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RATIONAL ARITHMETICAL FUNCTIONS OF ORDER (2,1) WITH RESPECT TO REGULAR CONVOLUTIONS

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Abstract: S.S. Pillai's arithmetical function $P(n)=\sum_{m \pmod n}(m,n)$ is an example of a rational arithmetical function of order $(2,1)$. We generalize $P(n)$ with respect to Narkiewicz's regular convolution and show that the generalized Pillai's function is an example of a rational arithmetical function of order $(2,1)$ with respect to Narkiewicz's regular convolution. We derive identities for rational arithmetical functions of order $(2,1)$ with respect to Narkiewicz's regular convolution and therefore also for Pillai's function and its generalization.

1 – Introduction

S.S. Pillai's [7] arithmetical function $P(n)$ is defined by

$$P(n) \;=\; \sum_{m \pmod n} (m,n) \;, \tag{1.1}$$

where (m,n) is the greatest common divisor of m and n. The structure of P is

$$P = I*I*e^{-1} = I*I*\mu = I*\varphi \;, \tag{1.2}$$

where $*$ is the Dirichlet convolution, $I(n)=n$, $e(n)=1$ $(n\geq 1)$, μ is the Möbius function, and φ is the Euler totient function. Equation (1.2) may be referred to as Cesàro's formula (see [2, p. 127]). The arithmetical function P is an example of a rational arithmetical function of order $(2,1)$ in the terminology of

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Vaidyanathaswamy [11], who called a multiplicative function f a rational arithmetical function of order (r, s) if there exist nonnegative integers r and s and completely multiplicative functions $g_1, g_2, ..., g_r,\ h_1, h_2, ..., h_s$ such that

$$f = g_1 * g_2 * \cdots * g_r * h_1^{-1} * h_2^{-1} * \cdots * h_s^{-1}\ , \tag{1.3}$$

where the inverses are under the Dirichlet convolution. By convention, the identity function e_0 is a rational arithmetical function of order $(0, 0)$, where $e_0(1) = 1$ and $e_0(n) = 0$ for $n > 1$.

Rational arithmetical functions of order $(2, 0)$ are said to be quadratics or specially multiplicative functions. The well-known identical equations [5, 9] for a quadratic f given as $f = g_1 * g_2$ are

$$f(m)\, f(n) = \sum_{d \mid (m,n)} f(mn/d^2)\, (g_1 g_2)\, (d)\ , \tag{1.4}$$

$$f(mn) = \sum_{d \mid (m,n)} f(m/d)\, f(n/d)\, \mu(d)\, (g_1 g_2)\, (d)\ . \tag{1.5}$$

A further well-known identity [9] for f is

$$f(m)\, (g_1 g_2)\, (n) = \sum_{d \mid n} f(n/d)\, f(mnd)\, \mu(d)\ . \tag{1.6}$$

Narkiewicz's [6] A-convolution $*_A$ is a well-known generalization of the Dirichlet convolution. Yocom's [12] A-multiplicative functions is a generalization of completely multiplicative functions in the setting of Narkiewicz's A-convolution. This suggests we define rational arithmetical functions in the setting of Narkiewicz's A-convolution. In fact, we define an arithmetical function f to be an A-rational arithmetical function of order (r, s) if there exist A-multiplicative functions $g_1, g_2, ..., g_r,\ h_1, h_2, ..., h_s$ such that

$$f = g_1 *_A g_2 *_A \cdots *_A g_r *_A h_1^{-1} *_A h_2^{-1} *_A \cdots *_A h_s^{-1}\ . \tag{1.7}$$

Some properties of A-rational arithmetical function of order $(2, 0)$ are given in [3] and [5]. In this paper we give properties of A-rational arithmetical functions of order $(2, 1)$. We motivate the study of these functions by an A-analogue of Pillai's function. This function appears to be a concrete example of an A-rational arithmetical function of order $(2, 1)$. All A-rational arithmetical functions of order $(2, 0)$ are also A-rational arithmetical functions of order $(2, 1)$.

We show that A-rational arithmetical functions of order $(2,1)$ satisfy identities of the types of (1.4), (1.5) and (1.6). Unfortunately, however, these identities then become restricted identities in the sense that these identities do not hold for all m and n. We also note that these identities serve as characterizations of A-rational arithmetical functions of order $(2,1)$.

2 – Preliminaries

In this section we introduce the concept of Narkiewicz's regular convolution. Background material on regular convolutions can be found e.g. in [5, Chapter 4] and [6]. We here review the concepts and notations which are needed in this paper.

