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ON THE HYPERBOLIC DIRICHLET TO NEUMANN FUNCTIONAL

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Abstract: We prove the injectivity of the linearization of the hyperbolic Dirichlet to Neumann functional associated to metrics near the euclidean one in a "small" bounded domain of $\mathbb{R}^3$, under some suitable transversality and geometric conditions.

1 – Introduction and statement of the results

Let $\mathcal{M}$ denote the set of all riemannian metrics g on $\mathbb{R}^n$ which coincide with the euclidean metric $\boldsymbol{e}$, outside a bounded domain Ω with smooth boundary $\partial\Omega$. We consider the anisotropic wave equation

$$\begin{aligned} \square_g u = \frac{\partial^2 u}{\partial t^2} - \Delta_g u = 0 \quad &\text{in } \ \Omega \times (0,T) \ , \\ u = f \quad \text{on } \ \Gamma = \partial\Omega \times (0,T), \quad &f \in C_0^\infty(\Gamma) \ , \\ u = \frac{\partial u}{\partial t} = 0 \quad &\text{in } \ \Omega \times \{0\} \ . \end{aligned} \tag{1.1}$$

There is a unique solution to (1.1); hence we may define the hyperbolic Dirichlet to Neumann map as the linear operator

$$\Lambda_g \colon \ C_0^\infty(\Gamma) \to C^\infty(\Gamma) \ , \tag{1.2}$$

$$\Lambda_g f = du \cdot \nu_g \Big|_\Gamma = \frac{\partial u}{\partial \nu_g}\Big|_\Gamma \ , \tag{1.3}$$

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where u is the unique solution to (1.1) and ν_g is the g-outward unit normal to $\partial\Omega$. The hyperbolic Dirichlet to Neumann Functional:

$$\begin{aligned} \Lambda\colon\ \mathcal{M} &\to O_p(\Gamma)\ , \\ g &\mapsto \Lambda_g\ , \end{aligned} \tag{1.4}$$

where $O_p(\Gamma)$ denotes the space of all linear operators from $C_0^\infty(\Gamma)$ into $C^\infty(\Gamma)$, is known to be invariantly defined on the orbit obtained by the action over $\mathcal{M}$, of the group $\mathcal{D}$ of all diffeomorphism ψ of $\overline{\Omega}$, each of which restricts to the identity on $\partial\Omega$. In fact, any such ψ can be used to construct a new metric, the pull-back metric, ψ^*g, such that $\Lambda_{\psi^*g} = \Lambda_g$. A natural conjecture is that this is the only obstruction to the uniqueness of Λ.

For fixed g, we consider the following map:

$$\psi \in \mathcal{D} \xrightarrow{A_g} \psi^* g \in \mathcal{M}\ . \tag{1.5}$$

It is easy to see that the tangent space $T_I\mathcal{D}$ of $\mathcal{D}$ at the identity mapping I is the vector space $\Gamma_0(T\overline{\Omega})$ of all smooth vector fields on $\overline{\Omega}$ which vanish on $\partial\Omega$. On the other hand, the tangent space $T_g\mathcal{M}$ of $\mathcal{M}$ at g is the vector space $\Gamma_0(S^2\overline{\Omega})$ of all smooth sections of symmetric 2-tensors on $\mathbb{R}^n$ which are supported on $\overline{\Omega}$. We introduce respectively on $\Gamma_0(T\overline{\Omega})$ and on $\Gamma_0(S^2\overline{\Omega})$, the inner products

$$\langle X, Y\rangle = \int_{\overline{\Omega}} g(X,Y)\, v_g\ , \qquad X, Y \in \Gamma_0(T\overline{\Omega})\ , \tag{1.6}$$

$$\langle\!\langle m, l\rangle\!\rangle = \frac{1}{n}\int_{\overline{\Omega}} \operatorname{tr}(\hat{m}\circ\hat{l})\, v_g\ , \qquad m, l \in \Gamma_0(S^2\overline{\Omega})\ , \tag{1.7}$$

where v_g (resp. tr) denote the volume element (resp. the trace) associated to g and $\hat{m}$ is the unique linear map (in fact a section of $\operatorname{End}(T\overline{\Omega})$) defined by

$$g(\hat{m}u, v) = m(u,v), \qquad \text{for all}\ \ u, v \in \Gamma(T\overline{\Omega})\ . \tag{1.8}$$

Of course, $\hat{g}$ is the identity on $\Gamma(T\overline{\Omega})$ and the factor $1/n$ in (1.7) is taken so as to have $\langle\!\langle g, g\rangle\!\rangle = \operatorname{vol}_g(\overline{\Omega})$.

Consider as in [1], the formal linearizations of A_g at I and of Λ at g, respectively:

$$A'_g[I] := A'_g\colon\ \Gamma_0(T\overline{\Omega}) \to \Gamma_0(S^2\overline{\Omega}) \tag{1.9}$$

and

$$\Lambda'_g\colon\ \Gamma_0(S^2\overline{\Omega}) \to O_p(\Gamma)\ . \tag{1.10}$$

Let $(A'_g)^*$ denote the formal adjoint of A'_g with respect to the inner product (1.6) and (1.7) and $\operatorname{diam}_g(\Omega)$ the diameter of Ω in the metric g. In [1] the authors stated the following

Conjecture 1. *Let $m \in \Gamma_0(S^2\overline{\Omega})$ and assume that*

a) $\Lambda'_g(m) = 0$,

b) $(A'_g)^*(m) = 0$ *and*

c) $\operatorname{diam}_g(\Omega) < T$ *is sufficiently small that the exponential map for g is a global diffeomorphism in $\overline{\Omega}$.*

Then m is identically zero.

