PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 2 – 1999

A NOTE ON MATRIX TRANSFORMATIONS OF HOLOMORPHIC DIRICHLET SERIES (*)

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Abstract: The aim of this note is to study matrix transformations of holomorphic Dirichlet series in bounded convex domains of $\mathbb{C}^n$. The problem considered here is motivated by the paper [1] of Borwein and Jakimovski for power series of one variable.

1 – Introduction

As is well-known the matrix transformation is one of the methods for summing series and sequences using an infinite matrix. Namely, having a matrix $[u_{jk}]_{j,k=1}^{\infty}$, a given series

$$\sum_{k=1}^{\infty} c_k \tag{1.1}$$

is transformed into the sequence $(\sigma_j)_{j=1}^{\infty}$ with

$$\sigma_j = \sum_{k=1}^{\infty} u_{jk}\, c_k \ . \tag{1.2}$$

The series (1.1) is said to be *summable to the sum* σ if, for all $j = 1, 2, ...$, the series on the right-hand side in (1.2) converges and

$$\lim_{j\to\infty} \sigma_j = \sigma \ .$$

The similar notion is also defined for functional series.

Received: May 19, 1997; *Revised*: March 4, 1998.
1991 Mathematics Subject Classification: 32A05, 47B37, 30B50.
Keywords: Multiple Dirichlet series, sequence space, matrix transformation.
(*) Supported in part by the Swedish Institute (Sweden) and the National Basic Research Program in Natural Sciences (Vietnam).

Matrix transformations of power series of one complex variable has been studied previously by several authors. Most of papers dealed with Nörlund matrices, i.e; triangular matrices of a special form (see, e.g., [13, 14]). For the general case of the matrices there seem to be very few articles. Recently Borwein and Jakimovski [1] considered matrix transformations of power series in the complex plane $\mathbb{C}$ and obtained some results on this direction.

In our previous article [12] we studied matrix transformations of the class of multiple Dirichlet series with complex frequencies that define entire functions in $\mathbb{C}^n$.

The present paper continues [12]. We shall be concerned with matrix transformations of holomorphic Dirichlet series in a bounded convex domain of $\mathbb{C}^n$.

It should be noted that the techniques used in [1] do not work for Dirichlet series considered in our article [12] as well as in this paper, because they are essentially one-dimensional and moreover, of power series.

Also, since every entire function as well as every holomorphic function in a convex domain can be represented in the form of Dirichlet series with complex frequencies (see, e.g., [6, 9]) a study of Dirichlet series attracts a great attention. Some problems for these series have already been studied [3, 4, 5, 10, 11].

2 – Holomorphic Dirichlet series in a domain

We recall some basic notation which will be used in this paper.

$\mathcal{O}(\Omega)$ (Ω being a domain in $\mathbb{C}^n$) denotes the space of holomorphic functions in Ω, with the topology of uniform convergence on compact subsets of Ω.

If $z, \zeta \in \mathbb{C}^n$ then $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$; $\langle z, \zeta\rangle = z_1\zeta_1 + \cdots + z_n\zeta_n$.

Let Ω be a bounded convex domain in $\mathbb{C}^n$, with the supporting function defined as follows

$$H_\Omega(\zeta) = \sup_{z\in\Omega} \operatorname{Re}\langle z, \zeta\rangle\,, \qquad \zeta \in \mathbb{C}^n\ .$$

Let further $(\lambda^k)_{k=1}^\infty$ be a sequence of complex vectors in $\mathbb{C}^n$.

For a Dirichlet series

$$\sum_{k=1}^\infty c_k\, e^{\langle\lambda^k, z\rangle}\,, \qquad z \in \Omega\ , \tag{2.1}$$

there is the following characterization of the coefficients of this series when it converges for the topology of $\mathcal{O}(\Omega)$ [10] which is important and necessary for further study.

Theorem 2.1. *If the multiple Dirichlet series (2.1) converges for the topology of $\mathcal{O}(\Omega)$ and $|\lambda^k| \to \infty$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} \frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leq 0 \ . \tag{2.2}$$

Conversely, if the coefficients of (2.1) satisfy condition (2.2) and if

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0 \ , \tag{2.3}$$

then the series (2.1) converges absolutely for the topology of $\mathcal{O}(\Omega)$.

From this theorem it follows that if (2.3) holds, then the series (2.1) converges for the topology of $\mathcal{O}(\Omega)$ if and only if it converges absolutely for the topology of $\mathcal{O}(\Omega)$.

