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ASYMPTOTIC DISTRIBUTION OF GUMBEL STATISTIC IN A SEMI-PARAMETRIC APPROACH (*)

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Abstract: This note is an answer to some open problems connected with recent developments for appropriate methodologies for making inferences on the tail of a distribution function (d.f.). Namely, in Fraga Alves and Gomes (1996), the *Gumbel statistic*, based on the top part of a sample, is used in a semi-parametric approach, in order to fit an appropriate tail to the underlying model to a data set. The problem of statistical inference about extremal observations is handled there according to a test for choosing the most appropriate domain of attraction for the tail distribution, which gives preference to the Gumbel domain for the null hypothesis. The asymptotic behaviour of the referred statistic is derived therein under that null hypothesis and here we present similar extended results under the alternative conditions, i.e., for d.f. that belongs to the other Generalized Extreme Value domains, as an accomplishment to the promise made in last chapters of Fraga Alves and Gomes (1995; 1996).

1 – Introduction

Suppose we are interested in making inferences about extremal values of some random variable, for which we have an available data set, in such a way that it is reasonable to identify it with $X_1, X_2, ..., X_n$, an independent, identically distributed (i.i.d.) sample from a d.f. $F(\cdot\,; \lambda, \delta)$, where $\lambda \in \mathbb{R}$ and $\delta > 0$ are eventually the location and the scale parameters, respectively.

There has been several approaches to accomplish the main objective of inferring about very extremal values of the random quantity under research, from

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which the extreme value theory plays a very important role. The standard *Generalized Extreme Value* $GEV(\gamma)$*-Model* given by

$$(1) \qquad G_\gamma(z) = \begin{cases} \exp\Big(-(1+\gamma z)^{-1/\gamma}\Big), & \text{for } \ 1+\gamma z>0, \quad \text{if } \ \gamma\neq 0 \\ \exp\Big(-\exp(-z)\Big), & \text{for } \ z\in\mathbb{R}, \quad \text{if } \ \gamma=0 \end{cases}$$

unifies the possible limit behaviours of the maximum conveniently normalized, i.e., if there are sequences $a_n > 0$ and b_n and some $\gamma \in \mathbb{R}$, such that $\lim_{n\to\infty} P\{(\max_{1\le i\le n} X_i - b_n)/a_n \le z\} = G_\gamma(z)$, for all z. This means that F belongs to some extremal domain of attraction and we denote that fact by $F \in \mathcal{D}(G_\gamma)$.

As a consequence, one possible way of handling the initial problem is modelling the upper tail of F with the $GEV(\gamma)$-Model or even with its own tail, the *Generalized Pareto Model,* $GP(\gamma)$*-Model*, defined for $y > 0$ by

$$(2) \qquad F_\gamma(y) = \begin{cases} 1-(1+\gamma\, y)^{-1/\gamma}, & \text{for } \ 1+\gamma\, y>0, \quad \text{if } \ \gamma\neq 0 \\ 1-\exp(-y), & \text{for } \ y>0, \quad \text{if } \ \gamma=0 \end{cases}$$

with the inclusion of eventual location and scale parameters.

Moreover, consider the top m values from the original sample of size n and denote the referred decreasing ordered sample by

$$\underset{\sim}{X} = \Big(X_{(1)},\, ...,\, X_{(r_m)},\, ...,\, X_{(m)}\Big)$$

to which corresponds the standardized sample

$$\underset{\sim}{Z} = \Big(Z_{(1)},\, ...,\, Z_{(r_m)},\, ...,\, Z_{(m)}\Big)\,, \qquad Z_{(j)} = (X_{(j)}-\lambda)/\delta, \quad j=1,...,m\ .$$

In what follows we will denote the k-th increasing o.s. associated to an i.i.d. sample of size n with a subscript ${}_{k:n}$. According to that notation we have the identity $X_{(k)} = X_{n-k+1:n}$, for instance.

We recall here that the Gumbel Statistic, *location/scale invariant*, is defined by

$$(3) \qquad G_m(\underset{\sim}{X}) = \frac{X_{(1)}-X_{(r_m)}}{X_{(r_m)}-X_{(m)}} \overset{d}{=} G_m(\underset{\sim}{Z})\ ,$$

where

$$r_m := \left[\frac{m+1}{2}\right] = \begin{cases} k, & \text{if } \ m=2k \\ k+1, & \text{if } \ m=2k+1 \end{cases}\,, \qquad k\in\mathbb{N}\ .$$

It is just in this set-up and inspired by an old paper of Gumbel (1965), that Gomes first used the *Gumbel Statistic*, with the objective of choosing between $H_0 : \gamma = 0$ vs. $H_1 : \gamma \neq 0$, (Gomes, (1982); Gomes and Van Montfort (1986)).

Here we will refer mainly two approaches that will be convenient for modelling the top m values from the original sample of size n ($m \ll n$).

In Section 2 (*parametric approach*) the upper observations will be modelled by the extremal process that corresponds to the limit behaviour of m top order statistics, m fixed and n increases to infinity. We present the results obtained so far in Gomes (1987), Gomes and Alpuim (1986), Fraga Alves and Gomes (1996), and the latest developments on the subject. The main idea of this section is to make an adequate bridge between both kind of set-ups under consideration (*parametric and semi-parametric*).

In Section 3 (*semi-parametric approach*) there is no fitted model, but only conditions of the type $F \in \mathcal{D}(G_\gamma)$, letting m increase with n, with some restrictions on the increasing rate, according to second order conditions on the tail of F. This is our main reason to rename this approach as *parametric on the tail*.

The main results now presented are connected with the asymptotic behaviour of the Gumbel Statistic, when $\gamma \neq 0$, since for the Gumbel domain of attraction similar conclusions were already available.

In fact, the problem of statistical inference about extremal observations is handled in Fraga Alves and Gomes (1996) according to a test for choosing the most appropriate domain of attraction for the tail distribution, which gives preference to the Gumbel domain for the null hypothesis. The asymptotic behaviour of the referred statistic is derived therein under that null hypothesis.

Here we present similar extended results under the alternative conditions, i.e., for d.f. that belongs to the other Generalized Extreme Value domains, which is useful to study the asymptotic power of such tests, for instance.

