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SUMS INVOLVING FIBONACCI AND PELL NUMBERS

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1 – Introduction

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$(1.1)\qquad \begin{cases} U_n = p\,U_{n-1} + U_{n-2}\,, & U_0 = 0,\ \ U_1 = 1\ , \\ V_n = p\,V_{n-1} + V_{n-2}\,, & V_0 = 2,\ \ V_1 = p\ . \end{cases}$$

For $p = 1$ we write $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$, which are the Fibonacci and Lucas numbers respectively. Their Binet forms, obtained by using standard techniques for solving linear recurrences, are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n\ ,$$

where α and β are the roots of $x^2 - x - 1 = 0$.

For $p = 2$ we write

$$(1.2)\qquad \begin{cases} P_n = 2\,P_{n-1} + P_{n-2}\,, & P_0 = 0,\ \ P_1 = 1\ , \\ Q_n = 2\,Q_{n-1} + Q_{n-2}\,, & Q_0 = 2,\ \ Q_1 = 2\ . \end{cases}$$

Here $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell–Lucas sequences respectively. Their Binet forms are given by

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n\ ,$$

where γ and δ are the roots of $x^2 - 2x - 1 = 0$.

Further details about the Pell and Pell–Lucas numbers can be found in [3].

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Melham and Shannon [8], motivated by the striking result of D.H. Lehmer [6],

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2i+1}}\right) = \frac{\pi}{4} ,$$

produced a host of similar sums, both finite and infinite, involving terms from the sequences (1.1). They also obtained results involving the arctanh function. Our first aim in this paper is to produce more results of a similar nature for the Fibonacci and Lucas sequences and also for the Pell and Pell–Lucas sequences. This is done in Section 2. The results involving F_n and L_n do not seem to have counterparts for P_n and Q_n, and vice-versa. We find this unusual.

Our second aim is to present an unpublished result of Mahon and to use it to produce a finite sum involving Fibonacci and Lucas numbers. This is done in Sections 3 and 4.

We require the following results which appear in [1] and [4]:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-x\,y}\right), \quad \text{if } x\,y < 1 , \tag{1.3}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+x\,y}\right), \quad \text{if } x\,y > -1 , \tag{1.4}$$

$$\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\frac{x+y}{1+x\,y}\right) , \tag{1.5}$$

$$\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-x\,y}\right) , \tag{1.6}$$

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \quad |x| < 1 . \tag{1.7}$$

We also require

$$F_n^2 - F_{n+k}\,F_{n-k} = (-1)^{n-k}\,F_k^2 , \tag{1.8}$$

which is the Catalan identity, and which generalizes the following Simson's identity:

$$F_n^2 - F_{n-1}\,F_{n+1} = (-1)^{n-1} . \tag{1.9}$$

2 – Sums for Fibonacci and Pell numbers

Our sums for the Fibonacci numbers stem from the following:

Lemma 1.

$$F_{n+2}^2 = F_n\,F_{n+4} + (-1)^n \ , \tag{2.1}$$

$$F_{n+4} - F_n = L_{n+2} \ , \tag{2.2}$$

$$F_{n+4} + F_n = 3\,F_{n+2} \ . \tag{2.3}$$

Identity (2.1) is a direct consequence of (1.8), while (2.2) and (2.3) follow from the recurrence relation for Fibonacci numbers, together with the fact that $L_n = F_{n+1} + F_{n-1}$.