From now on a bounded convex domain Ω in $\mathbb{C}^n$ with the supporting function $H_\Omega(\zeta)$ and a sequence $(\lambda^k)_{k=1}^\infty$ of complex vectors in $\mathbb{C}^n$ satisfying condition (2.3) are considered to be given.

By virtue of Theorem 2.1, without loss of generality, we can assume that $0 \in \Omega$. Then it is clear that

$$0 < \alpha = \inf_{|\zeta|=1} H_\Omega(\zeta) \leq \beta = \sup_{|\zeta|=1} H_\Omega(\zeta) < \infty \ ,$$

and, therefore

$$\alpha|\zeta| \leq H_\Omega(\zeta) \leq \beta|\zeta| \, , \qquad \forall\, \zeta \in \mathbb{C}^n \ .$$

Also, to the sequence $(\lambda^k)_{k=1}^\infty$ we can associate the following sequence space

$$E_\Omega = \Big\{ c = (c_k);\ \text{(2.2) satisfies} \Big\} \ .$$

Note that for any $c = (c_k) \in E_\Omega$ and $t \in (0,1)$

$$\sum_{k=1}^{\infty} |c_k|\, e^{tH_\Omega(\lambda^k)} < +\infty \ . \tag{2.4}$$

This inequality will be used very often in the sequel.

Several properties of the space E_Ω were studied in [10], in particular, the characterization of its Köthe dual was obtained. We recall this result that is needed in the next section.

Denote by E_Ω^α the Köthe dual of E_Ω, i.e.

$$E_\Omega^\alpha = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k\, u_k \text{ converges absolutely for all } (c_k) \in E_\Omega \right\} .$$

Lemma 2.2 ([10, Corollary 3.2]). *The Köthe dual of the space E_Ω can be defined as follows*

$$E_\Omega^\alpha = \left\{ (d_k);\ \limsup_{k\to\infty} \frac{\log |d_k|}{H_\Omega(\lambda^k)} < 1 \right\} .$$

Also we have the following result.

Lemma 2.3. *Let (a_k) be a sequence of real numbers. Suppose that*

$$\limsup_{k\to\infty} \left\{ a_k + \frac{Re\,\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \right\} < A < +\infty\,, \qquad \forall\, z \in \Omega\ . \tag{2.5}$$

Then

$$\limsup_{k\to\infty} a_k \leq A - 1\ .$$

Proof: As the function $\operatorname{Re}\langle \lambda^k, z\rangle$ is plurisubharmonic in Ω and we already have condition (2.5), it is desirable to apply Hartogs' lemma for the sequence

$$\varphi_k(z) = a_k + \frac{\operatorname{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)}\,, \qquad z \in \Omega,\quad k = 1, 2, \ldots\ .$$

In this case we have only to prove the local boundedness of the sequence $(\varphi_k(z))$. Indeed, it is clear that $\varphi_k(z) \leq a_k + 1$, $\forall\, z \in \Omega$. Moreover, from (2.5) it follows, in particular for $z = 0$, that $\limsup_{k\to\infty} a_k < A < +\infty$. These last two inequalities show that for each compact subset $K \subset \Omega$ there exists $M_K > 0$ such that

$$\varphi_k(z) \leq a_k + 1 \leq M_K\,, \qquad \forall\, z \in K,\quad \forall\, k \geq 1\ .$$

Now applying Hartogs' lemma (see, e.g. [7]) we obtain that if K is a compact in Ω and $\varepsilon > 0$ then for large k

$$\varphi_k(z) = a_k + \frac{\operatorname{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \leq A + \frac{\varepsilon}{2}\,, \qquad \forall\, z \in K\ ,$$

which implies that for large k

$$\sup_{z\in K} \varphi_k(z) \le A + \frac{\varepsilon}{2} \ . \tag{2.6}$$

Furthermore, for such an $\varepsilon > 0$ we can choose K so large that

$$\begin{aligned} \sup_{z\in K} \varphi_k(z) &= a_k + \sup_{z\in K} \frac{\operatorname{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} \\ &= a_k + \frac{H_K(\lambda^k)}{H_\Omega(\lambda^k)} \ge a_k + 1 - \frac{\varepsilon}{2} \ . \end{aligned} \tag{2.7}$$