2 – Parametric approach

2.1. Extremal process and Generalized Pareto model

Let us suppose that the largest m observations do not correspond to the largest o.s. associated to some i.i.d. sample, but instead the alternative asymptotic model that corresponds to *fixed* m, $n \to \infty$, conveniently describes their joint stochastic behaviour.

This means that the largest observations denoted now as $X_1 \geq \cdots \geq X_m$ are such that the standardized $Z_j = (X_j - \lambda)/\delta$, $j = 1, 2, ..., m$, have the joint density function

$$h_\gamma(z_1, z_2, ..., z_m) = g_\gamma(z_m) \prod_{j=1}^{m-1} \frac{g_\gamma(z_j)}{G_\gamma(z_j)}\,, \qquad z_1 > \cdots > z_m \;, \tag{4}$$

where $g_\gamma(z) = \partial G_\gamma(z)/\partial z$. This will be named by *$GEV(\gamma)$ Extremal Process*. The *main property* that Gomes (1987) used to obtain the exact distributional results for $G_m = (X_1 - X_{r_m})/(X_{r_m} - X_m)$ is the following:

The *normalized exceedances* in the *$GEV(\gamma)$ Extremal Process* (4)

$$(Z_i - Z_m)/(1 + \gamma\, Z_m) \stackrel{d}{=} Y_{m-i:m-1}, \qquad \text{for} \;\; i = 1, 2, ..., m-1 \;, \tag{5}$$

where $Y_{m-i:m-1}$, $i = 1, 2, ..., m-1$ are the corresponding increasing o.s. associated to an i.i.d. sample of size $m - 1$ from the *$GP(\gamma)$ Model* (2).

Notice that if we are working with an i.i.d. sample of *$GP(\gamma)$ Model* (2) with associated decreasing o.s. $\underset{\sim}{Y} \equiv (Y_{(1)}, ..., Y_{(m)}, ..., Y_{(n)})$ a similar property holds. In fact, it is very simple to show (cf. Appendix, Lemma 1) that, for all *subsamples* of size $m = 2, ..., n$, the *normalized exceedances*

$$(Y_{(i)} - Y_{(m)})/(1 + \gamma\, Y_{(m)}) \stackrel{d}{=} Y_{m-i:m-1}, \qquad \text{for} \;\; i = 1, 2, ..., m-1 \;.$$

As an important consequence, every methodology that is based on that distributional identity (5) or (6), makes the two approaches equivalent and if exact results are available for appropriate statistics, we produce goodness-of-fit tests for composed null hypothesis, as exponentiality tests, or more generally, tests for Pareto Model, in order to fit one of those models the top sample.

2.2. Exact results

For *fixed* m in the $GEV(\gamma)$ Extremal Process, for $\gamma = 0$, according to (5), we have

$$G_m \stackrel{d}{=} \frac{E_{s:s}}{E'_{m-s-1:m-1}}\,, \qquad s = r_m - 1 \;, \tag{7}$$

with $\{E_i\}_{i=1,...,s}$ an i.i.d. sample from EXP(1) Model and $E_{s:s} = \max_{1 \leq i \leq s} E_i$ independent from $E'_{m-s-1:m-1}$, the $(m-s-1)$-th increasing o.s. associated to the independent i.i.d. sample $\{E'_j\}_{j=1,...,m-1}$ from the EXP(1) Model.

In Gomes (1987), letting $X \stackrel{d}{=} E_{s:s}$ with d.f. given by $F_X(x) = (1-\exp(-x))^s$, $x > 0$ and $Y \stackrel{d}{=} E'_{m-s-1:m-1}$, with p.d.f.

$$f_Y(y) = \frac{1}{B(m-s-1,\, s+1)} \Big(1 - exp(-y)\Big)^{m-s-2} \Big(exp(-y)\Big)^{s+1}, \qquad y > 0 ,$$

X and Y independent r.v.'s., it is derived the exact d.f. of G_m

$$\begin{aligned} P[G_m \le u] &= \int_0^\infty F_X(uy)\, f_Y(y)\, dy \\ &= 1 + \sum_{j=1}^{s} \binom{s}{j} (-1)^j \prod_{i=1}^{m-s-1} \left\{1 + \frac{ju}{m-i}\right\}^{-1}, \qquad u > 0 . \end{aligned} \tag{8}$$

The exact percentage points for a normalized version of the Gumbel statistic, denoted then by $G_m^* = \ln 2G_m - \ln(r_m - 1)$, were obtained via numerical methods by inversion of (8) in Fraga Alves and Gomes (1996).

For *fixed* m in the $GEV(\gamma)$ Extremal Process, for $\gamma \neq 0$, according to (5) we have that

$$G_m \stackrel{d}{=} \frac{Y_{s:s}}{Y'_{m-s-1:m-1}}, \qquad s = r_m - 1 , \tag{9}$$

where $\{Y_i\}_{i=1,\dots,s}$ is an i.i.d. sample from GP(γ) Model (2), with $Y_{s:s} = \max_{1\le i\le s} Y_i$ independent from $Y'_{m-s-1:m-1}$, the $(m-s-1)$-th increasing o.s., associated to the independent i.i.d. sample $\{Y'_j\}_{j=1,\dots,m-1}$ from the GP$(-\gamma)$ Model.

Observation 2.1. Notice that if we let $\gamma \to 0$ in (9) result (7) is obtained.

2.3. Asymptotic results

Gomes and Alpuim (1986) derived the asymptotic behaviour of G_m, as $(m \to \infty)$.