Remark 1. The Condition b) in Conjecture 1.1 is obviously necessary. In fact, the range of A'_g is contained in the kernel of Λ'_g. Therefore, we should expect that Λ'_g be injective on a "transversal" subspace of the range of A'_g; hence we shall refer to Condition b) as the Transversality Condition. The Condition c) is necessary to avoid the appearance of caustics.

Remark 2. Cardoso and Mendoza, [1], proved that Conjecture 1.1 holds if $n \geq 2$ and g is the euclidean metric $\boldsymbol{e}$; they also proved the conjecture when $n = 2$ and g is near the euclidean metric in the C^3 topology.

The main result of this paper is:

Theorem 1. *Conjecture 1.1 holds if $n = 3$, g is near the euclidean metric in the C^3 topology and in addition, one of the following two conditions is true:*

I – *The g Levi-Civita connection commute with rotation, i.e., $\nabla^g \circ J = J \circ \nabla^g$ (see Section 4 for the definition of J).*

II – *The generalized gradients of solutions of the eikonal equation are g-Killing fields (see [2] for the definition).*

The article is organized as follows: In Section 2 and 3 we develop the necessary preliminaries dealing with invariant formulas for A'_g, $(A'_g)^*$ and Λ'_g and the generalized X-ray and Radon transform. In Section 4 we present the proof of Theorem 1.1 with condition I and in Section 5 we prove Theorem 1.1 with condition II.

2 – Invariant formulas

Cardoso and Mendoza, [1], proved the following two propositions:

Proposition 1. *If $X \in \Gamma_0(T\overline{\Omega})$ and $m \in \Gamma_0(S^2\overline{\Omega})$, then it follows that*

$$A'_g(X)(\,\cdot\,,\,\cdot\,) = g(\nabla X,\,\cdot\,) + g(\,\cdot\,,\nabla X) \;, \tag{2.1}$$

$$(A'_g)^*(m)(\,\cdot\,) = -\frac{2}{n}\sum_{i=1}^{n} \nabla_{e_i} m(\,\cdot\,,\boldsymbol{e}_i) \;. \tag{2.2}$$

In (2.1) ∇ denotes the g Levi-Civita connection on $\Gamma_0(T\overline{\Omega})$ and in (2.2) ∇ is the g Levi–Civita connection on $\Gamma_0(S^2\overline{\Omega})$ and $(\boldsymbol{e}_i)_{i=1,\dots,n} \in \Gamma(T\overline{\Omega})$, is a g orthonormal frame. We also observe that the right-hand side of (2.2) is independent of the chosen ortonormal frame.

We denote by $\tilde{m} \in \tilde{\Gamma}_0(S^2\overline{\Omega})$ the symmetric 2-tensor on $\Gamma(T^*\overline{\Omega})$ corresponding to m via g, i.e. $\tilde{m}(U^\#, V^\#) = m(U,V)$ for all $U, V \in \Gamma(T(\overline{\Omega}))$, where $U^\#(\,\cdot\,) = g(U,\,\cdot\,)$. We have the following:

Proposition 2. *The linearization of Λ at $g \in \mathcal{M}$, satisfies*

$$\Big\langle \Lambda'_g(m) f,\, h\Big\rangle_{L^2(\Gamma)} = \int_0^T\!\!\int_{\overline{\Omega}} \Big\{\tilde{m}(du,dv) + \frac{1}{2}\,\mathrm{tr}(\hat{m})\,\Big[\tilde{g}(du,dv) - u_t\,v_t\Big]\Big\}\, v_g\, dt \;, \tag{2.3}$$

for all $f, h \in C_0^\infty(\Gamma)$, where u is a solution of (1.1), v is a solution of

$$\Box_g v = 0 \;\; in\ \Omega\times(0,T)\,, \qquad v = v_t = 0 \;\; in\ \Omega\times\{T\}\,, \qquad v|_\Gamma = h \;, \tag{2.4}$$

and $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Gamma)}$ is the L^2-inner product in Γ with respect to the measure induced by the metric $g\otimes dt^2$.

3 – The geodesic X-ray transform and the Radon transform

Let $g \in \mathcal{M}$. We remind that g coincides with the euclidean metric, $\boldsymbol{e}$, outside $\overline{\Omega}$. We shall deal with sections of the following vector bundles

$$\begin{array}{cc} \mathcal{P} & Q \\ \downarrow & \downarrow \\ G & G \end{array}$$

where G denotes the manifold of geodesics (with respect to g), $\mathcal{P}$ the bundle of parallel vector fields and Q the bundle of quadratic forms on $\mathcal{P}$. The generalized X-ray transform $\mathcal{R}_g$ is the map

$$\mathcal{R}_g\colon\ \Gamma_0(S^2\overline{\Omega}) \to \Gamma(Q)\ , \tag{3.1}$$

defined by

$$\mathcal{R}_g(m)_\gamma\,(P_1,P_2) = \int_\gamma m(\gamma(t))\,\Big(P_1(\gamma(t)), P_2(\gamma(t))\Big)\ , \tag{3.2}$$

where $\gamma \in G$ and $P_1, P_2 \in \Gamma(\mathcal{P})$. There is a global non-vanishing section $T\colon G \to \mathcal{P}$, given by

$$T_\gamma(\gamma(t)) = \dot\gamma(t)\ , \tag{3.3}$$

since as it is well known if γ is a geodesic, then $\dot\gamma(t) = P_{\gamma,0,t}(\dot\gamma(0))$, where $P_{\gamma,0,t}$ is the parallel transport along γ, from 0 to t. In [1] it was proved the following:

Proposition 3. *Let $m \in \Gamma_0(S^2\overline{\Omega})$ satisfy $\Lambda'_g(m) = 0$ and $\operatorname{diam}_g(\Omega) < T$ be so small that the exponential map for g is a global diffeomorphism in $\overline{\Omega}$. Then*

$$\mathcal{R}_g(m)_\gamma\,(T_\gamma, T_\gamma) = 0\ , \tag{3.4}$$

for all g-geodesic γ.

Let $\mathcal{G}'$ denote the space of generalized hiperplanes $\Sigma = \Sigma_\phi^s = \Sigma(\phi,\eta,s)$, where $s \in \mathbb{R}$, $\eta \in S^{n-1}$ is a normal vector to Σ and $\phi(\,\cdot\,,\eta)$ is a solution of the eikonal equation

$$\begin{cases} g\Big(\nabla^g\phi(\,\cdot\,,\eta),\ \nabla^g\phi(\,\cdot\,,\eta)\Big) = 1\ , \\ \phi(\,\cdot\,,\eta)\Big|_\Sigma = s\ , \\ \nabla^g\phi(\,\cdot\,,\eta)\Big|_\Sigma = \eta\ . \end{cases} \tag{3.5}$$

We assume that

(**i**) the metric g satisfies Condition II of Theorem 1.1.

(**ii**) $\phi(x,tw) = t\,\phi(x,w)$, for all $(w,t) \in S^{n-1} \times \mathbb{R}$.

Remark 3. The generalized hiperplanes are closed submanifolds of dimension $n-1$. On the other hand taking into account (i) it is easy to see that they are totally geodesic submanifolds.

Let Q' denote the quadratic bundle over $\mathcal{G}'$. The generalized Radon transform R_g is the map

$$R_g\colon\ \Gamma_0(S^2\overline{\Omega}) \to \Gamma(Q') \ , \tag{3.6}$$

defined by

$$R_g(m)_\Sigma\,(X,Y) = \int_\Sigma m(X,Y)\,\mu_\Sigma \ , \tag{3.7}$$

where μ_Σ denotes the volume element induced on Σ by the metric g.

Corollary 1. *There is an orthonormal frame $T_1,...,T_{n-1},\ N$ of $T\mathbb{R}^n$ such that*

$$R_g(m)_\Sigma\,(T_i,T_j) = 0 \ , \tag{3.8}$$

for all $i,j\in\{1,...,n-1\}$.

Proof: Let $\Sigma=\Sigma^\lambda_\phi$ in $\mathcal{G}'$ and $\phi_i\in C^\infty(\mathbb{R}^n)$, $i=1,...,n-1$, such that

$$g(\nabla^g\phi,\ \nabla^g\phi_i) = 0 \ , \tag{3.9}$$

$$g(\nabla^g\phi_i,\ \nabla^g\phi_j) = \delta_{ij} \ . \tag{3.10}$$

Denoting $T_i:=\nabla^g\phi_i$ and $N:=\nabla^g\phi$, $i=1,...,n-1$, it follows from (3.9) and (3.10) that $T_i|_\Sigma\in\Gamma(T\Sigma)$ and $T_1,...,T_{n-1},\ N$ is an orthonormal frame of $T\mathbb{R}^n$.

We can assume that $\Sigma\cap\Sigma^0_{\phi_i}\cap\Sigma^0_{\phi_j}$ is not-empty for all $i,j\in\{1,...,n-1\}$ and denote

$$\mathcal{N}_i := \Sigma\cap\Sigma^0_{\phi_i} \ ,$$

$$\mathcal{N}_{ij} := \Sigma\cap\Sigma^0_{\phi_i}\cap\Sigma^0_{\phi_j} \ .$$

Let Φ^σ_i be the geodesic flow associated to the field $\nabla^g\phi_i$, then $\sigma\mapsto\Phi^\sigma_i(\,\cdot\,)$ are the geodesics which start at $\mathcal{N}_i$ and, using (3.4), we obtain

$$\begin{aligned} R_g(m)_\Sigma\,(T_i,T_j) &= \int_{-\infty}^{\infty}\int_{\mathcal{N}_i} m(\Phi^\sigma_i(y))\left(\dot\Phi^\sigma_i(y),\dot\Phi^\sigma_i(y)\right)dS_y\,d\sigma \\ &= \int_{\mathcal{N}_i}\mathcal{R}_g(m)_{\Phi^{(\bullet)}_i(y)}\left(\dot\Phi^{(\bullet)}_i(y),\ \dot\Phi^{(\bullet)}_i(y)\right)dS_y \\ &= 0 \ . \end{aligned}$$