As a consequence of (1.3)–(1.6) and Lemma 1 we have:

Theorem 1. *If n is a positive integer, then*

$$\tan^{-1} F_{n+4} - \tan^{-1} F_n = \tan^{-1}\left(\frac{L_{n+2}}{F_{n+2}^2}\right), \qquad n \text{ even} \ , \tag{2.4}$$

$$\tan^{-1}\left(\frac{1}{F_n}\right) + \tan^{-1}\left(\frac{1}{F_{n+4}}\right) = \tan^{-1}\left(\frac{3}{F_{n+2}}\right), \qquad n \text{ odd} \ , \tag{2.5}$$

$$\tanh^{-1}\left(\frac{1}{F_n}\right) + \tanh^{-1}\left(\frac{1}{F_{n+4}}\right) = \tanh^{-1}\left(\frac{3}{F_{n+2}}\right), \qquad n \text{ even}, \quad n > 2 \ , \tag{2.6}$$

$$\tanh^{-1}\left(\frac{1}{F_n}\right) - \tanh^{-1}\left(\frac{1}{F_{n+4}}\right) = \tanh^{-1}\left(\frac{L_{n+2}}{F_{n+2}^2}\right), \qquad n \text{ odd}, \quad n > 1 \ . \tag{2.7}$$

In (2.4)–(2.7) we replace n by k, $k+4$, $k+8$, ..., $k+4n-4$ to obtain sums which telescope to yield respectively

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{L_{k+4i-2}}{F_{k+4i-2}^2}\right) = \tan^{-1} F_{k+4n} - \tan^{-1} F_k \,, \qquad k \text{ even} \ , \tag{2.8}$$

$$\sum_{i=1}^{n} (-1)^{i-1} \tan^{-1}\left(\frac{3}{F_{k+4i-2}}\right) = \tan^{-1}\left(\frac{1}{F_k}\right) + (-1)^{n-1} \tan^{-1}\left(\frac{1}{F_{k+4n}}\right), \qquad k \text{ odd} \ , \tag{2.9}$$

$$\sum_{i=1}^{n}(-1)^{i-1}\tanh^{-1}\left(\frac{3}{F_{k+4i-2}}\right)=\tanh^{-1}\left(\frac{1}{F_k}\right)+(-1)^{n-1}\tanh^{-1}\left(\frac{1}{F_{k+4n}}\right),\quad k \text{ even},\quad k>2\ , \tag{2.10}$$

$$\sum_{i=1}^{n}\tanh^{-1}\left(\frac{L_{k+4i-2}}{F^2_{k+4i-2}}\right)=\tanh^{-1}\left(\frac{1}{F_k}\right)-\tanh^{-1}\left(\frac{1}{F_{k+4n}}\right),\quad k \text{ odd},\quad k>1\ . \tag{2.11}$$

The infinite sums which arise from (2.8)–(2.11) are respectively

$$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{L_{k+4i-2}}{F^2_{k+4i-2}}\right)=\frac{\pi}{2}-\tan^{-1}F_k\,,\qquad k \text{ even}\ , \tag{2.12}$$

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tan^{-1}\left(\frac{3}{F_{k+4i-2}}\right)=\tan^{-1}\left(\frac{1}{F_k}\right),\qquad k \text{ odd}\ , \tag{2.13}$$

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{3}{F_{k+4i-2}}\right)=\tanh^{-1}\left(\frac{1}{F_k}\right),\qquad k \text{ even},\quad k>2\ , \tag{2.14}$$

$$\sum_{i=1}^{\infty}\tanh^{-1}\left(\frac{L_{k+4i-2}}{F^2_{k+4i-2}}\right)=\tanh^{-1}\left(\frac{1}{F_k}\right),\qquad k \text{ odd},\quad k>1\ . \tag{2.15}$$

The summations (2.14) and (2.15) can be expressed, using (1.7), as

$$\prod_{i=1}^{\infty}\frac{F_{k+4i-2}+(-1)^{i-1}3}{F_{k+4i-2}+(-1)^{i}3}=\frac{F_k+1}{F_k-1}\,,\qquad k \text{ even},\quad k>2\ , \tag{2.16}$$

$$\prod_{i=1}^{\infty}\frac{F^2_{k+4i-2}+L_{k+4i-2}}{F^2_{k+4i-2}-L_{k+4i-2}}=\frac{F_k+1}{F_k-1}\,,\qquad k \text{ odd},\quad k>1\ . \tag{2.17}$$

The next lemma gives identities for the Pell and Pell–Lucas sequences which do not have succinct counterparts for the Fibonacci and Lucas sequences or the sequences (1.1). We use these identities to obtain summation identities involving the Pell and Pell–Lucas numbers.