For $\gamma < -1/2$,

$$G_m^* = \frac{G_m - b_m(\gamma)}{a_m(\gamma)} \;\underset{m\to\infty}{\overset{w}{\longrightarrow}}\; \aleph(0,1) \tag{10}$$

with normalizing sequences

$$a_m(\gamma) = \frac{\gamma 2^{-\gamma}}{(2^{-\gamma} - 1)^2} \frac{1}{\sqrt{2r_m}} \tag{11}$$

and

$$b_m(\gamma) = \frac{1-(r_m-1)^\gamma}{2^{-\gamma}-1} \; . \tag{12}$$

On the other hand, for $\gamma > -1/2$

$$G_m^* = \frac{G_m - b_m(\gamma)}{a_m(\gamma)} \quad \mathop{\longrightarrow}^{w}_{m\to\infty} \quad GEV(\gamma) \tag{13}$$

where the normalizing contants are now

$$a_m(\gamma) = \frac{\gamma(r_m-1)^\gamma}{1-2^{-\gamma}} \tag{14}$$

and

$$b_m(\gamma) = \frac{1-(r_m-1)^\gamma}{2^{-\gamma}-1} \; . \tag{15}$$

Observation 2.2. The normalizing sequences are such that

$$a_m(\gamma) \mathop{\longrightarrow}_{\gamma\to 0} a_m(0) = 1/\ln 2 \quad \text{and} \quad b_m(\gamma) \mathop{\longrightarrow}_{\gamma\to 0} b_m(0) = \ln(r_m-1)/\ln 2 \; .$$

3 – Semi-parametric or parametric on the tail

Let us now relax the underlying conditions of Section 2 and consider instead that the top o.s. of the sample, $\underset{\sim}{X} = (X_{(1)}, X_{(2)}, ..., X_{(m)})$, are such that

$$F \in \mathcal{D}(\gamma) \quad \text{and} \quad m(n) \to \infty,\; m(n)/n \to 0, \quad \text{as } n\to\infty \; . \tag{16}$$

Moreover, under the validity of second order conditions for $F \in \mathcal{D}(G_\gamma)$ and extra conditions on the rate for $m(n) \to \infty$, as $n\to\infty$, we still achieve the results on the asymptotic distribution of G_m^*, suitably normalized with the sequences, for each case, properly choosen in (10)–(15).

Let $U := (\frac{1}{1-F})^{\leftarrow}$, where the arrow ($\leftarrow$) stands for the generalized inverse function. We are going to establish the second order conditions for F in terms of $U(\cdot)$.

3.1. Second order π-variation conditions

Suppose that $F \in \mathcal{D}(G_\gamma)$, and F has a positive density F' so that U' exists. Moreover, suppose that

$$\pm t^{1-\gamma}\, U'(t) \in \Pi(a), \qquad \text{for a positive function } \ a(t) \ . \tag{17}$$

[second order π-variation, (*Dekkers and De Haan, 1989*)].

For $F \in \mathcal{D}(G_0)$ the following result was obtained by Fraga Alves and Gomes (1996).

Theorem 3.1. *Suppose that condition (17) holds for $\gamma = 0$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n \to \infty$,*

$$\frac{1}{2}\, b\Big(\frac{n}{r_0}\Big)\, \ln^2(r_0 - 1) \underset{n\to\infty}{\longrightarrow} C_0, \qquad \textit{with } C_0 \textit{ a positive constant and } \ b(t) := \frac{a(t)}{t\, U'(t)} \ .$$

Then:

- (**i**) *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n \to \infty$, $G^*_m \overset{d}{=} W_k + o_p(1)$, for a r.v. $W_k \cap G_0$, i.e., G^*_m has asymptotically a standard Gumbel distribution.*
- (**ii**) *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m \sim m_0$, as $n \to \infty$, $G^*_m \overset{d}{=} W_k \pm C_0 + o_p(1)$, for a r.v. $W_k \cap G_0$, i.e., G^*_m is asymptotically a Gumbel r.v. with location $\pm C_0$.*

The next theorems state similar results under the general condition of $F \in \mathcal{D}(G_\gamma)$, with second order behaviour of type (17), for $\gamma \neq 0$.

Theorem 3.2. *Suppose that condition (17) holds for $\gamma < -1/2$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n \to \infty$,*

$$b\Big(\frac{n}{r_0}\Big)\, \sqrt{r_0} \underset{n\to\infty}{\longrightarrow} C_0, \qquad \textit{with } C_0 \textit{ a positive constant and } \ b(t) := \frac{a(t)}{t^{1-\gamma}\, U'(t)} \ .$$

Then:

- (**i**) *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n \to \infty$, $G^*_m \overset{d}{=} Q_{n,2} + o_p(1)$, for a r.v. $Q_{n,2} \cap \aleph(0,1)$, i.e., G^*_m has asymptotically a standard normal distribution.*

(**ii**) *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m \sim m_0$, as $n \to \infty$, $G^*_m \overset{d}{=} Q_{n,2} \pm \sqrt{2}\,\gamma^{-1} \log 2C_0 + o_p(1)$, for a r.v. $Q_{n,2} \cap \aleph(0,1)$, i.e., G^*_m is asymptotically normal distributed, for some location.*

Proof: In the following we will use freely the notation of Fraga Alves and Gomes (1996). Also for simplicity of proofs and exposition, we state the results for even $m \equiv 2k$, i.e., we consider

$$G_m \equiv G_{2k} = \frac{X_{(1)} - X_{(k)}}{X_{(k)} - X_{(2k)}}, \quad \text{for} \quad k \in \mathbb{N}$$

and

$$G^*_m \equiv G^*_{2k} = \frac{G_{2k} - b_{2k}(\gamma)}{a_{2k}(\gamma)} \ ,$$

where $a_{2k}(\gamma) = \frac{\gamma.2^{-\gamma}}{(2^{-\gamma}-1)^2} \frac{1}{\sqrt{2k}}$ and $b_{2k}(\gamma) = \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \sim \frac{1}{2^{-\gamma}-1}$ as $n \to \infty$, from (11) and (12).

Now we refer some basic results (see Fraga Alves (1995), Fraga Alves and Gomes (1996) or references therein). Namely, for $i = 1, ..., n$ the descending o.s. associated to the original sample satisfy the distributional identity $X_{(i)} \overset{d}{=} U(Y_{(i)})$ for i.i.d. r.v.'s $Y_1, Y_2, ..., Y_n$ with d.f. $F_Y(y) = 1 - y^{-1}$, $y \geq 1$.