For each n, let $A(n)$ be a subset of the set of positive divisors of n. The elements of $A(n)$ are said to be the A-divisors of n. The A-convolution of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)\, g(n/d) \ .$$

Narkiewicz [6] defines an A-convolution to be regular if

(**a**) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the A-convolution,

(**b**) the A-convolution of multiplicative functions is multiplicative,

(**c**) the constant function 1 has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

The inverse of an arithmetical function f such that $f(1) \neq 0$ with respect to the A-convolution is defined by

$$f *_A f^{-1} = f^{-1} *_A f = e_0 \ .$$

It can be proved [6] that an A-convolution is regular if and only if

(**i**) $A(mn) = \{de\colon\ d \in A(m),\ e \in A(n)\}$ whenever $(m,n) = 1$,

(**ii**) for each prime power $p^a\,(>1)$ there exists a divisor $t = \tau_A(p^a)$ of a such that

$$A(p^a) = \{1, p^t, p^{2t}, ..., p^{rt}\} \ ,$$

where $rt = a$, and

$$A(p^{it}) = \{1,\ p^t, p^{2t}, ..., p^{it}\}\,, \qquad 0 \le i < r \ .$$

The positive integer $t = \tau_A(p^a)$ in item (ii) is said to be the A-type of p^a. A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. The A-primitive numbers are 1 and p^t, where p runs through the primes and t runs through the A-types of the prime powers p^a with $a \geq 1$. The order of an A-primitive number $p^t\,(>1)$ is defined by

$$o(p^t) = \sup\Big\{ s \in \mathbb{Z}^+ \colon\ \tau_A(p^{st}) = t \Big\}\ .$$

For all n, let $D(n)$ be the set of all positive divisors of n and let $U(n)$ be the set of all unitary divisors of n, that is,

$$U(n) = \Big\{ d > 0 \colon\ d \,|\, n,\ (d, n/d) = 1 \Big\} = \Big\{ d > 0 \colon\ d \| n \Big\}\ .$$

The D-convolution is the classical Dirichlet convolution and the U-convolution is the unitary convolution [1]. These convolutions are regular with $\tau_D(p^a) = 1$ and $\tau_U(p^a) = a$ for all prime powers $p^a\,(>1)$. Further, if $A = D$, then $o(p) = \infty$ for all primes p, and if $A = U$, then $o(p^a) = 1$ for all prime powers $p^a\,(>1)$.

For a positive integer k, the A_k-convolution is defined by $A_k(n) = \{d : d^k \in A(n^k)\}$. It is known [8] that the A_k-convolution is regular whenever the A-convolution is regular. The symbol $(m,n)_{A,k}$ denotes the greatest k-th power divisor of m which belongs to $A(n)$. Note that $(m,n)_{D,1}$ is the usual greatest common divisor of m and n.

The function $\gamma_A(n)$ is defined as the product of the A-primitive divisors of n. The function $\gamma_D(n)$ is the product of the distinct prime divisors of n with the convention that $\gamma_D(1) = 1$. Under the usual notation $\gamma_D(n) = \gamma(n)$. Further, $\gamma_U(n) = n$ for all n.

The A-analogue μ_A of the Möbius function is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a\,(>1) \text{ is } A\text{-primitive}, \\ 0 & \text{if } p^a \text{ is non-}A\text{-primitive}\ . \end{cases}$$

In particular, $\mu_D = \mu$, the classical Möbius function, and $\mu_U = \mu^*$, the unitary analogue of the Möbius function [1].

Yocom [12] defines an arithmetical function f to be A-multiplicative if f is not identically zero and

$$f(n) = f(d)\, f(n/d)$$

whenever $d \in A(n)$. In particular, D-multiplicative functions are the usual completely multiplicative functions and U-multiplicative functions are the usual multiplicative functions. It is known [12] that a multiplicative function f is A-multiplicative if and only if

$$f^{-1} = \mu_A f\ .$$

It is also known [12] that an A-multiplicative function f is completely determined by its values at A-primitive prime powers p^t. In fact,

$$f(n) = \prod_{p^t \in A(n)} f(p^t)^{\partial_p(n)/t} ,$$

where $n = \prod_p p^{\partial_p(n)}$ is the canonical factorization of n. Further properties of A-multiplicative functions can be found in [12].