A similar calculation holds for Φ^σ_{ij}, the geodesic flow associated to the field $\nabla^g\xi_{ij}$, where $\xi_{ij}=(\phi_i+\phi_j)/\sqrt{2}$, taking into account that m is symmetric and (3.4). We obtain

$$R_g(m)_\Sigma\,(T_i,T_j) = \frac{1}{2}\,R_g(m)_\Sigma\,(T_i+T_j,\ T_i+T_j) = 0 \ ,$$

which concludes the proof. ∎

4 – Proof of Theorem 1.1 with condition I

In this section $\overline{\Omega}$ will be a smooth bounded domain in $\mathbb{R}^3$. It is convenient that $\overline{\Omega}$ be placed in the open first octant in $\mathbb{R}^3$. Consider the vector bundles $\mathcal{P}$ and Q over G as in the beginning of Section 3. A section $m = (m_{ij})_{i,j=1,2,3} \in \Gamma_0(S^2\overline{\Omega})$ belongs to $L^2(S^2\overline{\Omega})$ if

$$\|m\|_0^2 = \int_{\overline{\Omega}} \Big(m_{11}^2 + m_{22}^2 + m_{33}^2 + 2\,(m_{13}^2 + m_{12}^2 + m_{23}^2)\Big)\, dx_1\, dx_2\, dx_3 < \infty \ ,$$

where

$$m_{ij} = m\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) \ ,$$

and x_1, x_2 and x_3 are the standard euclidean coordinates in $\mathbb{R}^3$. The corresponding Sobolev space based on $L^2(S^2\overline{\Omega})$ will be denoted by $H^s(S^2\overline{\Omega})$.

There is a natural frame for $\mathcal{P}$, manely

$$T = \dot\gamma\,, \quad N = J\dot\gamma\,, \quad M \ , \tag{4.1}$$

where $\dot\gamma$ is the unit tangent vector to the g-geodesic γ, J denotes the $\frac{\pi}{2}$ clockwise rotation (with respect to g) in the plane generated by T and the axis oz and M is the parallel transport along γ of the vector product of the euclidean counterparts of T and N. We say that a section H of Q is in $L^2(Q)$ if

$$\|H\|_0^2 = \int_G \Big(H_{TT}^2 + H_{NN}^2 + H_{MM}^2 + 2\,(H_{TN}^2 + H_{TM}^2 + H_{NM}^2)\Big)\, d\mu \ < \ \infty \ ,$$

where $H_{AB} := H(A,B)$, and $d\mu$ represents the naturally defined Liouville measure on G. The corresponding Sobolev space will be denoted by $H^s(Q)$. We shall need the following (see [5]):

Lemma 1. *If g is near the euclidean metric in the C^3 topology, then*

$$\mathcal{R}_g\colon\, H^s_{\rm comp}(S^2\overline{\Omega}) \to H^{s+1/2}_{\rm loc}(Q)$$

is a bounded linear operator with a bounded inverse.

We shall also introduce local coordinates in G, parametrizing a geodesic by $(x_1, x_2, \theta, \varphi)$ or $(x_2, x_3, \theta, \varphi)$ or $(x_1, x_3, \theta, \varphi)$, where (x_1, x_2) (resp. (x_2, x_3), resp. (x_1, x_3)) is the point of intersection of the geodesic with the x_1x_2 (resp. x_2x_3, resp. x_1x_3) plane and (θ, φ) is the spacial position of the speed vector. Let

$$T = T(x_1, x_2, \theta, \varphi, t), \quad N = N(x_1, x_2, \theta, \varphi, t), \quad M = M(x_1, x_2, \theta, \varphi, t) \ , \tag{4.2}$$

be the orthonormal frame with respect to g, defined by (4.1). We assume that

$$t \to M(x_1, x_2, \theta, \varphi, t) \quad \text{and} \quad t \to N(x_1, x_2, \theta, \varphi, t) \tag{4.3}$$

are extended as odd functions for $t \leq 0$. We also denote

$$\begin{aligned} \Theta_1 &:= \gamma_*\Big(\frac{\partial}{\partial\theta}\Big) = \beta_0\, N \ , \\ \Theta_2 &:= \gamma_*\Big(\frac{\partial}{\partial\varphi}\Big) = \delta_0\, M \ , \\ X_i &:= \gamma_*\Big(\frac{\partial}{\partial x}\Big) = \alpha_i\, T + \beta_i\, N + \delta_i\, M\, , \qquad i = 1, 2 \ , \end{aligned} \tag{4.4}$$

where β_j and δ_j, $j = 0, 1, 2$ are functions that depend of the variables $x_1, x_2, \theta, \varphi, t$ and α_1, α_2 only depend on the variables x_1, x_2, θ and φ. To see this, we note that

$$Tg(\Theta_i, T) = g(\nabla_T \Theta_i, T) = g(\nabla_{\Theta_i} T, T) = \frac{1}{2}\, \Theta_i\, g(T, T) = 0 \ ,$$

and

$$Tg(X_i, T) = g(\nabla_T X_i, T) = g(\nabla_{X_i} T, T) = \frac{1}{2}\, X_i\, g(T, T) = 0 \ .$$

Now, at $t = 0$,

$$\Theta_1\Big|_{t=0} = \frac{\partial\dot{\gamma}}{\partial\theta} = -N \ ,$$

and

$$\Theta_2\Big|_{t=0} = \frac{\partial\dot{\gamma}}{\partial\varphi} = \cos\theta\, M \ .$$

If g is the euclidean metric, geodesics are straight lines and in this case $T = \cos\,\theta \boldsymbol{e}^{i\varphi} + \sin\,\theta e_3$, $N = \sin\,\boldsymbol{e}^{i\varphi} - \cos\,\theta e_3$ and $M = i\, \boldsymbol{e}^{i\varphi}$, where $(e_i)_{i=1,2,3}$ is the canonical basis in $\mathbb{R}^3$.