Combining (2.6)–(2.7) gives that for each $\varepsilon > 0$ there exists k_0 such that

$$a_k \le A - 1 + \varepsilon\,, \qquad \forall\, k \ge k_0 \ ,$$

which means that $\limsup_{k\to\infty} a_k \le A - 1$. The proof is complete. ∎

3 – Matrix transformations of holomorphic Dirichlet series

Denote by $E_\Omega(\mathcal{U})$ the class of all matrices $[u_{jk}]_{j,k=1}^\infty$ having the property that whenever the sequence $c = (c_k) \in E_\Omega$ the sequence of functions $(f_j(z))_{j=1}^\infty$ given by

$$f_j(z) := \sum_{k=1}^\infty u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}\,, \qquad j = 1, 2, \dots \ , \tag{3.1}$$

converges locally uniformly in Ω, each Dirichlet series $\sum_{k=1}^\infty u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}$ being convergent in Ω, $j = 1, 2, \dots$.

We shall study conditions for a given matrix $[u_{jk}]_{j,k=1}^\infty$ to belong to the class $E_\Omega(\mathcal{U})$.

Theorem 3.1. *If the following conditions hold:*

$$\exists \lim_{j\to\infty} u_{jk} = u_k\,, \qquad k = 1, 2, \dots \ , \tag{3.2}$$

and

$$\limsup_{k\to\infty} \left(\sup_{j\ge 1} \frac{\log |u_{jk}|}{H_\Omega(\lambda^k)} \right) \le 0 \ , \tag{3.3}$$

then the matrix $[u_{jk}]$ belongs to $E_\Omega(\mathcal{U})$.

Proof: Assume that conditions (3.2) and (3.3) hold. Let $c = (c_k) \in E_\Omega$. Take an arbitrary compact subset K of Ω. Then $K \subset s\,\Omega$ for some $s \in (0,1)$.

Due to condition (3.2), for every $k \in \mathbb{N}$ the sequence $(u_{jk})_{j=1}^{\infty}$ is bounded and therefore,

$$Q_k := \sup_{j\geq 1} \log |u_{jk}| < +\infty\,, \qquad \forall\, k \geq 1\ .$$

Hence,

$$|u_{jk}| \leq e^{Q_k}\,, \qquad \forall\, k \geq 1,\ \ \forall\, j \geq 1\ .$$

Furthermore, by condition (3.3), for $s_1 = (1-s)/2$, there exists $N(s)$ such that

$$\frac{\log |u_{jk}|}{H_\Omega(\lambda^k)} \leq s_1\,, \qquad \forall\, k > N(s),\ \ \forall\, j \geq 1\ ,$$

or equivalently,

$$|u_{jk}| \leq e^{s_1 H_\Omega(\lambda^k)}\,, \qquad \forall\, k > N(s),\ \ \forall\, j \geq 1\ . \tag{3.4}$$

Then we have for all $j \geq 1$

$$\begin{aligned}\sup_{z\in K}\Big|\sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}\Big| &\leq \sum_{k=1}^{\infty} |u_{jk}\, c_k| \sup_{z\in s\Omega} |e^{\langle \lambda^k, z\rangle}| \\ &= \sum_{k=1}^{N(s)} |u_{jk}\, c_k|\, e^{sH_\Omega(\lambda^k)} + \sum_{k=N(s)+1}^{\infty} |u_{jk}\, c_k|\, e^{sH_\Omega(\lambda^k)} \\ &\leq \sum_{k=1}^{N(s)} |c_k|\, e^{Q_k + sH_\Omega(\lambda^k)} + \sum_{k=N(s)+1}^{\infty} |c_k|\, e^{(s_1+s)H_\Omega(\lambda^k)} < +\infty\ ,\end{aligned}$$

due to (2.4).

Thus, each series $\sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}$, $j = 1, 2, ...,$ converges absolutely for the topology of the space $\mathcal{O}(\Omega)$ and therefore, represents a holomorphic function $f_j(z)$ in Ω.

We now prove that the sequence (f_j) converges uniformly on K.