An arithmetical function f is said [3, 5] to be an A-specially multiplicative function if $f = g_1 *_A g_2$, where g_1 and g_2 are A-multiplicative functions. In the language of A-rational arithmetical functions (see (1.7)), A-specially multiplicative functions are A-rational arithmetical functions of order $(2,0)$ or A-quadratics.

It is known [3, 5] that generalizations of (1.4) and (1.5) can be written as

$$f(m)\, f(n) = \sum_{d \in A((m,n))} f(mn/d^2)\, (g_1 g_2)\, (d) \,, \tag{2.1}$$

$$f(mn) = \sum_{d \in A((m,n))} f(m/d)\, f(n/d)\, \mu_A(d)\, (g_1 g_2)\, (d) \,, \tag{2.2}$$

where $m, n \in A(mn)$. Since (2.1) and (2.2) do not hold for all m and n, these identities are referred to as "restricted" identities.

If $f = g_1 *_A g_2$, then the function $g_1 g_2$ is referred to as the A-multiplicative function associated with f and is denoted briefly as f'. It can be shown that an A-specially multiplicative function f is completely determined by the values of f and f' at A-primitive prime powers p^t.

3 – Pillai's function in regular convolution rings

We recall that Pillai's function $P(n)$ is defined as

$$P(n) = \sum_{m \ (\mathrm{mod}\ n)} (m, n) \,. \tag{3.1}$$

Now, let f be an arithmetical function, let A be a regular convolution and let k be a positive integer. We define the generalized Pillai's function $P^f_{A,k}(n)$ as

$$P^f_{A,k}(n) = \sum_{m \ (\mathrm{mod}\ n^k)} f\Big((m, n^k)_{A,k}\Big) \,. \tag{3.2}$$

If $f = I$, $A = D$ and $k = 1$, then $P^f_{A,k}(n) = P(n)$. If $f = I$, $A = U$ and $k = 1$, then $P^f_{A,k}(n)$ is the unitary analogue of Pillai's function (see [10]).

We show that the structure of $P^f_{A,k}(n)$ depends on the generalized Euler function $\varphi_{A,k}(n)$ which is defined as the number of integers m (mod n^k) such that $(m,n^k)_{A,k}=1$. It is known that

$$\varphi_{A,k}(n) = (I^k *_{A_k} \mu_{A_k})\,(n) ,$$

where $I^k(n)=n^k$ (see [8]). It is clear that $\varphi_{D,1}$ is the classical Euler totient function.

Theorem 3.1. *We have*

$$P^f_{A,k}(n) = \sum_{d\in A_k(n)} f(d^k)\,\varphi_{A,k}(n/d) . \tag{3.3}$$

Proof: Let $S=\{1,2,...,n\}$. We write

$$S = \bigcup_{d\in A_k(n)} S_d ,$$

where $m\in S_d$ if and only if $(m,n^k)_{A,k}=d^k$. It is clear that this is a partition of S, and $m\in S_d$ if and only if $m=d^k j$, $1\le j\le n^k/d^k$, $(j,n^k/d^k)_{A,k}=1$. Therefore $|\,S_d\,| = \varphi_{A,k}(n/d)$. We thus arrive at our result. ∎

Now suppose that f is an A-multiplicative function. Define f_k by $f_k(n)=f(n^k)$. It can be verified that f_k is A_k-multiplicative. The structure of $P^f_{A,k}$ can be written as

$$P^f_{A,k} = f_k *_{A_k} I^k *_{A_k} \mu_{A_k} = f_k *_{A_k} \varphi_{A_k} . \tag{3.4}$$

Thus $P^f_{A,k}$ is an A_k-rational arithmetical function of order $(2,1)$.

In particular, if $f=I$, then $P^f_{A,k}\equiv P_{A,k} = I^k *_{A_k} I^k *_{A_k} \mu_{A_k} = I^k\,\tau_{A_k} *_{A_k} \mu_{A_k}$, where τ_{A_k} is the number of A_k-divisors of n. The function g given by $g=I^k *_{A_k} I^k = I^k\,\tau_{A_k}$ is an A_k-quadratic with $g'=I^{2k}$.

4 – Identities

In this section we derive analogues of the identities (1.4), (1.5) and (1.6) for A-rational arithmetical functions of order $(2,1)$. Throughout this section we write an A-rational arithmetical function f of order $(2,1)$ in the form

$$f = g_1 *_A g_2 *_A h^{-1} = g *_A h^{-1} , \tag{4.1}$$

where g_1, g_2 and h are A-multiplicative functions.