Moreover, $Y_{(k)} \overset{p}{\sim} \frac{n}{k}$ and $\frac{Y_{(2k)}}{Y_{(k)}} \overset{p}{\sim} \frac{1}{2}$, as $n \to \infty$; $\frac{Y_{(1)}}{Y_{(k)}} \overset{d}{=} \max_{1 \leq i \leq k-1} Y_i$ with $\{Y_i\}_{i=1,...,k-1}$ an i.i.d. sample from F_Y; $\frac{Y_{(2k)}}{Y_{(k)}} \overset{d}{=} \frac{1}{2} \exp\{-Q_{n,2}/\sqrt{2k}\}$ and $\frac{Y_{(1)}}{Y_{(k)}} \overset{d}{=} (k-1)\, S_k$ are independent, where $Q_{n,2}$ is asymptotically standard normal and $S_k := (\max_{1 \leq i \leq k-1} Y_i)/(k-1)$ has asymptotically a Fréchet d.f. with unit parameter, $\Phi(y) = \exp(-y^{-1})$, $y > 0$; and finally,

$$\Big\{U(Y_{(k)}) - U(Y_{(2k)})\Big\} \,/\, \Big(Y_{(k)}\, U'(Y_{(k)})\Big) \overset{p}{\sim} (1-2^{-\gamma})/\gamma, \quad \text{as} \ n \to \infty \ .$$

Notice that the second-order π-variation condition $\pm\, t^{1-\gamma}\, U'(t) \in \Pi(a)$ was considered in Theorem 2.3 of Dekkers and De Haan (1989), for deriving the asymptotic distribution of Pickands estimator (Pickands (1975)). Note also that $\lim_{t \to \infty} b(t) = 0$, with $|b(t)|$ a slow varying function. For further information about these class of functions and second order conditions in extreme value theory consult Geluk and De Haan (1987), for example. For the proof we use the following equivalence: $\pm\, t^{1-\gamma}\, U'(t) \in \Pi(a)$ iff, for $x > 0$,

$$\text{(18)} \qquad \frac{U(t\,x) - U(t)}{t\,U'(t)} = \frac{x^\gamma - 1}{\gamma} \pm \frac{b(t)}{\gamma^2} \Big\{x^\gamma\,(\log x^\gamma - 1) + 1\Big\}\,(1 + o(1)) \ ,$$

as $t \to \infty$ with $b(t) := \frac{a(t)}{t^{1-\gamma}\,U'(t)}$ (cf. Appendix, Lemma 2).

Condition (18) is a particular case of class functions (35) referred in Appendix A in Dekkers and De Haan (1993) and of Theorem 1 in De Haan and Stadtmüller (1996, pp. 383–387).

$$G_{2k} - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \overset{d}{=}$$

$$\overset{d}{=} \frac{U\left(\frac{Y_{(1)}}{Y_{(k)}}\,Y_{(k)}\right) - U(Y_{(k)}) - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1}\left\{\left[U(Y_{(k)}) - U\left(\frac{Y_{(2k)}}{Y_{(k)}}\,Y_{(k)}\right)\right]\right\}}{U(Y_{(k)}) - U(Y_{(2k)})}$$

$$\overset{p}{\sim} \frac{\gamma}{1-2^{-\gamma}}\,\frac{1}{Y_{(k)}\,U'(Y_{(k)})}\left\{U\Big((k-1)\,S_k\,Y_{(k)}\Big) - U(Y_{(k)}) \right.$$
$$\left. - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1}\left[U(Y_{(k)}) - U\left(2^{-1}\exp\left(\frac{-Q_{n,2}}{\sqrt{2k}}\right)Y_{(k)}\right)\right]\right\},$$

which, using (18) implies that,

$$G_{2k} - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \overset{d}{=}$$

$$\overset{d}{=} \frac{\gamma}{1-2^{-\gamma}}\left\{\frac{(k-1)^\gamma\,{S_k}^\gamma}{\gamma} - \frac{1}{\gamma} - \frac{1-(k-1)^\gamma}{1-2^{-\gamma}}\left(\frac{2^{-\gamma}}{\gamma}\exp\left(\frac{-\gamma\,Q_{n,2}}{\sqrt{2k}}\right) - \frac{1}{\gamma}\right)\right.$$
$$(19) \qquad \pm \frac{b\left(\frac{n}{k}\right)}{\gamma^2}\left[(k-1)^\gamma\,{S_k}^\gamma\left[\gamma\,\log\Big((k-1)\,S_k\Big) - 1\right] + 1\right.$$
$$\left. + \frac{(k-1)^\gamma - 1}{1-2^{-\gamma}}\left(2^{-\gamma}\exp\left(\frac{-\gamma\,Q_{n,2}}{\sqrt{2k}}\right)\left(-\gamma\log 2 - \frac{\gamma\,Q_{n,2}}{\sqrt{2k}} + o_p\left(\frac{1}{\sqrt{k}}\right) - 1\right) + 1\right)\right]$$
$$\left. \cdot\,(1+o_p(1))\right\}.$$

Reminding that if $\gamma < 0$, then $\frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \sim \frac{1}{2^{-\gamma}-1}$, as $n \to \infty$, the following is a very convenient representation for the results, in the range $\gamma < -1/2$:

$$\frac{(2^{-\gamma}-1)^2\,\sqrt{2k}}{\gamma\,2^{-\gamma}}\left\{G_{2k} - \frac{1}{2^{-\gamma}-1}\right\} \overset{d}{=}$$

$$\overset{d}{=} Q_{n,2} + o_p(1) \pm \frac{b\left(\frac{n}{k}\right)}{\gamma^2}\left[\sqrt{2k}\,\gamma\log 2 + O_p(k^{\gamma+1/2}\log k) + O_p(k^{\gamma+1/2}) + O_p(1) + o_p(1)\right].$$

Let $m_0(n) \equiv 2\,k_0(n)$ be a sequence such that, $b(\frac{n}{k_0})\sqrt{k_0} \sim C_0$, as $n \to \infty$, with C_0 a positive constant. Then results (i) and (ii) follow, for the appropriate sequences $m \equiv 2\,k(n)$. ∎

Observation 3.1. In the expression (19) the ratio $\frac{1-(k-1)^\gamma}{2^{-\gamma}-1}$ was kept, because that will be used in the following theorems.