Lemma 2. *For $\{P_n\}$ and $\{Q_n\}$*

$$P_n^2+P_{n-1}P_{n+1}=\frac{Q_n^2}{4}\ , \tag{2.18}$$

$$Q_n^2+Q_{n-1}Q_{n+1}=16\,P_n^2\ . \tag{2.19}$$

Proof: Use Binet forms. ∎

Now, using the same techniques as previously, we can prove the following with the aid of Lemma 2:

Theorem 2. *For* $\{P_n\}_{n=1}^{\infty}$ *and* $\{Q_n\}_{n=1}^{\infty}$,

$$\tan^{-1}\left(\frac{P_{n-1}}{P_n}\right)+\tan^{-1}\left(\frac{P_n}{P_{n+1}}\right)=\tan^{-1}\left(\frac{Q_n^2}{8\,P_n^2}\right)\,, \tag{2.20}$$

$$\tan^{-1}\left(\frac{Q_{n-1}}{Q_n}\right)+\tan^{-1}\left(\frac{Q_n}{Q_{n+1}}\right)=\tan^{-1}\left(\frac{8\,P_n^2}{Q_n^2}\right)\,, \tag{2.21}$$

$$\tanh^{-1}\left(\frac{P_{n-1}}{P_n}\right)+\tanh^{-1}\left(\frac{P_n}{P_{n+1}}\right)=\tanh^{-1}\left(\frac{Q_n}{4\,P_n}\right)\,, \tag{2.22}$$

$$\tanh^{-1}\left(\frac{Q_{n-1}}{Q_n}\right)+\tanh^{-1}\left(\frac{Q_n}{Q_{n+1}}\right)=\tanh^{-1}\left(\frac{2\,P_n}{Q_n}\right), \qquad n\geq 2\,. \tag{2.23}$$

Identities (2.20)–(2.23) yield respectively

$$\sum_{i=1}^{n}(-1)^{i-1}\tan^{-1}\left(\frac{Q_i^2}{8\,P_i^2}\right)=(-1)^{n-1}\tan^{-1}\left(\frac{P_n}{P_{n+1}}\right)\,, \tag{2.24}$$

$$\sum_{i=1}^{n}(-1)^{i-1}\tan^{-1}\left(\frac{8\,P_i^2}{Q_i^2}\right)=\frac{\pi}{4}+(-1)^{n-1}\tan^{-1}\left(\frac{Q_n}{Q_{n+1}}\right)\,, \tag{2.25}$$

$$\sum_{i=1}^{n}(-1)^{i-1}\tanh^{-1}\left(\frac{Q_i}{4\,P_i}\right)=(-1)^{n-1}\tanh^{-1}\left(\frac{P_n}{P_{n+1}}\right)\,, \tag{2.26}$$

$$\sum_{i=2}^{n}(-1)^{i}\tanh^{-1}\left(\frac{2\,P_i}{Q_i}\right)=\frac{1}{2}\log_e 2+(-1)^{n}\tanh^{-1}\left(\frac{Q_n}{Q_{n+1}}\right)\,. \tag{2.27}$$

Each of (2.24)–(2.27) is an oscillating series which does not converge. If we note that

$$\lim_{n\to\infty}\frac{P_n}{P_{n+1}}=\lim_{n\to\infty}\frac{Q_n}{Q_{n+1}}=\sqrt{2}-1 \qquad \text{and} \qquad \tan^{-1}(\sqrt{2}-1)=\frac{\pi}{8}\,,$$

we see that in (2.24) and (2.25) the length of each oscillation approaches $\frac{\pi}{4}$. Similar observations can be made about (2.26) and (2.27).