It is easy to see that when g is nearly euclidean i.e. $\|g - \boldsymbol{e}\|_{C^k(\overline{\Omega})} \leq \delta$, then T, N, M, Θ_i and X_i are close to their euclidean counterparts, so that we may assume that

$$\sup_{\substack{0 \leq x, y, t \leq L \\ 0 \leq \theta, \varphi \leq \pi/4}} \left\{ \sum_{i=0}^{2} \left(\left| \frac{\partial\beta_i}{\partial t} \right| + \left| \frac{\partial\delta_i}{\partial t} \right| \right) + |N(\beta_0)| + |M(\delta_0)| \right\} \leq \varepsilon \ , \tag{4.5}$$

where L is the length of the sides of the isosceles triangles with sides on the coordinates axis whose faces generate a prisme which completely encloses $\overline{\Omega}$. We shall need the

Lemma 2. *The following identities hold:*

$$\nabla_{\Theta_1} M = \nabla_N M = 0 \ , \tag{4.6}$$

$$\nabla_{\Theta_1} N = -T(\beta_0)\, T; \quad \nabla_N N = -\frac{T(\beta_0)}{\beta_0}\, T \ , \tag{4.7}$$

$$\nabla_{\Theta_2} N = T(\delta_0)\, M; \quad \nabla_M N = \frac{T(\delta_0)}{\delta_0}\, M \ , \tag{4.8}$$

$$\nabla_{\Theta_2} M = -T(\delta_0)\,(T+N); \quad \nabla_M M = -\frac{T(\delta_0)}{\delta_0}\,(T+N) \ , \tag{4.9}$$

$$\nabla_M T = \frac{T(\delta_0)}{\delta_0}\, M; \quad \nabla_N T = \frac{T(\beta_0)}{\beta_0}\, N \ , \tag{4.10}$$

$$\nabla_{X_i} N = -T(\beta_i)\, T + T(\delta_i)\, M; \quad i=1,2 \ , \tag{4.11}$$

$$\nabla_{X_i} M = -T(\delta_i)\,(T+N); \quad i=1,2 \ , \tag{4.12}$$

$$\frac{T(\delta_i)}{\delta_i} = \frac{T(\delta_j)}{\delta_j}; \quad \frac{T(\beta_i)}{\beta_i} = \frac{T(\beta_j)}{\beta_j}; \quad i,j=0,1,2 \ , \tag{4.13}$$

$$M(\beta_i) = 0; \quad T(\delta_i) = N(\delta_i); \quad i=0,1,2 \ , \tag{4.14}$$

$$N(\alpha_i) = M(\alpha_i) = 0; \quad i=1,2 \ , \tag{4.15}$$

$$\frac{N(\beta_i)}{\beta_i} = \frac{N(\beta_j)}{\beta_j}; \quad \frac{M(\delta_i)}{\delta_i} = \frac{M(\delta_j)}{\delta_j}; \quad i,j=0,1,2 \ . \tag{4.16}$$

Proof: Because (taking into account Condition I)

$$\nabla_{\Theta_1} N = \nabla_{\Theta_1} JT = J\,\nabla_{\Theta_1} T = J\,\nabla_T \Theta_1 = J\,T(\beta_0)\,N = -T(\beta_0)\,T \ ,$$

we obtain (4.7).

To establish (4.12), write

$$\nabla_{X_i} M = a\,T + b\,N \ ,$$

where we have

$$\begin{aligned} a &= -g(M, \nabla_T X_i) \\ &= -T(\beta_i)\, g(M,N) - T(\delta_i)\, g(M,M) \\ &= -T(\delta_i) \ , \end{aligned}$$

and

$$
\begin{aligned}
b &= -g(M, \nabla_{X_i} N) \\
&= T(\beta_i)\, g(M,T) - T(\delta_i)\, g(M,M) \\
&= -T(\delta_i)\ .
\end{aligned}
$$

To establish (4.14), (4.15) and (4.26), we note that $[\Theta_1, \Theta_2] = 0$; in this way,

$$
\begin{aligned}
\nabla_N M - \nabla_M N &= [N, M] \\
&= \frac{M(\beta_0)}{\beta_0}\, N - \frac{N(\delta_0)}{\delta_0}\, M\ .
\end{aligned}
$$

Using (4.6) we obtain

$$
\nabla_M N = \frac{N(\delta_0)}{\delta_0}\, M - \frac{M(\beta_0)}{\beta_0}\, N\ ,
$$

and if we compare with (4.8), we conclude that $M(\beta_0) = 0$ and $T(\delta_0) = N(\delta_0)$.