Theorem 4.1. *Let f be an A-rational arithmetical function of order $(2,1)$ given as in* (4.1). *Then*

$$f(p^{it}) = g(p^t)\, f(p^{(i-1)t}) - (g_1 g_2)(p^t)\, f(p^{(i-2)t}) \tag{4.2}$$

for all A-primitive prime powers p^t and $i = 2, 3, ..., o(p^t)$.

Proof: Let $i = 2$. Then

$$\begin{aligned}
g(p^t)\, f(p^t) - (g_1 g_2)(p^t) &= \Big(g_1(p^t)+g_2(p^t)\Big)\, \Big(g_1(p^t)+g_2(p^t)-h(p^t)\Big) - (g_1 g_2)(p^t) \\
&= g_1(p^t)^2 + g_1(p^t)\, g_2(p^t) + g_2(p^t)^2 - \Big(g_1(p^t)+g_2(p^t)\Big)\, h(p^t) \\
&= g(p^{2t}) - g(p^t)\, h(p^t) \\
&= (g * h^{-1})(p^{2t}) \\
&= f(p^{2t}) \ .
\end{aligned}$$

Let $i \geq 3$. Then

$$\begin{aligned}
& g(p^t)\, f(p^{(i-1)t}) - (g_1 g_2)(p^t)\, f(p^{(i-2)t}) \ = \\
&= \Big(g_1(p^t) + g_2(p^t)\Big) \Big[(g_1 * g_2)(p^{(i-1)t}) - (g_1 * g_2)(p^{(i-2)t})\, h(p^t)\Big] \\
&\quad - g_1(p^t)\, g_2(p^t) \Big[(g_1 * g_2)(p^{(i-2)t}) - (g_1 * g_2)(p^{(i-3)t})\, h(p^t)\Big] \\
&= \sum_{j=0}^{i-1} g_1(p^{(j+1)t})\, g_2(p^{(i-1-j)t}) + \sum_{j=0}^{i-1} g_1(p^{jt})\, g_2(p^{(i-j)t}) - \sum_{j=0}^{i-2} g_1(p^{(j+1)t})\, g_2(p^{(i-1-j)t}) \\
&\quad - h(p^t) \Bigg(\sum_{j=0}^{i-2} g_1(p^{(j+1)t})\, g_2(p^{(i-2-j)t}) + \sum_{j=0}^{i-2} g_1(p^{jt})\, g_2(p^{(i-1-j)t}) \\
&\qquad\qquad - \sum_{j=0}^{i-3} g_1(p^{(j+1)t})\, g_2(p^{(i-2-j)t})\Bigg) \\
&= \sum_{j=0}^{i} g_1(p^{jt})\, g_2(p^{(i-j)t}) - h(p^t) \sum_{j=0}^{i-1} g_1(p^{jt})\, g_2(p^{(i-1-j)t}) \\
&= g(p^{it}) - h(p^t)\, g(p^{(i-1)t}) \\
&= (g * h^{-1})(p^{it}) \\
&= f(p^{it}) \ .
\end{aligned}$$

This completes the proof of Theorem 4.2. ∎

Example 4.1. If $P_{A,k}$ is as given in Section 3, then

$$P_{A,k}(p^{it}) = 2\,p^{kt}P_{A,k}(p^{(i-1)t}) - p^{2kt}P_{A,k}(p^{(i-2)t})$$

for all A_k-primitive prime powers p^t and $i = 2, 3, ..., o(p^t)$.

Theorem 4.2. *Let f be an A-rational arithmetical function of order $(2,1)$ given as in (4.1). Then*

$$f(mn) = \sum_{d\in A((m,n))} g(m/d)\, f(n/d)\, \mu_A(d)\, g'(d) \tag{4.3}$$

whenever $\gamma(m)\,|\,\gamma(n)$ and $m, n \in A(mn)$.

Proof: By multiplicativity, it is enough to prove that for all A-primitive prime powers p^t (>1)

$$f(p^{(a+b)t}) = g(p^{at})\, f(p^{bt}) - g(p^{(a-1)t})\, f(p^{(b-1)t})\, g'(p^t) \tag{4.4}$$

whenever $a+b \le o(p^t)$, $a, b \ge 1$. We proceed by induction on a. By Theorem 4.1, (4.4) holds for $a = 1$. Suppose that (4.4) holds for $a < n$, where $2 \le n \le o(p^t)-1$. Then