3 – An identity of Mahon

In this section we state and prove an identity of Mahon [7, ch. 4]. It is an extension to third order sequences of Simson's identity for the Fibonacci numbers.

We have not seen the identity elsewhere in the literature, and we believe that its statement and Mahon's elegant proof deserve to be more widely known. In the next section we use the identity to derive a finite sum involving Fibonacci numbers, which we believe to be new.

Consider the third order recurrence

$$R_n = a_1 R_{n-1} + a_2 R_{n-2} + a_3 R_{n-3}\,, \qquad a_3 \neq 0\,, \tag{3.1}$$

in which a_1, a_2 and a_3 are complex numbers. Then using (3.1) we define the sequence $\{r_n\}$ for all integers n by

$$r_n = a_1 r_{n-1} + a_2 r_{n-2} + a_3 r_{n-3}\,, \qquad (r_{-1}, r_0, r_1) = (0, 0, 1)\,. \tag{3.2}$$

If α_1, α_2 and α_3 are the roots, assumed distinct, of $x^3 - a_1 x^2 - a_2 x - a_3 = 0$, we have

$$r_n = \frac{\alpha_1^{n+1}}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{\alpha_2^{n+1}}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{\alpha_3^{n+1}}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}\,. \tag{3.3}$$

This can be found in Jarden [5], where it is stated differently due to a shift in the initial values.

An important feature of $\{r_n\}$ is that it is an analogue of the Fibonacci sequence for the recurrence (3.1). Indeed, the same can be said about the two sequences generated by (3.1) whose initial values are $(0,1,0)$ and $(1,0,0)$. An early reference which details properties of these sequences is Bell [2].

Following Shannon and Horadam [10], we define the matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,.$$

Interestingly, A also occurs as M_1 in a sequence of matrices $\{M_n\}_{-\infty}^{\infty}$ defined much earlier by Ward [12], who obtained properties of the solutions of (3.1) using field extensions. We require the following, which occurs essentially as (4.1) in [10]:

$$\begin{pmatrix} r_{n+k} \\ r_{n+k-1} \\ r_{n+k-2} \end{pmatrix} = A^n \begin{pmatrix} r_k \\ r_{k-1} \\ r_{k-2} \end{pmatrix}\,. \tag{3.4}$$

Theorem 3 (Mahon). *For the sequence* $\{r_n\}$,

$$r_n^2 - r_{n-1} r_{n+1} = a_3^n r_{-(n+1)}\,. \tag{3.5}$$

Proof:

$$r_n^2 - r_{n-1}\,r_{n+1} = \begin{vmatrix} r_{n+1} & r_{n+2} & 1 \\ r_n & r_{n+1} & 0 \\ r_{n-1} & r_n & 0 \end{vmatrix}$$

$$= \left| A^n \begin{pmatrix} r_1 \\ r_0 \\ r_{-1} \end{pmatrix} \quad A^n \begin{pmatrix} r_2 \\ r_1 \\ r_0 \end{pmatrix} \quad A^n \cdot A^{-n} \begin{pmatrix} r_1 \\ r_0 \\ r_{-1} \end{pmatrix} \right| \qquad \text{from (3.4)}$$

$$= |A^n| \begin{vmatrix} r_1 & r_2 & r_{-n+1} \\ r_0 & r_1 & r_{-n} \\ r_{-1} & r_0 & r_{-n-1} \end{vmatrix} \qquad \text{from (3.4)}$$

$$= a_3^n \begin{vmatrix} 1 & a_1 & r_{-n+1} \\ 0 & 1 & r_{-n} \\ 0 & 0 & r_{-n-1} \end{vmatrix}$$

$$= a_3^n \cdot r_{-(n+1)} \ . \ \blacksquare$$

To construct this proof, Mahon adapted pioneering work of Waddill [11] for second order sequences. Actually, the sequence which Mahon considered was a special case of $\{r_n\}$, but the proof translates immediately to $\{r_n\}$. Our use of (3.5) in the next section stems from the following simple observation:

$$\sum_{i=1}^{n} \frac{a_3^i\, r_{-(i+1)}}{r_i\, r_{i+1}} = \sum_{i=1}^{n} \left(\frac{r_i}{r_{i+1}} - \frac{r_{i-1}}{r_i} \right) = \frac{r_n}{r_{n+1}} \ . \tag{3.6}$$