Theorem 3.3. *Suppose that condition (17) holds for $-1/2 < \gamma < 0$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n \to \infty$, $b(\frac{n}{r_0})(r_0-1)^{-\gamma} \underset{n\to\infty}{\longrightarrow} C_0$, with C_0 a positive constant and $b(t) := \frac{a(t)}{t^{1-\gamma}\,U'(t)}$. Then:*

- **(i)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n \to \infty$, $G^*_m \stackrel{d}{=} S^*_k + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m has asymptotically a $GEV(\gamma)$ distribution.*
- **(ii)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m \sim m_0$, as $n \to \infty$, $G^*_m \stackrel{d}{=} S^*_k \pm \frac{2^{-\gamma}\log 2}{\gamma(2^{-\gamma}-1)}C_0 + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m is asymptotically a $GEV(\gamma)$, for some location.*

Proof: For $\gamma > -1/2$, from (19), we obtain:

$$\frac{1-2^{-\gamma}}{\gamma(k-1)^\gamma}\left(G_{2k} - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1}\right) \stackrel{d}{=}$$

$$\stackrel{d}{=} (S_k^\gamma-1)/\gamma + o_p(1) \pm \frac{b\left(\frac{n}{k}\right)}{\gamma^2}\left\{\gamma\, S_k^\gamma \log(k-1) + \frac{2^{-\gamma}\,\gamma\log 2}{2^{-\gamma}-1}\,(k-1)^{-\gamma} + O_p(1) + o_p(1)\right\}.$$

So, with $S^*_k := (S^\gamma_k-1)/\gamma$ asymptotically $GEV(\gamma)$ and $G^*_{2k} = \frac{G_{2k}-b_{2k}(\gamma)}{a_{2k}(\gamma)}$, where from (14) and (15)

$$a_{2k}(\gamma) = \frac{\gamma(k-1)^\gamma}{1-2^{-\gamma}}, \quad b_{2k}(\gamma) = \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \sim \frac{1}{2^{-\gamma}-1}, \qquad \text{as } n \to \infty\,,$$

we have

$$\begin{aligned} G^*_{2k} &\stackrel{d}{=} S^*_k + o_p(1) \pm b\left(\frac{n}{k}\right)\,\log(k-1)\,S^\gamma_k/\gamma \pm b\left(\frac{n}{k}\right)\frac{2^{-\gamma}\log 2}{\gamma(2^{-\gamma}-1)}\,(k-1)^{-\gamma} \\ &\quad + O_p\left(b\left(\frac{n}{k}\right)\right)\left(1+o_p(1)\right). \end{aligned} \tag{20}$$

Let $m_0(n) \equiv 2\,k_0(n)$ be a sequence such that, $b(\frac{n}{k_0})(k_0-1)^{-\gamma} \sim C_0$, as $n \to \infty$, with C_0 a positive constant. Then results (i) and (ii) follow, for the appropriate sequences $m \equiv 2\,k(n)$. ∎

Theorem 3.4. *Suppose that condition (17) holds for $\gamma > 0$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n \to \infty$,*

$$b\left(\frac{n}{r_0}\right) \ln(r_0 - 1) \underset{n\to\infty}{\longrightarrow} C_0, \quad \text{with } C_0 \text{ a positive constant} \quad \text{and} \quad b(t) := \frac{a(t)}{t^{1-\gamma}\, U'(t)} \,.$$

Then:

(**i**) *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n \to \infty$, $G^*_m \overset{d}{=} S^*_k + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m has asymptotically a $GEV(\gamma)$ distribution.*

(**ii**) *If $m \equiv m(n)$ is a sequence of integers satisfying(16), such that $m \sim m_0$, as $n \to \infty$, $G^*_m \overset{d}{=} S^*_k \pm C_0(S^*_k + 1/\gamma) + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m is asymptotically a $GEV(\gamma)$, for some location.*

Proof: For $\gamma > 0$, from (20), we obtain:

$$G^*_{2k} \overset{d}{=} S^*_k + o_p(1) \pm b\left(\frac{n}{k}\right) \log(k-1)\, S^\gamma_k/\gamma + O_p\left(b\left(\frac{n}{k}\right)\right) \Big(1 + o_p(1)\Big) \,.$$

Let $m_0(n) \equiv 2\,k_0(n)$ be a sequence such that, $b(\frac{n}{k_0}) \log(k_0 - 1) \sim C_0$, as $n \to \infty$, with C_0 a positive constant. Then results (i) and (ii) follow, for the appropriate sequences $m \equiv 2\,k(n)$. ∎

3.2. Second order regular variation conditions

In this paragraph the underlying conditions for d.f. are chosen in (21) or (22) according to negative or positive γ.

Suppose that $F \in \mathcal{D}(G_\gamma)$, such that, for positive constants ρ and c,

(21) $$\text{for } \gamma < 0\,, \quad \mp\Big\{t^{-\gamma}\Big[U(\infty) - U(t)\Big] - c^\gamma\Big\} \in RV_{\rho\gamma} \,,$$

or

(22) $$\text{for } \gamma > 0\,, \quad \pm\Big\{t^{-\gamma}\, U(t) - c^\gamma\Big\} \in RV_{-\rho\gamma} \,.$$

[second order regular variation, (*Dekkers and De Haan, 1993*)].

Similar assumptions about the second order behaviour of F have been considered in Fraga Alves (1995).

Theorem 3.5. *Suppose that $F \in \mathcal{D}(G_\gamma)$, for $\gamma > 0$, verifying (22). Then, if $m \equiv m(n)$ is a sequence of integers satisfying (16) $G^*_m \overset{d}{=} S^*_k + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m has asymptotically a $GEV(\gamma)$ distribution.*

Proof: First notice that under condition (22)

$$\frac{U(tx)}{U(t)} = x^\gamma + x^\gamma \,(x^{-\rho\gamma} - 1)\, a(t)\,(1 + o(1)) \;, \tag{23}$$

where $a(t) = 1 - \frac{(ct)^\gamma}{U(t)}$, with $|a(t)| \in RV_{-\rho\gamma}$ (see Theorem 2.3 in Fraga Alves (1995)). It will be considered, as previously, for m an even integer.