$$\begin{aligned}
f(p^{(n+b)t}) &= f(p^{(n-1+b+1)t}) \\
&= g(p^{(n-1)t})\, f(p^{(b+1)t}) - g(p^{(n-2)t})\, f(p^{bt})\, g'(p^t) \\
&= g(p^{(n-1)t}) \Big[g(p^t)\, f(p^{bt}) - g'(p^t)\, f(p^{(b-1)t})\Big] - g(p^{(n-2)t})\, f(p^{bt})\, g'(p^t) \\
&= g(p^{(n-1)t})\, g(p^t)\, f(p^{bt}) - g(p^{(n-1)t})\, f(p^{(b-1)t})\, g'(p^t) \\
&\quad - g(p^{(n-2)t})\, f(p^{bt})\, g'(p^t) \\
&= \Big[g(p^{nt}) + g(p^{(n-2)t})\, g'(p^t)\Big]\, f(p^{bt}) - g(p^{(n-1)t})\, f(p^{(b-1)t})\, g'(p^t) \\
&\quad - g(p^{(n-2)t})\, f(p^{bt})\, g'(p^t) \\
&= g(p^{nt})\, f(p^{bt}) - g(p^{(n-1)t})\, f(p^{(b-1)t})\, g'(p^t)\ .
\end{aligned}$$

This completes the proof of Theorem 4.2. ∎

Example 4.2. We have

$$P_{A,k}(mn) = \sum_{d\in A_k((m,n))} m^k\, \tau_{A_k}(m/d)\, P_{A,k}(n/d)\, \mu_{A_k}(d)\, d^k$$

whenever $\gamma(m)\,|\,\gamma(n)$ and $m, n \in A_k(mn)$.

Theorem 4.3. *Let f be an A-rational arithmetical function of order* $(2,1)$ *given as in* (4.1)*. Then*

$$g(m)\,f(n) \;=\; \sum_{d\in A((m,n))} f(mn/d^2)\,g'(d) \tag{4.5}$$

whenever $\gamma(m)\,|\,\gamma(n)$ *and* $m,n\in A(mn)$.

Proof: By (4.3), we have

$$\begin{aligned}
\sum_{d\in A((m,n))} f(mn/d^2)\,g'(d) \;=\; & \\
&= \sum_{d\in A((m,n))}\;\sum_{e\in A((m/d,n/d))} g(m/(de))\,f(n/(de))\,\mu_A(e)\,g'(e)\,g'(d)\\
&= \sum_{d\in A((m,n))}\;\sum_{\substack{\delta\in A((m,n))\\ d\in A(\delta)}} g(m/\delta)\,f(n/\delta)\,\mu_A(\delta/d)\,g'(\delta)\\
&= \sum_{\delta\in A((m,n))} g(m/\delta)\,f(n/\delta)\,g'(\delta)\;\sum_{d\in A(\delta)}\mu_A(\delta/d)\\
&= g(m)\,f(n)\;,
\end{aligned}$$

that is, (4.5) holds. ∎

Example 4.3. We have

$$m^k\,\tau_{A_k}(m)\,P_{A,k}(n) \;=\; \sum_{d\in A_k((m,n))} P_{A,k}(mn/d^2)\,d^{2k}$$

whenever $\gamma(m)\,|\,\gamma(n)$ and $m,n\in A_k(mn)$.

It is known [4] that generalized Ramanujan sums can be involved in the identical equations (1.4) and (1.5). In Theorems 4.4 and 4.5 we show that generalized Ramanujan sums can also be involved in equations (4.3) and (4.5).

Let α and β be arithmetical functions. We use $S^{\alpha,\beta}_{A,k}(m,n)$ to denote the generalized Ramanujan sum defined by

$$S^{\alpha,\beta}_{A,k}(m,n) \;=\; \sum_{\substack{d\in A_k(n)\\ d^k\,|\,m}} \alpha(d)\,\beta(n/d)\;, \tag{4.6}$$

where m is a nonnegative integer and n is a positive integer. With $\alpha(n)=I(n)=n$ for all n, $\beta=\mu$, $A=D$ and $k=1$, the sum $S^{\alpha,\beta}_{A,k}(m,n)$ reduces to the classical

Ramanujan sum $C(m,n)$. With $\alpha = I^k$ and $\beta = \mu_{A_k}$, the sum $S^{\alpha,\beta}_{A,k}(m,n)$ becomes the generalized Ramanujan sum $C_{A,k}(m,n)$ by Sita Ramaiah [8].