On the other hand, using the fact that $[X_i, \Theta_1] = 0$, we obtain

$$
\nabla_{X_i} N - \nabla_N X_i = [X_i, N] = -\frac{X_i(\beta_0)}{\beta_0}\, N\ ;
$$

thus

$$
\begin{aligned}
\nabla_{X_i} N &= -\frac{X_i(\beta_0)}{\beta_0}\, N + \nabla_N X_i \\
&= \Big(-\frac{X_i(\beta_0)}{\beta_0} + \alpha_i\, \frac{T(\beta_0)}{\beta_0} + N(\beta_i)\Big) N + \Big(N(\alpha_i) - \beta_i\, \frac{T(\beta_0)}{\beta_0}\Big) T + N(\delta_i)\, M \\
&= \Big(N(\beta_i) - \beta_i\, \frac{N(\beta_0)}{\beta_0}\Big) N + \Big(N(\alpha_i) - T(\beta_i)\Big)\, T + N(\delta_i)\, M\ .
\end{aligned}
$$

Now comparing with (4.11), we obtain

$$
N(\alpha_i) = 0, \quad T(\delta_i) = N(\delta_i), \quad \frac{N(\beta_i)}{\beta_i} = \frac{N(\beta_0)}{\beta_0}\,, \qquad i = 1, 2\ .
$$

It follows from similar computations (taking into account that $[X_i, \Theta_2] = 0$) that

$$
M(\beta_i) = M(\alpha_i) = 0, \quad \frac{M(\delta_i)}{\delta_i} = \frac{M(\delta_0)}{\delta_0}\,, \qquad i = 1, 2\,;
$$

this concludes the proof. ∎

We introduce the following notation:

$$
A_i := \frac{\beta_i}{\delta_0}\, T(\delta_0)\,, \quad B_i := \frac{\delta_i}{\delta_0}\, M(\delta_0)\,, \quad C_i := \frac{\delta_i}{\beta_0}\, T(\delta_0)\,, \quad D_i := \frac{\beta_i}{\beta_0}\, N(\beta_0)\ .
$$

We shall need the

Proposition 4. *If m satisfies the hypotheses of Theorem 1.1 with Condition I and $H = \mathcal{R}_g(m)$, then the following system of equations holds:*

$$\Theta_i(H_{TT}) = X_i(H_{TT}) = 0\,, \quad i = 1,2\ , \tag{4.17}$$

$$\Theta_1(H_{NN}) = -\int_\gamma T(\beta_0)\, m(N,T) + A_0\Big(m(N,T+N) - m(M,M)\Big)\ , \tag{4.18}$$

$$\Theta_2(H_{NN}) = -\int_\gamma M(\delta_0)\, m(N,N)\ , \tag{4.19}$$

$$\begin{aligned} X_i(H_{NN}) = -\int_\gamma & T(\beta_i)\, m(N,T) + A_i\Big(m(N,T+N) - m(M,M)\Big) \\ &+ B_i m(N,N)\ , \end{aligned} \tag{4.20}$$

$$\Theta_1(H_{MM}) = -\int_\gamma N(\beta_0)\, m(M,M)\ , \tag{4.21}$$

$$\Theta_2(H_{MM}) = -\int_\gamma T(\delta_0)\, m(M,T+N) + C_0\, m(N,T)\ , \tag{4.22}$$

$$X_i(H_{MM}) = -\int_\gamma T(\delta_i)\, m(M,T+N) + C_i m(M,T) + D_i m(M,M)\ , \tag{4.23}$$

$$\Theta_1(H_{TN}) = \int_\gamma T(\beta_0)\, m(N,N) + A_0\Big(m(T,T+N) - m(M,M)\Big)\ , \tag{4.24}$$

$$\Theta_2(H_{TN}) = -\int_\gamma M(\delta_0)\, m(T,N)\ , \tag{4.25}$$

$$\begin{aligned} X_i(H_{TN}) = \int_\gamma & T(\beta_i)\, m(N,N) - A_i\Big(m(T,T+N) - m(M,M)\Big) \\ &- B_i m(T,N)\ , \end{aligned} \tag{4.26}$$

$$\Theta_1(H_{NM}) = 2\int_\gamma A_0\, m(T+N,M)\ , \tag{4.27}$$

$$\Theta_2(H_{NM}) = \int_\gamma T(\delta_0)\, m(M,M) - 2\,C_0\, m(T,N)\ , \tag{4.28}$$

$$X_i(H_{NM}) = \int_\gamma T(\delta_i)\, m(M,M) - 2\,C_i\, m(T,N) - D_i\, m(N,M)\ , \tag{4.29}$$

$$\Theta_1(H_{TM}) = -\int_\gamma N(\beta_0)\, m(T,M) \ , \tag{4.30}$$

$$\Theta_2(H_{TM}) = \int_\gamma T(\delta_0)\, m(M,M) - C_0\Big(m(T,T) - m(N,N)\Big) \ , \tag{4.31}$$

$$\begin{aligned} X_i(H_{TM}) = \int_\gamma & T(\delta_i)\, m(M,M) - C_i\Big(m(T,T) - m(N,N)\Big) \\ & - D_i\, m(T,N) \ . \end{aligned} \tag{4.32}$$

