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CLINIC OBSTRUCTIONS AND CLINIC COHOMOLOGICAL GROUPS

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1 – Introduction

In [2], P. Hall established an important method for the classification of groups, mainly the classification of groups in isoclinic classes, then within each class the classification up to isomorphism. This viewpoint has been used by many authors, see for instance [1, 5, 3, 8].

Given a group G, we denote by $\overline{G}$ the quotient $G/Z(G)$ of G by its center, and by G' the commutator subgroup $[G,G]$ of G. A mapping $f\colon G \to H$ from a group G into a group H respecting identity elements is said to be *homoclinic* if there are two group homomorphisms $\overline{f}$ and f' such that the following diagrams commute:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"] \arrow[dr, phantom, "\text{(a)}"] & H \arrow[d, "p"] \\ \overline{G} \arrow[r, "\overline{f}"'] & \overline{H} \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[r, "f^2"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\text{(b)}"] & H \times H \arrow[d, "\alpha"] \\ G' \arrow[r, "f'"'] & H' \end{tikzcd}$$

where the p are canonical projections and the α commutator mappings, that is to say $\alpha(x,y) = [x,y] = x^{-1}y^{-1}xy$. In case $\overline{f}$ and f' are isomorphisms, we say that f is *isoclinic* and we also say that G and H are *isoclinic*.

According to the definition, if G and H are isoclinic, then they have the same inner automorphism group, i.e., there is an isomorphism $\overline{G} \to \overline{H}$. We also say that this isomorphism is raised to an isoclinic $G \to H$. Consider now a class consisting of all groups with a certain inner automorphism group. We want to

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classify isoclinically the groups in the class. In general, we consider the following problem:

Let $\varphi\colon \overline{G} \to \overline{H}$ be a given group homomorphism.
Under what conditions may φ be raised to a homoclinic $G \to H$?

The given homomorphism induces a group homomorphism $\varphi' : \overline{G}' \to \overline{H}'$. Let $\mathrm{Obst}(\varphi') \in H^2(G', Z(H))$ be the obstruction to raising φ' to a group homomorphism $G' \to H'$. When $\mathrm{Obst}(\varphi') = 0$, there is a *clinic obstruction*, $\mathrm{Obst.c}(\varphi) \in H_c^2(G, Z(H))$ to raising φ to a homoclinic, where H_c^2 is an Abelian group and is called the *2-dimensional clinic cohomological group*.

Theorem. *Let $\varphi\colon \overline{G} \to \overline{H}$ be a group homomorphism such that* $\mathrm{Obst}(\varphi') = 0$ *Then φ raises to a homoclinic if and only if* $\mathrm{Obst.c}(\varphi) = 0$.

2 – Clinic representative

Let $\varphi : \overline{G} \to \overline{H}$ be a given group homomorphism, where usually G is a multiplicative group and H an additive group. A mapping $f\colon G \to H$ such that $f(1) = 0$ and $\varphi(xZ(G)) = f(x) + Z(H)$ for all $x \in G$ is called a *representative of* φ and we indicate this by writing $f \in \varphi$. Of course, if $f(Z(G)) \subseteq Z(H)$, then f induces a mapping $\overline{f}\colon \overline{G} \to \overline{H}$ such that $\overline{f}(xZ(G)) = f(x) + Z(H)$. In other words, $\overline{f} = \varphi$.

A representative $f \in \varphi$ is called a *clinic representative* if f is homoclinic. By definition, $\overline{f} = \varphi$ and there is a group homomorphism $f'\colon G' \to H'$ such that the diagrams (a) and (b) commute.

Proposition 1. *If φ has a clinic representative, then any of its representatives is clinic and the induced homomorphism $G' \to H'$ is uniquely determined by φ, that is it is independent of the choice of representative for φ.*

Proof: Let $f, g \in \varphi$. By the definition of representative of φ, for every $x \in G$, there is $k(x) \in Z(H)$ such that $g(x) = f(x) + k(x)$. Taking commutators, we obtain

$$\Big[g(x), g(y)\Big] = \Big[f(x) + k(x),\ f(y) + k(y)\Big] = \Big[f(x), f(y)\Big]$$

for all $x, y \in G$. This means that, when $f \in \varphi$ is homoclinic with induced homomorphism $f' : G' \to H'$ defined by $f'[x,y] = [f(x), f(y)]$, then $g \in \varphi$ is also clinic with induced homomorphism $g' = f'$. This completes the proof of the proposition. ∎

In particular, if $f \in \varphi$ is a group homomorphism, then $f(Z(G)) \subseteq Z(H)$ and $f(G') \subseteq H'$. In other words, the restriction of f to G' determines a homomorphism $G' \to H'$ and the diagrams (a) and (b) commute. Hence any group homomorphism $f \in \varphi$ is homoclinic. If $f \in \varphi$ is an isomorphism, then it is also isoclinic.