Theorem 4.4. *Let f be an A-rational arithmetical function of order $(2,1)$ given as in (4.1), let α be an arithmetical function and let c be a nonnegative integer. Then*

$$\sum_{d\in A_k((m,n))} g'(d)\, g(m/d)\, f(n/d)\, S^{\alpha,\mu}_{A,k}(c,d) \;=\; \sum_{\substack{d\in A_k((m,n))\\ d^k \mid c}} \alpha(d)\, g'(d)\, f(mn/d^2) \tag{4.7}$$

whenever $\gamma(m)\mid\gamma(n)$ and $m,n\in A_k(mn)$.

Proof: By Theorem 4.2,

$$\begin{aligned}
\sum_{d\in A_k((m,n))} g'(d)\, g(m/d)\, f(n/d) \sum_{\substack{e\in A_k(d)\\ e^k\mid c}} \alpha(e)\,\mu(d/e) \;=& \\
=& \sum_{\substack{e\in A_k((m,n))\\ e^k\mid c}} \alpha(e) \sum_{\substack{d\in A_k((m,n))\\ e\mid d}} g(m/d)\, f(n/d)\, g'(d)\,\mu(d/e)\\
=& \sum_{\substack{e\in A_k((m,n))\\ e^k\mid c}} \alpha(e)\, g'(e) \sum_{\delta\in A_k((m/e,n/e))} g\Big((m/e)/\delta\Big)\, f\Big((n/e)/\delta\Big)\, g'(\delta)\,\mu(\delta)\\
=& \sum_{\substack{e\in A_k((m,n))\\ e^k\mid c}} \alpha(e)\, g'(e)\, f(mn/e^2)
\end{aligned}$$

whenever $\gamma(m)\mid\gamma(n)$ and $m,n\in A_k(mn)$. ∎

Example 4.4. We have

$$\sum_{d\in A_k((m,n))} d^k\, \tau_{A_k}(m/d)\, P_{A,k}(n/d)\, C_{A,k}(c,d) \;=\; \sum_{\substack{d\in A_k((m,n))\\ d^k\mid c}} d^{3k}\, P_{A,k}(mn/d^2)$$

whenever $\gamma(m)\mid\gamma(n)$ and $m,n\in A_k(mn)$.

Theorem 4.5. *Let f be an A-rational arithmetical function of order $(2,1)$ given as in (4.1), let α be an arithmetical function and let c be a nonnegative integer. Then*

$$\sum_{d\in A_k((m,n))} f(mn/d^2)\, g'(d)\, S^{\alpha,I}_{A,k}(c,d) \;=\; \sum_{\substack{d\in A_k((m,n))\\ d^k\mid c}} \alpha(d)\, g'(d)\, g(m/d)\, f(n/d) \tag{4.8}$$

whenever $\gamma(m)\mid\gamma(n)$ and $m,n\in A_k(mn)$.

Proof: Theorem 4.5 follows by Theorem 4.2 in a way similar to as Theorem 4.4 follows by Theorem 4.2. ∎

Lemma 4.1. *Let f be an A-rational arithmetical function of order* $(2,1)$ *given as in* (4.1). *Then*

$$g'(p^{it}) \;=\; g(p^{it})\, f(p^{it}) - g(p^{(i-1)t})\, f(p^{(i+1)t}) \tag{4.9}$$

for all A-primitive prime powers p^t and $i = 1, 2, ..., o(p^t)$.

Proof: If $i = 1$, then (4.9) holds by Theorem 4.1 (with $i = 2$). Assume that (4.9) holds for $i = n-1$. Applying Theorem 4.1 to $g(p^{nt})$ with $h = e_0$ and to $f(p^{(n+1)t})$ we obtain

$$\begin{aligned} g(p^{nt})\, f(p^{nt}) - g(p^{(n-1)t})\, f(p^{(n+1)t}) \;=\; & \\ =\; & \Big[g(p^t)\, g(p^{(n-1)t}) - g'(p^t)\, g(p^{(n-2)t})\Big]\, f(p^{nt}) \\ & - g(p^{(n-1)t})\, \Big[g(p^t)\, f(p^{nt}) - g'(p^t)\, f(p^{(n-1)t})\Big] \\ =\; & g'(p^t)\, \Big[g(p^{(n-1)t})\, f(p^{(n-1)t}) - g(p^{(n-2)t})\, f(p^{nt})\Big] \;. \end{aligned}$$

Now, by the induction hypothesis,

$$g(p^{nt})\, f(p^{nt}) - g(p^{(n-1)t})\, f(p^{(n+1)t}) \;=\; g'(p^{nt}) \;.$$

This completes the proof of Lemma 4.1. ∎

Theorem 4.6. *Let f be an A-rational arithmetical function of order* $(2,1)$ *given as in* (4.1). *Then*

$$f(m)\, g'(n) \;=\; \sum_{d \in A(n)} g(n/d)\, f(mnd)\, \mu_A(d) \tag{4.10}$$

whenever $m, n \in A(mn\, \gamma_A(n))$.