Reminding the *location/scale invariant* property (3) we obtain

$$G_{2k} = \frac{X_{(1)} - X_{(k)}}{X_{(k)} - X_{(2k)}} \overset{d}{=} \frac{\dfrac{Z_{(1)}}{Z_{(k)}} - 1}{1 - \dfrac{Z_{(2k)}}{Z_{(k)}}} \;. \tag{24}$$

Then, taking into account (23),

$$\begin{aligned}
\frac{Z_{(1)}}{Z_{(k)}} &\overset{d}{=} \frac{U\left(\dfrac{Y_{(1)}}{Y_{(k)}}\, Y_{(k)}\right)}{U(Y_{(k)})} \\
&\overset{d}{=} \left(\frac{Y_{(1)}}{Y_{(k)}}\right)^\gamma + \left(\frac{Y_{(1)}}{Y_{(k)}}\right)^\gamma \left\{\left(\frac{Y_{(1)}}{Y_{(k)}}\right)^{-\rho\gamma} - 1\right\} a\left(\frac{n}{k}\right) \Big(1 + o_p(1)\Big) \\
&\overset{d}{=} \Big((k-1)\, S_k\Big)^\gamma + \Big((k-1)\, S_k\Big)^\gamma \left\{\Big((k-1)\, S_k\Big)^{-\rho\gamma} - 1\right\} a\left(\frac{n}{k}\right) \Big(1 + o_p(1)\Big)
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\frac{Z_{(2k)}}{Z_{(k)}} &\overset{d}{=} \frac{U\left(\dfrac{Y_{(2k)}}{Y_{(k)}} Y_{(k)}\right)}{U(Y_{(k)})} \\
&\overset{d}{=} \left(\frac{Y_{(2k)}}{Y_{(k)}}\right)^\gamma + \left(\frac{Y_{(2k)}}{Y_{(k)}}\right)^\gamma \left\{\left(\frac{Y_{(2k)}}{Y_{(k)}}\right)^{-\rho\gamma} - 1\right\} a\left(\frac{n}{k}\right) \Big(1 + o_p(1)\Big) \\
&\overset{d}{=} 2^{-\gamma} \exp\left(-\frac{\gamma\, Q_{n,2}}{\sqrt{2\,k}}\right) \\
&\quad + 2^{-\gamma} \exp\left(-\frac{\gamma\, Q_{n,2}}{\sqrt{2\,k}}\right) \left\{2^{\rho\gamma} \exp\left(\frac{\gamma\,\rho\, Q_{n,2}}{\sqrt{2\,k}}\right) - 1\right\} a\left(\frac{n}{k}\right) \Big(1 + o_p(1)\Big) \;.
\end{aligned} \tag{26}$$

Notice that it is valid the following convenient representation

$$G_{2k} - \frac{1 - (k-1)^\gamma}{2^{-\gamma} - 1} \overset{p}{\sim} \frac{1}{1 - 2^{-\gamma}} \left\{\left[\frac{Z_{(1)}}{Z_{(k)}} - 1\right] - \frac{1 - (k-1)^\gamma}{2^{-\gamma} - 1} \left[1 - \frac{Z_{(2k)}}{Z_{(k)}}\right]\right\} ,$$

which after replacing (25) and (26) leads us to the final conclusion that, for every sequence $m(n) \equiv 2\,k(n)$ satisfying (16)

$$G^*_{2k} \stackrel{d}{=} S^*_k + o_p(1)$$

with $G^*_{2k} = \frac{G_{2k}-b_{2k}(\gamma)}{a_{2k}(\gamma)}$, with $a_{2k}(\gamma) = \frac{\gamma(k-1)^\gamma}{1-2^{-\gamma}}$ and $b_{2k}(\gamma) = \frac{1-(k-1)^\gamma}{2^{-\gamma}-1}$ from (14) and (15), respectively. ∎

Theorem 3.6. *Suppose that $F \in \mathcal{D}(G_\gamma)$, for $-1/2 < \gamma < 0$, holding condition (21). Let $a(t) := 1 - \frac{(ct)^{-\gamma}}{U^*(t)}$, with $U^*(t) \equiv 1/[U(\infty) - U(t)]$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n \to \infty$,*

$$\frac{1-2^{-\rho\gamma}}{\gamma\,(1-2^\gamma)}\, a\Big(\frac{n}{r_0}\Big)\,(r_0-1)^{-\gamma} \underset{n\to\infty}{\longrightarrow} C_0\,, \qquad \text{with } C_0 \text{ a constant}\,.$$

Then:

- **(i)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n \to \infty$, $G^*_m \stackrel{d}{=} S^*_k + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m has asymptotically a $GEV(\gamma)$ distribution.*
- **(ii)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m \sim m_0$, as $n \to \infty$, $G^*_m \stackrel{d}{=} S^*_k + C_0 + o_p(1)$, for a r.v. $S^*_k \cap G_\gamma$, i.e., G^*_m is asymptotically a $GEV(\gamma)$, for some location C_0.*

Proof: Notice that

$$G_{2k} \;=\; \frac{X_{(1)}-X_{(k)}}{X_{(k)}-X_{(2k)}} \;\stackrel{d}{=}\; \frac{1-\dfrac{U(\infty)-U(Y_{(1)})}{U(\infty)-U(Y_{(k)})}}{\dfrac{U(\infty)-U(Y_{(2k)})}{U(\infty)-U(Y_{(k)})}-1} \;=\; \frac{1-\dfrac{Z^*_{(k)}}{Z^*_{(1)}}}{\dfrac{Z^*_{(k)}}{Z^*_{(2k)}}-1} \tag{27}$$

where $Z^*_{(i)} = 1/[U(\infty)-U(Y_{(i)})] \stackrel{d}{=} 1/[U(\infty)-X_{(i)}]$, $i=1,...,n$ are the associated decreasing o.s.'s to the i.i.d. sample $(Z^*_1,...,Z^*_n)$ with d.f. $H \in \mathcal{D}(G_{-\gamma})$, where $H(z) = F(U(\infty)-z^{-1})$, for $z>0$ (for details see Theorem 3.1 in Fraga Alves (1995)).