4 – Another finite sum

The main result in this section, a finite sum involving Fibonacci numbers, relies on the following:

Lemma 3. *If $k \neq 0$ is an integer and $\{r_n\}$ is defined by*

$$\begin{aligned} & r_n = 2\,L_k\, r_{n-1} - \Big(L_{2k} + 3\,(-1)^k\Big)\, r_{n-2} + (-1)^k\, L_k\, r_{n-3} \ , \\ & (r_{-1}, r_0, r_1) = (0,0,1) \ , \end{aligned} \tag{4.1}$$

then

$$r_n = \frac{(-1)^k\, (F_k\, L_k^{n+1} - F_{k(n+2)})}{F_k} \ . \tag{4.2}$$

Proof: By substitution it is easy to check that the auxiliary equation associated with (4.1) has roots α^k, β^k and $\alpha^k+\beta^k$, where α and β were defined earlier. We now make use of (3.3), and the Binet forms for F_n and L_n, to obtain (4.2). ∎

We are now in a position to state our finite sum. It is

$$(4.3)\qquad \sum_{i=1}^{n} \frac{L_k^i\,F_{k(i-1)} + (-1)^{k(i+1)}\,F_k}{(F_k\,L_k^{i+1} - F_{k(i+2)})\,(F_k\,L_k^{i+2} - F_{k(i+3)})} = \frac{F_k\,L_k^{n+1} - F_{k(n+2)}}{F_k(F_k\,L_k^{n+2} - F_{k(n+3)})}\;.$$

Proof of (4.3): We take $a_3=(-1)^k\,L_k$ and recall that $F_{-n}=(-1)^{n+1}\,F_n$. The result now follows if we use (4.2) to substitute into (3.6). ∎

For $k=1$ and $k=2$ (4.3) becomes, respectively

$$\sum_{i=1}^{n} \frac{F_{i-1} + (-1)^{i+1}}{(F_{i+2}-1)\,(F_{i+3}-1)} = \frac{F_{n+2}-1}{F_{n+3}-1}$$

and

$$\sum_{i=1}^{n} \frac{3^i\,F_{2i-2} + 1}{(3^{i+1} - F_{2i+4})\,(3^{i+2} - F_{2i+6})} = \frac{3^{n+1} - F_{2n+4}}{3^{n+2} - F_{2n+6}}\;.$$

5 – Concluding comments

Our method of obtaining (4.3) is similar to the approach used in [9], where third order sequences were also used to obtain reciprocal sums for second order sequences. We remark that (4.3) also holds for the sequences (1.1). We simply replace F by U and L by V. This becomes evident if we trace through the arguments of Section 4.

Finally, since the sums in [8] and in section 2 of the present paper were inspired by a result of D.H. Lehmer, we feel that it is appropriate to mention some of his achievements. The following short account was kindly provided to me by the referee. In 1931, D.H. Lehmer, son of the mathematician D.N. Lehmer (1867–1938), proved that the Mersenne number M_{257} is not prime. He was the first mathematician to find an even pseudoprime — the number 161 038 — in 1954. He corrected a result announced by the Danish mathematician Bertelsen, in 1983, that the number of primes below 10^9 was 50 847 478, showing that the correct number is 50 847 534. In collaboration with Emma Lehmer, he found, in 1958, that there are 152 892 pairs of twin primes less than 30 000 000.

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