Proof: The meaning of (3.4) in Proposition 3.1 is that $H_{TT}=0$, and hence (4.17) holds. The Transversality Condition satisfied by m means that

$$\nabla_T\, m(T,T) + \nabla_N\, m(T,N) + \nabla_M\, m(T,M) = 0 \ , \tag{4.33}$$

$$\nabla_T\, m(N,T) + \nabla_N\, m(N,N) + \nabla_M\, m(N,M) = 0 \ , \tag{4.34}$$

$$\nabla_T\, m(M,T) + \nabla_N\, m(M,N) + \nabla_M\, m(M,M) = 0 \ . \tag{4.35}$$

We begin by computing (4.18). Since $\nabla_{\Theta_1} N = -T(\beta_0)\, T$, using (4.34), after integrating by parts (taking into account (4.3), (4.8), (4.9), (4.15) and that m is compactly supported in $\overline{\Omega}$), we obtain

$$\begin{aligned} \Theta_1(H_{NN}) &= \int_{-\infty}^{\infty} \Theta_1\, m(N,N) \\ &= \int_{-\infty}^{\infty} \nabla_{\Theta_1} m(N,N) + 2\, m(\nabla_{\Theta_1} N, N) \\ &= \int_{-\infty}^{\infty} -\beta_0\, Tm(T,N) - 2\, T(\beta_0)\, m(T,N) - \beta_0 M m(N,N) \\ &\quad + \int_{-\infty}^{\infty} \beta_0\, m(\nabla_M M, N) + \beta_0\, m(M, \nabla_M N) \\ &= \int_{-\infty}^{\infty} -T(\beta_0)\, m(T,N) + M(\beta_0)\, m(N,N) + A_0\Big(m(M,M) - m(T+N,N)\Big) \\ &= -\int_{-\infty}^{\infty} T(\beta_0)\, m(T,N) + A_0\Big(m(T+N,N) - m(M,M)\Big) \ . \end{aligned}$$

To establish (4.20) we proceed in a similar way. In fact,

$$X_i(H_{NN}) = \int_{-\infty}^{\infty} \beta_i\, \nabla_N m(N,N) + \delta_i \nabla_M m(N,N) + 2\, m(\nabla_{X_i} N, N) =$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} -\beta_i\, Tm(N,T) - \beta_i\, Mm(N,M) + \delta_i\, Mm(N,N) - 2\,T(\beta_i)\, m(T,N) \\
&\quad + \int_{-\infty}^{\infty} -2\,\delta_i\, m(\nabla_M N, N) + 2\,T(\delta_i)\, m(M,N) \\
&\quad + \int_{-\infty}^{\infty} \beta_i\, m(\nabla_M N, M) + \beta_i\, m(N, \nabla_M M) \\
&= \int_{-\infty}^{\infty} -T(\beta_i)\, m(T,N) + M(\beta_i)\, m(N,M) - M(\delta_i)\, m(N,N) \\
&\quad + \int_{-\infty}^{\infty} A_i\Big(m(M,M) - m(N, T{+}N)\Big) \\
&= -\int_{-\infty}^{\infty} T(\beta_i)\, m(T,N) + B_i\, m(N,N) + A_i\Big(m(N,T{+}N) - m(M,M)\Big)\ .
\end{aligned}
$$

The remaining equations follow from analogous computations. ∎

Proof of Theorem 1.1 with Condition I: It follows from Proposition 4.1 and (4.5) that there is a constant $C_1 > 0$ such that

$$\|dH\|_{L^2(Q)} \;\le\; C_1\, \varepsilon\, \|m\|_{L^2(S^2\overline{\Omega})}\ , \tag{4.36}$$

where ε can be made arbitrarily small by requiring that g be sufficiently close to the euclidean metric. On the other hand, by the Poincaré inequality, we obtain that there is a constant $C_2 > 0$ such that

$$\|H\|_{H^1(Q)} \;\le\; C_2\, \|dH\|_{L^2(Q)}\ , \tag{4.37}$$

and using Lemma 4.1, it follows that there is a constant $C_3 > 0$ such that

$$\|m\|_{L^2(S^2\overline{\Omega})} \;\le\; C_3\, \|H\|_{H^{1/2}(Q)} \;\le\; C_3\, \|H\|_{H^1(Q)}\ . \tag{4.38}$$

Using (4.36)–(4.38) we get $H = 0$ if ε is chosen small enough and, consequently, $m = 0$. ∎

5 – Proof of Theorem 1.1 with condition II

In this section $\overline{\Omega}$ will be a smooth domain in $\mathbb{R}^3$. It is convenient that $\overline{\Omega}$ be placed in the open first octant in $\mathbb{R}^3$. Consider the quadratic bundle Q' over $\mathcal{G}'$ as in the beginning of Section 3. A section M of Q' belongs to $L^2(Q')$ if

$$\|M\|_0^2 = \int_{\mathcal{G}'} \Big(M_{T_1T_1}^2 + M_{T_2T_2}^2 + M_{NN}^2 + 2\,(M_{T_1N}^2 + M_{T_2N}^2 + M_{T_1T_2}^2)\Big)\, d\mu \;<\; \infty\ , \tag{5.1}$$

where $M_{AB} := M(A,B)$, $d\mu$ represents the naturally defined Liouville measure on $\mathcal{G}'$ and T_1, T_2, N are the vector fields given by Corollary 3.1. The corresponding Sobolev space based on $L^2(Q')$ will be denoted by $H^s(Q')$. We shall need the

Lemma 3. *If g is near the euclidean metric in the C^3 topology, then*

$$R_g\colon\, H^s_{\mathrm{comp}}(S^2\overline{\Omega}) \to H^{s+\frac{n-1}{2}}_{\mathrm{loc}}(Q')$$

is a bounded linear operator with a bounded inverse.

