forall j \geq 1.$$

Then we have for all $j \geq 1$

$$\sup_{z \in \Omega} \left| \sum_{k=1}^\infty u_{jk} c_k e^{(\lambda^k z)} \right| \leq \sum_{k=1}^{N(s)} |u_{jk} c_k| e^{(\lambda^k z)} + \sum_{k=N(s)+1}^\infty |u_{jk} c_k| e^{sH_\Omega(\lambda^k)}$$

$$\leq \sum_{k=1}^{N(s)} |c_k| e^{Q_k + sH_\Omega(\lambda^k)} + \sum_{k=N(s)+1}^\infty |c_k| e^{(s_1 + s)H_\Omega(\lambda^k)} < +\infty,$$

due to (2.4).

Thus, each series $\sum_{k=1}^\infty u_{jk} c_k e^{(\lambda^k z)}$, $j = 1, 2, \ldots$, converges absolutely for the topology of the space $\mathcal{O}(\Omega)$ and therefore, represents a holomorphic function $f_j(z)$ in $\Omega$.

We now prove that the sequence $(f_j)$ converges uniformly on $K$.

Let $\varepsilon$ be any positive number. We choose $N_1 \geq N(s)$ so that

$$(3.5) \quad \sum_{k=N_1+1}^\infty |c_k| e^{(s_1 + s)H_\Omega(\lambda^k)} < \frac{\varepsilon}{4}.$$

Denote

$$(3.6) \quad C(N_1) := \sum_{k=1}^{N_1+1} |c_k| e^{sH_\Omega(\lambda^k)}.$$
Consider the first $N_1$ columns of the matrix $[u_{jk}]$. From condition (3.2) it follows that there exists $N_2$ such that

$$|u_{pk} - u_{qk}| < \frac{\varepsilon}{2C(N_1)}, \quad \forall k = 1, 2, \ldots, N_1, \forall \ p, q > N_2. \quad (3.7)$$

Then, for all $p, q > N_2$, we get

$$\sup_{z \in K} |f_p(z) - f_q(z)| \leq \sum_{k=1}^{N_1} (|u_{pk} - u_{qk}|) c_k e^{sH_\Omega(\lambda^k)}$$

$$= \sum_{k=1}^{N_1} (|u_{pk} - u_{qk}|) c_k e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty} (|u_{pk} - u_{qk}|) c_k e^{sH_\Omega(\lambda^k)}$$

$$\leq \frac{\varepsilon}{2C(N_1)} \sum_{k=1}^{N_1} |c_k e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty} (|u_{pk} - u_{qk}|) |c_k e^{sH_\Omega(\lambda^k)}$$

$$= \frac{\varepsilon}{2} + \sum_{k=N_1+1}^{\infty} (|u_{pk} - u_{qk}|) |c_k e^{sH_\Omega(\lambda^k)},$$

due to (3.6)–(3.7).

Concerning the last series, by virtue of (3.4)–(3.5) we have

$$\sum_{k=N_1+1}^{\infty} (|u_{pk} - u_{qk}|) |c_k e^{sH_\Omega(\lambda^k)} \leq 2 \sum_{k=N_1+1}^{\infty} |c_k e^{(s_1+s)H_\Omega(\lambda^k)} \leq \frac{\varepsilon}{2}.$$

The theorem is proved. 

**Theorem 3.2.** If the matrix $[u_{jk}]$ belongs to $E_\Omega(U)$, then the condition (3.2) and the following condition

$$\limsup_{k \to \infty} \frac{\log |u_{jk}|}{H_\Omega(\lambda^k)} \leq 0, \quad \forall j = 1, 2, \ldots, \quad (3.8)$$

must necessarily hold.

This theorem is a consequence of two results given below. Namely, the first part of the theorem follows from Proposition 3.3, while the second one is a consequence of Proposition 3.4 applying for $x_k = u_{jk}, \ j = 1, 2, \ldots.$

**Proposition 3.3.** Suppose that for all “unit vectors” $a^{(m)}, \ m = 1, 2, \ldots,$ in $E_\Omega$ with

$$a_k^{(m)} = \begin{cases} 
1, & \text{if } k = m, \\
0, & \text{otherwise}, 
\end{cases}$$

\[\text{Then, for all } p, q > N_2, \text{ we get} \]
the sequence \((j_j^{(m)}(z))_{j=1}^{\infty}\) defined by

\[
(j_j^{(m)}(z) := \sum_{k=1}^{\infty} u_{jk} a_k^{(m)} e^{(\lambda^k, z)}, \quad j = 1, 2, \ldots ,
\]

converges at the point \(z = 0\). Then condition (3.2) is satisfied.

**Proposition 3.4.** Let \((x_k)\) be a given sequence of complex numbers. Suppose that whenever \((c_k) \in E_\Omega\) the series \(\sum_{k=1}^{\infty} x_k c_k e^{(\lambda^k, z)}\) converges in \(\Omega\). Then

\[
\limsup_{k \to \infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \leq 0.
\]

**Proof of Proposition 3.3:** Obviously, for each “unit vector” \(a^{(m)}\) of the space \(E_\Omega\) the sequence (3.9) is well defined. Furthermore, from a convergence of the sequence \((j_j^{(m)}(0))_{j=1}^{\infty}\), which in this case has a form \((u_{jm})_{j=1}^{\infty}\), it follows that \(u_m = \lim_{j \to \infty} u_{jm}, m \in \mathbb{N}\), exists. Thus condition (3.2) is satisfied. □

**Proof of Proposition 3.4:** From the assumption in the Proposition it follows that \((x_k e^{(\lambda^k, z)})_{k=1}^{\infty} \in E_\Omega^\infty, \forall z \in \Omega, \forall j \geq 1\). By Lemma 2.2 we have

\[
\limsup_{k \to \infty} \frac{\log |x_k| + Re(\lambda^k, z)}{H_\Omega(\lambda^k)} < 1, \quad \forall z \in \Omega.
\]

Applying Lemma 2.3 gives

\[
\limsup_{k \to \infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \leq 0.
\]

The proof is completed. □

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