Proof: By multiplicativity, it is enough to consider the case in which m and n are prime powers. Let $m = p^{at}$ and $n = p^{bt}$, where p^t is A-primitive and $a + b + 1 \le o(p^t)$.

If $b = 0$, then both sides of (4.10) reduce to $f(p^{at})$.

If $a = 0$ and $b > 0$, then we obtain by Lemma 4.1

$$\sum_{d \in A(p^{bt})} g(p^{bt}/d)\, f(p^{bt} d)\, \mu_A(d) \;=\; g(p^{bt})\, f(p^{bt}) - g(p^{(b-1)t})\, f(p^{(b+1)t}) \;=\; g'(p^{bt}) \;,$$

that is, (4.10) holds.

If $a, b > 0$, then by (4.4) and Lemma 4.1

$$\begin{aligned}\sum_{d\in A(p^{bt})} g(p^{bt}/d)\, f(p^{(a+b)t}d)\, \mu_A(d) \;=&\\ &= g(p^{bt})\, f(p^{(a+b)t}) - g(p^{(b-1)t})\, f(p^{(a+b+1)t})\\ &= g(p^{bt}) \left[g(p^{bt})\, f(p^{at}) - g(p^{(b-1)t})\, f(p^{(a-1)t})\, g'(p^t)\right]\\ &\quad - g(p^{(b-1)t}) \left[g(p^{(b+1)t})\, f(p^{at}) - g(p^{bt})\, f(p^{(a-1)t})\, g'(p^t)\right]\\ &= f(p^{at}) \left[g^2(p^{bt}) - g(p^{(b-1)t})\, g(p^{(b+1)t})\right]\\ &= f(p^{at})\, g'(p^{bt}) \ ,\end{aligned}$$

that is, (4.10) holds. ∎

Example 4.5. We have

$$P_{A,k}(m)\, n^k \;=\; \sum_{d\in A_k(n)} \tau_{A_k}(n/d)\, P_{A,k}(mnd)\, \mu_{A_k}(d)/d^k$$

whenever $m, n \in A_k(mn\, \gamma_{A_k}(n))$.

Remark 4.1. If f is an A-quadratic, then f is an A-rational arithmetical function of order $(2,1)$ with $f = g$ under the notation of (4.1). Clearly Theorems 4.1–4.6 hold in this case with the replacement $f = g$, and it can be verified that then the condition $\gamma(m) \mid \gamma(n)$ may be left out, cf. (2.1) and (2.2).

Remark 4.2. If $A = D$, the Dirichlet convolution, then the conditions $m, n \in A_k(mn)$, $m, n \in A_k(mn\,\gamma(n))$ hold for all m and n, and therefore these conditions may be left out. If $A = U$, then the identities in this section are trivial.

Remark 4.3. For Pillai's function $P(n)$, Examples 4.1–4.4 reduce to

$$P(p^i) \;=\; 2\,p\,P(p^{i-1}) - p^2\, P(p^{i-2}) \tag{4.11}$$

for all primes p and integers $i \geq 2$,

$$P(mn) \;=\; \sum_{d \mid (m,n)} m\, \tau(m/d)\, P(n/d)\, \mu(d)\, d\,, \qquad \gamma(m) \mid \gamma(n) \ , \tag{4.12}$$

$$m\, \tau(m)\, P(n) \;=\; \sum_{d \mid (m,n)} P(mn/d^2)\, d\,, \qquad \gamma(m) \mid \gamma(n) \ , \tag{4.13}$$

$$(4.14)\qquad \sum_{d\,|\,(m,n)} d\,\tau(m/d)\,P(n/d)\,C(c,d) \;=\; \sum_{\substack{d\,|\,(m,n)\\ d\,|\,c}} d^3\,P(mn/d^2)\,, \qquad \gamma(m)\,|\,\gamma(n)\ ,$$

$$(4.15)\qquad P(m)\,n \;=\; \sum_{d\,|\,n} \tau(n/d)\,P(mnd)\,\mu(d)/d\ ,$$

where $\tau(n)$ is the number of divisors of n.