3 – Obstruction to raising a group homomorphism to a group homomorphism

Consider the following diagram of group homomorphisms

$$\begin{tikzcd} & & & G \arrow[d, "\alpha"] & \\ 0 \arrow[r] & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r] & 1, \end{tikzcd}$$

where the row is a central extension. Since v is an epimorphism, there is a mapping $t\colon G \to B$ such that $t(1) = 0$ and $v\,t = \alpha$. By definition, for all $x, y \in G$, there is an element $k(x, y) \in A$ such that $t(x) + t(y) = k(x, y) + t(x\,y)$. This equality determines a mapping $k\colon G \times G \to A$. By associativity of the operation in B, it follows that k is a 2-cocycle, $k \in Z^2(G, A)$. The cohomological class $[k] \in H^2(G, A)$ is uniquely determined by α, independently of the choice of t. We write $\mathrm{Obst}(\alpha) = [k]$ and we call $[k]$ the *obstruction to raising* α *to a group homomorphism* $G \to B$. Then (see [6]):

Proposition 2. *The homomorphism* α *may be raised to a group homomorphism* $G \to B$ *if and only if* $\mathrm{Obst}(\alpha) = 0$.

4 – Clinic obstruction

Consider a group homomorphism $\varphi\colon \overline{G} \to \overline{H}$. Then

$$\overline{G}' = [\overline{G}, \overline{G}] = [G, G]Z(G)/Z(G) \simeq G'/G' \cap Z(G)\ .$$

So the restriction of φ to $\overline{G}'$ induces a group homomorphism $\varphi'\colon \overline{G}' \to \overline{H}'$ and we have a diagram of group homomorphisms

$$
\text{(c)} \qquad
\begin{tikzcd}
 & & & G' \arrow[d, "\varphi' p"] & \\
0 \arrow[r] & H' \cap Z(H) \arrow[r] & H' \arrow[r, "p"] & \overline{H}' \arrow[r] & 0
\end{tikzcd}
$$

where p is the canonical projection and the row is an extension. By the result of section 2, there is an obstruction of $\varphi' p$, $\mathrm{Obst}(\varphi' p) \in H^2(G', Z(H))$.

Suppose that $f \in \varphi$ is a clinic representative. Then there is a homomorphism $f'\colon G' \to H'$ such that $f'[x,y] = [f(x), f(y)]$ for all $x, y \in G$. This means that, in diagram (c), the homomorphism $\varphi' p$ is raised to the homomorphism f' and then, by Proposition 2, $\mathrm{Obst}(\varphi' p) = 0$.

Conversely, let $\mathrm{Obst}(\varphi' p) = 0$, that is $\varphi' p$ is raised to a homomorphism $f'\colon G' \to H'$. This does not mean that (φ, f') determines a homoclinic. Thus, for each $f \in \varphi$, we have $\varphi' p[x,y] = pf'[x,y]$ for all $x, y \in G$. So, for each pair of elements $x, y \in G$, there is an element $\beta_f(x,y) \in Z(H)$ such that

$$\Big[f(x), f(y)\Big] = f'[x,y] + \beta_f(x,y) \ . \tag{1}$$

We denote by $\beta_f\colon G \times G \to Z(H)$ the mapping defined by (1).

Lemma 1.

(i) *$\beta_f(x,y) = 0$ if $x \in Z(G)$ or $y \in Z(G)$.*

(ii) *$\beta_f(y^x, z^x) + \beta_f(x, yz) = \beta_f(x,y) + \beta_f(xy, z)$ for all $x, y, z \in G$, where $y^x = x^{-1} y x$.*

Proof:

(i) Let $x \in Z(G)$ or $y \in Z(G)$. Since $f \in \varphi$, then $f(x) \in Z(H)$ or $f(y) \in Z(H)$. Hence $[f(x), f(y)] = 0$. On the other hand, $[x,y] = 1$ and so we have $f'[x,y] = 0$. Now (i) follows from (1).

(ii) Computing the commutators, we obtain

$$[y^x, z^x]\,[x, yz] = [x,y]\,[xy, z] \qquad (x,y,z \in G) \ . \tag{2}$$

Letting f' act on both sides of (2), we get

$$f'[y^x, z^x] + f'[x, yz] = f'[x,y] + f'[xy, z] \qquad (x,y,z \in G) \ . \tag{3}$$

On the other hand, since $f \in \varphi$, for each $x, y \in G$ we have $f(xy) = f(x) + f(y) + a_{xy}$ for a certain $a_{xy} \in Z(H)$. Hence

$$\Big[f(y^x), f(z^x)\Big] = \Big[f(y)^{f(x)}, f(z)^{f(x)}\Big] \ . \tag{4}$$

In view of (4), and applying (2) to $f(x)$, $f(y)$, and $f(z)$, we obtain

$$\Big[f(y^x), f(z^x)\Big] + \Big[f(x), f(yz)\Big] = \Big[f(x), f(y)\Big] + \Big[f(xy), f(z)\Big] \ . \tag{5}$$