Denoting by $U^* \equiv (\frac{1}{1-H})^{\leftarrow}$ we have

$$\frac{Z^*_{(k)}}{Z^*_{(1)}} \stackrel{d}{=} \frac{U^*(Y_{(k)})}{U^*(Y_{(1)})} \quad\text{and}\quad \frac{Z^*_{(k)}}{Z^*_{(2k)}} \stackrel{d}{=} \frac{U^*(Y_{(k)})}{U^*(Y_{(2k)})}\,.$$

Note that $U*$ satisfies (22) and we can apply similar computations to those of the previous case and we get, for $\gamma < 0$,

$$
\begin{aligned}
G_{2k} - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} &\overset{p}{\sim} \\
&\overset{p}{\sim} \frac{1}{2^{-\gamma}-1}\left\{\left[1-\frac{Z^*_{(k)}}{Z^*_{(1)}}\right] - \frac{1-(k-1)^\gamma}{2^{-\gamma}-1}\left[\frac{Z^*_{(k)}}{Z^*_{(2k)}}-1\right]\right\} \\
&\overset{d}{=} \frac{\gamma(k-1)^\gamma}{1-2^{-\gamma}}\,S^*_k + O_p\!\left(k^\gamma\, a\!\left(\frac{n}{k}\right)\right) + o_p\!\left(k^\gamma\, a\!\left(\frac{n}{k}\right)\right) \\
&\quad + \frac{\gamma 2^{-\gamma}}{(2^{-\gamma}-1)^2}\,\frac{Q_{n,2}}{\sqrt{2\,k}} + o_p\!\left(\frac{1}{\sqrt{k}}\right) - \frac{1}{(2^{-\gamma}-1)^2}\, a\!\left(\frac{n}{k}\right) 2^{-\gamma}(1-2^{-\rho\gamma}) \\
&\quad + O_p\!\left(\frac{1}{\sqrt{k}}\, a\!\left(\frac{n}{k}\right)\right) + o_p\!\left(\frac{1}{\sqrt{k}}\, a\!\left(\frac{n}{k}\right)\right).
\end{aligned}
\tag{28}
$$

For $-\frac{1}{2} < \gamma < 0$,

$$G^*_{2k} \overset{d}{=} S^*_k + \frac{1-2^{-\rho\gamma}}{\gamma(1-2^\gamma)}\,(k-1)^{-\gamma}\, a\!\left(\frac{n}{k}\right) + o_p(1)$$

with $G^*_{2k} = \frac{G_{2k}-b_{2k}(\gamma)}{a_{2k}(\gamma)}$, with $a_{2k}(\gamma) = \frac{\gamma(k-1)^\gamma}{1-2^{-\gamma}}$ and $b_{2k}(\gamma) = \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \sim \frac{1}{2^{-\gamma}-1}$, as $n\to\infty$, from (14) and (15), respectively.

Let $m_0(n) \equiv 2\,k_0(n)$ be a sequence such that, $\frac{1-2^{-\rho\gamma}}{\gamma(1-2^\gamma)}\, a(\frac{n}{k_0})\,(k_0-1)^{-\gamma} \sim C_0$, as $n\to\infty$, with C_0 a constant. Then results (i) and (ii) follow, for the appropriate sequences $m \equiv 2\,k(n)$. ∎

Observation 3.2. Notice that sequences $m_0(n) = O(n^{\rho/(1+\rho)})$ as a consequence of $|a(t)| \in RV_{\rho\gamma}$.

Theorem 3.7. *Suppose that $F \in \mathcal{D}(G_\gamma)$, for $\gamma < -1/2$, holding condition (21). Let $a(t) := 1 - \frac{(ct)^{-\gamma}}{U^*(t)}$, with $U^*(t) \equiv 1/[U(\infty)-U(t)]$. Let $m_0 \equiv m_0(n)$ and $r_0 \equiv r_{m_0}$, be sequences such that, as $n\to\infty$, $a(\frac{n}{r_0})\,\sqrt{r_0} \underset{n\to\infty}{\longrightarrow} C_0$, with C_0 a constant. Then:*

- **(i)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m = o(m_0)$ as $n\to\infty$, $G^*_m \overset{d}{=} Q_{n,2} + o_p(1)$, for a r.v. $Q_{n,2} \cap \aleph(0,1)$, i.e., G^*_m has asymptotically a standard normal distribution.*
- **(ii)** *If $m \equiv m(n)$ is a sequence of integers satisfying (16), such that $m \sim m_0$, as $n\to\infty$, $G^*_m \overset{d}{=} Q_{n,2} + \sqrt{2}\,\gamma^{-1}(2^{-\rho\gamma}-1)\,C_0 + o_p(1)$, for a r.v. $Q_{n,2} \cap \aleph(0,1)$, i.e., G^*_m is asymptotically normal distributed, for some location.*

Proof: If we take $\gamma < -\frac{1}{2}$ in expression (28), we obtain

$$G^*_{2k} \stackrel{d}{=} Q_{n,2} + \frac{\sqrt{2}\,(2^{-\rho\gamma}-1)}{\gamma}\,\sqrt{k}\,a\Big(\frac{n}{k}\Big) + o_p(1)$$

with $G^*_{2k} = \frac{G_{2k}-b_{2k}(\gamma)}{a_{2k}(\gamma)}$, where the normalizing sequences are, respectively, $a_{2k}(\gamma) = \frac{\gamma\,2^{-\gamma}}{(2^{-\gamma}-1)^2}\,\frac{1}{\sqrt{2k}}$, and $b_{2k}(\gamma) = \frac{1-(k-1)^\gamma}{2^{-\gamma}-1} \sim \frac{1}{2^{-\gamma}-1}$, as $n \to \infty$, from (11) and (12).

Let $m_0(n) \equiv 2\,k_0(n)$ be a sequence such that, $a(\frac{n}{k_0})\sqrt{k_0} \sim C_0$, as $n \to \infty$, with C_0 a constant. Then results (i) and (ii) follow, for the appropriate sequences $m \equiv 2\,k(n)$. ∎

Observation 3.3. Notice that sequences $m_0(n) = O(n^{-2\rho\gamma/(1-2\rho\gamma)})$, as a consequence of $|a(t)| \in RV_{\rho\gamma}$.