5 – Converse forms of Theorems 4.1–4.6

In this section we show that the identities in Theorems 4.1–4.6 serve as characterizations of A-rational arithmetical functions of order $(2,1)$. To prove this it is enough to show that the converse forms of Theorems 4.1–4.6 hold.

Theorem 5.1. *Let p^t be an A-primitive prime power. Suppose that there exist complex numbers $G(p^t)$ and $G'(p^t)$ such that*

$$(5.1)\qquad f(p^{it}) = G(p^t)\,f(p^{(i-1)t}) - G'(p^t)\,f(p^{(i-2)t})$$

for all $i = 2,3,...,o(p^t)$. Then

$$(5.2)\qquad f(p^{it}) = (g *_A h^{-1})(p^{it})$$

for all $i = 0,1,...,o(p^t)$, where g is an A-quadratic such that $g(p^t) = G(p^t)$, $g'(p^t) = G'(p^t)$ and h is an A-multiplicative function such that $h(p^t) = G(p^t) - f(p^t)$.

Proof: Let g_1 and g_2 be A-multiplicative functions such that

$$g_1(p^t) = \Big(G(p^t) + z(p^t)\Big)\,/\,2\ ,$$

$$g_2(p^t) = \Big(G(p^t) - z(p^t)\Big)\,/\,2\ ,$$

where $z(p^t)$ is one of the values of

$$\sqrt{G(p^t)^2 - 4\,G'(p^t)}\ .$$

Then

$$g_1(p^t) + g_2(p^t) = G(p^t)\ ,$$

$$g_1(p^t)\,g_2(p^t) = G'(p^t)\ .$$

We thus may write $g = g_1 *_A g_2$.

We aim to prove that (5.2) holds. If $i=0$, then both sides of (5.2) reduce to 1. If $i=1$, then (5.2) reduces to $f(p^t) = g(p^t) - h(p^t)$, which holds by the definition of h. Let $i=2$. Then

$$\begin{aligned} f(p^{it}) &= f(p^{2t}) = G(p^t)\,f(p^t) - G'(p^t) \\ &= \Big(g_1(p^t)+g_2(p^t)\Big)\,\Big(g_1(p^t)+g_2(p^t)-h(p^t)\Big) - g_1(p^t)\,g_2(p^t)\ . \end{aligned}$$

Now proceeding on the lines of the proof of Theorem 4.1 we obtain

$$f(p^{2t}) = (g*h^{-1})(p^{2t})\ .$$

Suppose that (5.2) holds for $i<n$, where $3 \le n \le o(p^t)$. Then

$$\begin{aligned} f(p^{nt}) &= G(p^t)\,g(p^{(n-1)t}) - G'(p^t)\,f(p^{(n-2)t}) \\ &= \Big(g_1(p^t)+g_2(p^t)\Big)\,\Big[(g_1*g_2)(p^{(n-1)t}) - (g_1*g_2)(p^{(n-2)t})\,h(p^t)\Big] \\ &\quad - g_1(p^t)\,g_2(p^t)\,\Big[(g_1*g_2)(p^{(n-2)t}) - (g_1*g_2)(p^{(n-3)t})\,h(p^t)\Big]\ . \end{aligned}$$

Proceeding on the lines of the proof of Theorem 4.1 we obtain

$$f(p^{nt}) = (g*h^{-1})(p^{nt})\ .$$

This completes the proof. ∎

Theorem 5.2. *If f is multiplicative and if there exist multiplicative functions G and G' such that*

$$f(mn) = \sum_{d\in A((m,n))} G(m/d)\,f(n/d)\,\mu_A(d)\,G'(d) \tag{5.3}$$

whenever $\gamma(m)\,|\,\gamma(n)$ and $m,n \in A(mn)$, then f is an A-rational arithmetical function of order $(2,1)$ and f is given as $f = g*h^{-1}$, where g is the A-quadratic given by $g(p^t) = G(p^t)$, $g'(p^t) = G'(p^t)$ and h is the A-multiplicative function given by $h(p^t) = G(p^t) - f(p^t)$ for all A-primitives p^t.

Proof: Taking $m=p^t$ and $n=p^{(i-1)t}$ in (5.3) we obtain (5.1). Now, applying multiplicativity and Theorem 5.1 we obtain Theorem 5.2. ∎

Remark 5.1. The converse forms of Theorems 5.3–5.6 are similar to the converse form of Theorem 5.2 in character. We therefore omit the details.

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