Let ε be any positive number. We choose $N_1 \geq N(s)$ so that

$$\sum_{k=N_1+1}^{\infty} |c_k|\, e^{(s_1+s)H_\Omega(\lambda^k)} < \frac{\varepsilon}{4}\ . \tag{3.5}$$

Denote

$$C(N_1) := \sum_{k=1}^{N_1+1} |c_k|\, e^{sH_\Omega(\lambda^k)}\ . \tag{3.6}$$

Consider the first N_1 columns of the matrix $[u_{jk}]$. From condition (3.2) it follows that there exists N_2 such that

$$(3.7) \qquad |u_{pk} - u_{qk}| < \frac{\varepsilon}{2\,C(N_1)}\,, \qquad \forall\, k = 1,2,...,N_1, \quad \forall\, p,q > N_2\ .$$

Then, for all $p,q > N_2$, we get

$$\begin{aligned}
\sup_{z\in K}\Big|f_p(z) - f_q(z)\Big| &\le \sum_{k=1}^{\infty}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)} \\
&= \sum_{k=1}^{N_1}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty}\Big|(u_{pk} - u_{qk})\,c_k\Big|\,e^{sH_\Omega(\lambda^k)} \\
&\le \frac{\varepsilon}{2\,C(N_1)}\sum_{k=1}^{N_1}|c_k|\,e^{sH_\Omega(\lambda^k)} + \sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)} \\
&= \frac{\varepsilon}{2} + \sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)}\ ,
\end{aligned}$$

due to (3.6)–(3.7).

Concerning the last series, by virtue of (3.4)–(3.5) we have

$$\sum_{k=N_1+1}^{\infty}\Big(|u_{pk}| + |u_{qk}|\Big)\,|c_k|\,e^{sH_\Omega(\lambda^k)} \ \le\ 2\sum_{k=N_1+1}^{\infty}|c_k|\,e^{(s_1+s)H_\Omega(\lambda^k)} \ <\ \frac{\varepsilon}{2}\ .$$

The theorem is proved. ∎

Theorem 3.2. *If the matrix $[u_{jk}]$ belongs to $E_\Omega(\mathcal{U})$, then the condition (3.2) and the following condition*

$$(3.8) \qquad \limsup_{k\to\infty}\left(\frac{\log|u_{jk}|}{H_\Omega(\lambda^k)}\right) \le 0\,, \qquad \forall\, j = 1,2,\dots\ ,$$

must necessarily hold.

This theorem is a consequence of two results given below. Namely, the first part of the theorem follows from Proposition 3.3, while the second one is a consequence of Proposition 3.4 applying for $x_k = u_{jk}$, $j = 1,2,\dots$.

Proposition 3.3. *Suppose that for all "unit vectors" $a^{(m)}$, $m = 1,2,...$, in E_Ω with*

$$a_k^{(m)} = \begin{cases} 1, & \textit{if } k = m, \\ 0, & \textit{otherwise}\ , \end{cases}$$

the sequence $(f_j^{(m)}(z))_{j=1}^{\infty}$ *defined by*

$$f_j^{(m)}(z) := \sum_{k=1}^{\infty} u_{jk}\, a_k^{(m)}\, e^{\langle \lambda^k, z\rangle}\,, \qquad j = 1, 2, \dots\,, \tag{3.9}$$

converges at the point $z = 0$. *Then condition (3.2) is satisfied.*

Proposition 3.4. *Let* (x_k) *be a given sequence of complex numbers. Suppose that whenever* $(c_k) \in E_\Omega$ *the series* $\sum_{k=1}^{\infty} x_k\, c_k\, e^{\langle \lambda^k, z\rangle}$ *converges in* Ω. *Then*

$$\limsup_{k\to\infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \le 0\ .$$

Proof of Proposition 3.3: Obviously, for each "unit vector" $a^{(m)}$ of the space E_Ω the sequence (3.9) is well defined. Furthermore, from a convergence of the sequence $(f_j^{(m)}(0))_{j=1}^{\infty}$, which in this case has a form $(u_{jm})_{j=1}^{\infty}$, it follows that $u_m = \lim_{j\to\infty} u_{jm}$, $m \in \mathbb{N}$, exists. Thus condition (3.2) is satisfied. ∎

Proof of Proposition 3.4: From the assumption in the Proposition it follows that $(x_k\, e^{\langle \lambda^k, z\rangle})_{k=1}^{\infty} \in E_\Omega^\alpha$, $\forall\, z \in \Omega$, $\forall\, j \ge 1$. By Lemma 2.2 we have

$$\limsup_{k\to\infty} \frac{\log |x_k| + \mathrm{Re}\langle \lambda^k, z\rangle}{H_\Omega(\lambda^k)} < 1\,, \qquad \forall\, z \in \Omega\ .$$

Applying Lemma 2.3 gives

$$\limsup_{k\to\infty} \frac{\log |x_k|}{H_\Omega(\lambda^k)} \le 0\ .$$

The proof is completed. ∎

ACKNOWLEDGEMENTS – The author would like to thank Professor C.O. Kiselman for stimulating discussions. Also he thanks the referee for useful remarks.

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