Comparing (5), (3) and (1), we deduce (ii). ∎

Let $g\colon G \to H$ be another representative of φ and let $\beta_g\colon G \times G \to Z(H)$ be the mapping determined by (1), that is

$$[g(x), g(y)] = g'[x, y] + \beta_g(x, y) \qquad (x, y \in G) \ .$$

Since $f, g \in \varphi$, then $[f(x), f(y)] = [g(x), g(y)]$ and we have

$$f'[x, y] + \beta_f(x, y) = g'[x, y] + \beta_g(x, y) \qquad (x, y \in G) \ . \tag{6}$$

This means that β_f does not depend of the choice of $f \in \varphi$ and so depends only on f'. Clearly the mapping $u\colon G' \to Z(H)$ defined by $u[x, y] = f'[x, y] - g'[x, y]$ is a group homomorphism and (6) becomes

$$\beta_g(x, y) - \beta_f(x, y) = u\,\alpha(x, y) \qquad (x, y \in G) \ , \tag{7}$$

where α is the commutator mapping. This leads to the following lemma and definition.

Lemma 2. *Let G be a group, A an Abelian group, $Z_c^2(G, A)$ the set of all mappings $\beta\colon G \times G \to A$ satisfying the conditions (i) and (ii) of Lemma 1, and $B_c^2(G, A)$ the set of all mappings of the form $\beta = u \circ \alpha$, where $u\colon G' \to A$ is a group homomorphism and $\alpha\colon G \times G \to G'$ is the commutator mapping. Then:*

(i) *$Z_c^2(G, A)$ is an Abelian group under the operation*

$$(\beta_1 + \beta_2)(x, y) = \beta_1(x, y) + \beta_2(x, y) \ ;$$

(ii) *$B_c^2(G, A)$ is a (normal) subgroup of $Z_c^2(G, A)$.*

An element of Z_c^2 is called a clinic 2-cocycle and an element of B_c^2 is called a clinic 2-coboundary. The quotient group

$$H_c^2(G, A) = Z_c^2(G, A)/B_c^2(G, A)$$

is called the 2-dimensional clinic cohomological group of the group G with coefficients in A. The cohomological class of $\beta \in Z_c^2$ is denoted by $[\beta]$.

Proof: This is an immediate corollary of the definition. ∎

According to Lemma 1 and the equality (7), we have

$$[\beta_f] = [\beta_g] \in H_c^2(G, Z(H)) \qquad (f, g \in \varphi) \ .$$

This means that the homomorphism φ determines a clinic cohomological class $[\beta_f]$. This class is independent of the choice of any representatives $f \in \varphi$ and $f' \in \varphi'$. We denote this class by $\mathrm{Obst.c}(\varphi)$ and we call it the *clinic obstruction* of φ.

We can now prove the theorem stated in the introduction.

5 – Proof of the Theorem

For a given homomorphism $\varphi\colon \overline{G} \to \overline{H}$, by the result of section 2, there is an obstruction $\mathrm{Obst}(\varphi' p) \in H^2(G', Z(H))$ to raising $\varphi' p\colon G' \to \overline{H}'$ to a homomorphism $\varphi'\colon G' \to H'$.

Suppose that $\mathrm{Obst}(\varphi' p) = 0$. Then, by Proposition 2, we may raise $\varphi' p$ to φ'. When φ has a clinic representative $g\colon G \to H$, according to Lemma 2 the clinic obstruction $\mathrm{Obst.c}(\varphi)$ is independent of the choice of representative of φ, and so $\mathrm{Obst.c}(\varphi) = [\beta_g]$. Since g is homoclinic, it follows that there is a group homomorphism $g'\colon\ G' \to H'$ such that $g'[x,y] = [g(x), g(y)]$ for all $x, y \in G$. From (1) we then deduce that $\beta_g = 0$ and so $\mathrm{Obst.c}(\varphi) = [\beta_g] = 0$.

Conversely, suppose that $\mathrm{Obst.c}(\varphi) = 0$ and $\mathrm{Obst.c}(\varphi) = [\beta_f]$ with $f \in \varphi$ such that f verifies (1). Applying (7), we conclude that there is a group homomorphism $u\colon G' \to Z(H)$ such that $\beta_f = u\,\alpha$ where α is the commutator mapping. Hence $\mathrm{Im}\, u \subseteq H' \cap Z(H)$ and the mapping $g'\colon G' \to H'$ given by $g'(x) = f'(x) + u(x)$ is a group homomorphism. So we have $g'[x,y] = f'[x,y] + u[x,y] = f'[x,y] + \beta_f(x,y) = [f(x), f(y)]$ by (1). Hence f induces the homomorphism $\overline{f} = \varphi$ and g' satisfies the conditions to be homoclinic. This completes the proof of the theorem. ∎

6 – Remark

As shown above, the concept of 2-dimensional clinic cohomological group appears when we consider the clinic obstruction to the clinic raising. In the forthcoming work, we will develop the theory of clinic extensions and establish a relationship between these notions. Later, we will construct the n-dimensional clinic cohomological groups.

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