4 – Final comments

The results presented in Section 3, are not unexpected. In fact, for suitable sequences $m(n)$ the asymptotic results achieved in the semi-parametric set-up (*part (i) of the theorems 3.1–3.7*) are similar to those already obtained by Gomes and Alpuim (1986) in the parametric approach, fitting the extremal process to the top part of the original sample. The case $\gamma = -1/2$ is not handled here explicitly, but asymptotic results follow in a similar way as before in Gomes and Alpuim (1986); namely, a convolution of Normal and $GEV(-1/2)$ distributions is obtained, for appropriate sequences $m(n)$.

As a final remark, notice that the asymptotic (and also the exact) behaviour of Gumbel statistic is much influenced by the maximum of the sample, which appears explicitly in the expression of G_m. This is a great contrast with the behaviour of the well known Pickands statistic (Pickands (1975)), which does not distinguish the tails as well as Gumbel statistic does in the present context (cf. Fraga Alves and Gomes (1996, pp. 793).

Appendix – Analytical Results

Lemma 1. *Let* $\underset{\sim}{Y} \equiv (Y_{(1)}, ..., Y_{(n)})$ *be the associated decreasing o.s. to an i.i.d. sample of* $GP(\gamma)$ *Model, (2). Then, for all subsamples of size* $m = 2, ..., n$, *the normalized exceedences*

$$(Y_{(i)} - Y_{(m)})/(1+\gamma\,Y_{(m)}) \stackrel{d}{=} Y_{m-i:m-1}, \qquad \textit{for} \;\; i = 1, 2, ..., m-1 \;.$$

Proof: Denoting by $Y_{i:m}$ and $U_{i:m}$, as usual, the increasing o.s. associated to i.i.d. samples from $GP(\gamma)$ and from the uniform $U(0,1)$ *Models*, respectively, the following relations remain true,

$$Y_{(i)} := Y_{m-i+1:m} \overset{d}{=} \Big[(1-U_{m-i+:m})^{-\gamma} - 1\Big]/\gamma \quad \text{and} \quad 1+\gamma Y_{(m)} \overset{d}{=} (1-U_{1:m})^{-\gamma}$$

which imply that

$$\begin{aligned}
\frac{Y_{(i)} - Y_{(m)}}{1+\gamma\, Y_{(m)}} &\overset{d}{=} \frac{(1-U_{m-i+1:m})^{-\gamma} - (1-U_{1:m})^{-\gamma}}{\gamma\,(1-U_{1:m})^{-\gamma}} \\
&\overset{d}{=} \left\{\frac{(1-U_{m-i+1:m})^{-\gamma}}{(1-U_{1:m})^{-\gamma}} - 1\right\}/\gamma \\
&\overset{d}{=} \left\{\left(\frac{U_{i:m}}{U_{m:m}}\right)^{-\gamma} - 1\right\}/\gamma \\
&\overset{d}{=} \Big\{(U_{i:m-1})^{-\gamma} - 1\Big\}/\gamma \\
&\overset{d}{=} \Big[(1-U_{m-i:m-1})^{-\gamma} - 1\Big]/\gamma \\
&\overset{d}{=} Y_{m-i:m-1}\ .
\end{aligned}$$

Notice that the distributional identity $\frac{U_{i:m}}{U_{m:m}} \overset{d}{=} U_{i:m-1}$ is an obvious consequence of $\frac{U_{i:m}}{U_{m:m}} \overset{d}{=} \exp(-(E_{m-i+1:m} - E_{1:m})) \overset{d}{=} \exp(-E_{m-i:m-1}))$, using the *Rényi (1953) representation.* ∎

Lemma 2. *Suppose that $F \in \mathcal{D}(G_\gamma)$, and F has a positive density F' so that U' exists. Then, for a positive function $a(t)$, $\pm\, t^{1-\gamma}\, U'(t) \in \Pi(a)$ iff*

$$\frac{U(t\,x) - U(t)}{t\,U'(t)} = \frac{x^\gamma - 1}{\gamma} \pm \frac{b(t)}{\gamma^2} \Big\{x^\gamma(\log x^\gamma - 1) + 1\Big\} \Big(1 + o(1)\Big)\ ,$$

for $x > 0$, as $t \to \infty$, where $b(t) := \frac{a(t)}{t^{1-\gamma}\, U'(t)}$.

Proof:

$$\pm\, t^{1-\gamma}\, U'(t) \in \Pi(a) \quad \Longleftrightarrow \quad \lim_{t\to\infty} \Big\{(ts)^{1-\gamma}\, U'(ts) - t^{1-\gamma}\, U'(t)\Big\}/a(t) = \pm \log s\ ,$$

uniformly for $s > 0$. Consequently, as $t \to \infty$,

$$\Big\{(ts)^{1-\gamma}\, U'(ts) - t^{1-\gamma}\, U'(t)\Big\} = \pm\, a(t)\, \log s \Big(1 + o(1)\Big)$$

and

$$\frac{(ts)^{1-\gamma}\, U'(ts) - t^{1-\gamma}\, U'(t)}{U'(t)} = \pm\, \frac{a(t)}{U'(t)}\, \log s \Big(1 + o(1)\Big)$$

which is equivalent to

$$\frac{U'(ts)}{U'(t)} = s^{\gamma-1} \pm \frac{a(t)}{t^{1-\gamma}\, U'(t)}\, s^{\gamma-1}\, \log s\Big(1+o(1)\Big) \ .$$

Then, with $b(t) := \frac{a(t)}{t^{1-\gamma}\, U'(t)}$,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\dfrac{U(tx)-U(t)}{t\,U'(t)} - \dfrac{x^\gamma-1}{\gamma}}{b(t)} &= \lim_{t\to\infty} \frac{\displaystyle\int_1^x \left(\frac{U'(ts)}{U'(t)} - s^{\gamma-1}\right) ds}{b(t)} \\
&= \pm \int_1^x s^{\gamma-1} \log s\, ds \\
&= \pm \left[\frac{x^\gamma\,\log x}{\gamma} - \frac{x^\gamma-1}{\gamma^2}\right]
\end{aligned}$$

and we conclude that, as $t\to\infty$,

$$\frac{U(tx)-U(t)}{t\,U'(t)} = \frac{x^\gamma-1}{\gamma} \pm \frac{b(t)}{\gamma^2}\Big\{x^\gamma\,(\log x^\gamma - 1) + 1\Big\}\,\Big(1+o(1)\Big) \ . \blacksquare$$

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