Proof: The adjoint, R_g^*, of R_g is given by

$$R_g^* h(x) = \int_{S^{n-1}} h\Big(\omega, \phi(x,\omega)\Big)\, d\omega \ .$$

Let $P = (2\pi)^{1-n} R_g^*\, \partial_s^{n-1} R_g$. Using Fourier inversion formula, making $t\,\omega = \xi$ and observing that $dt\, d\omega = |\xi|^{1-n} d\xi$, we obtain

$$Pf(x) \;=\; (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \boldsymbol{e}^{i(\phi(x,\xi)-\phi(y,\xi))} f(y)\ d\xi\, dy \ . \tag{5.2}$$

By Taylor's formula with integral remainder, we get

$$\phi(x,\xi) - \phi(y,\xi) = \Big\langle x-y,\ k_{x,y}(\xi)\Big\rangle \ , \tag{5.3}$$

where

$$k_{x,y}(\xi) = \int_0^1 D_x\phi\Big(y + s(x-y),\, \xi\Big)\, ds \ . \tag{5.4}$$

We observe that since g is near the euclidean metric, the function $k_{x,y}$ is a global diffeomorphism. If we substitute (5.3) in (5.2) we obtain that P is a pseudodifferential operator of order zero, with amplitude function given by

$$a(x,y,\xi) = \frac{1}{\det\Big[D_\xi k_{x,y}(k^{-1}_{x,y}(\xi))\Big]} \ . \tag{5.5}$$

Hence, we obtain from the standard estimates for pseudodifferential operators (see [4], Proposition 9.2) and the fact that $D_\xi k_{x,y}(\xi)$ is near the identity, that $\|P - I\|_{\mathcal{L}(L^2)} < 1$ and, consequently, R_g is inversible. ∎

Let us consider the following open set

$$\mathcal{U} = \Big\{\Sigma \in \mathcal{G}'\colon\, \Sigma \text{ is transversal to the } x\text{-axis}\Big\} \ .$$

We may parametrize a generalized hiperplane $\Sigma \in \mathcal{U}$ by $\Sigma = \Sigma(x,\theta,\varphi)$, where $x\boldsymbol{e}_1$ is the point of intersection of Σ with the x-axis and $N^{\boldsymbol{e}} = N^{\boldsymbol{e}}(\theta,\varphi)$ is the representation of the normal vector of Σ in spherical coordinates. We remind that the generalized hiperplanes are totally geodesic by Condition II. We consider the following map

$$\mathrm{Exp}\colon \mathbb{R}^2 \to \Sigma \ ,$$

given by

$$\mathrm{Exp}(x_1,x_2) = \mathrm{Exp}_{x\boldsymbol{e}_1}(x_1 T_1^{\boldsymbol{e}} + x_2 T_2^{\boldsymbol{e}}) \ ,$$

where $T_1^{\boldsymbol{e}} = \frac{\partial N^{\boldsymbol{e}}}{\partial\theta}$, $T_2^{\boldsymbol{e}} = \frac{\partial N^{\boldsymbol{e}}}{\partial\varphi}$ and $\mathrm{Exp}_{x\boldsymbol{e}_1}$ denote the exponential map at $x\boldsymbol{e}_1$. We write

$$M = R_g(m)\,, \qquad (\mathrm{Exp})^*(\mu_\Sigma) = \delta\, dx_1\, dx_2 \ .$$

Here μ_Σ is the volume element on Σ induced by the metric g. We may finally write:

$$M(X,Y) = \int_{\mathbb{R}^2} m\Big(\mathrm{Exp}(x)\Big)\Big(X\circ \mathrm{Exp}(x),\, Y\circ\mathrm{Exp}(x)\Big)\,\delta(x)\,dx_1\,dx_2 \ .$$

Let $\gamma(t,x,\theta,\varphi) = \mathrm{Exp}(t(x_1,x_2))$ be the g-geodesic through x with tangent vector $x_1T_1^{\boldsymbol{e}} + x_2T_2^{\boldsymbol{e}}$, and consider the g-orthonormal fields

$$N = \nabla^g\phi\,, \qquad T_1 = \nabla^g\phi_1\,, \qquad T_2 = \nabla^g\phi_2 \ ,$$

given by Corollary 3.1, along the g-geodesic γ. Let

$$\begin{aligned} N &:= N(t,x,\theta,\varphi) = N(\gamma(t))\,, & T_1 &:= T_1(t,x,\theta,\varphi) = T_1(\gamma(t)) \ , \\ T_2 &:= T_2(t,x,\theta,\varphi) = T_2(\gamma(t))\,, & T &:= T(t,x,\theta,\varphi) = \dot\gamma(t) \ . \end{aligned} \tag{5.6}$$

We note that

$$\begin{aligned} \Theta_1 &:= \gamma_*\left(\frac{\partial}{\partial\theta}\right) = \beta_0 N \ , \\ \Theta_2 &:= \gamma_*\left(\frac{\partial}{\partial\varphi}\right) = \beta_1T_1 + \beta_2T_2 + \beta_3N \ , \\ X &:= \gamma_*\left(\frac{\partial}{\partial x}\right) = \alpha_1T_1 + \alpha_2T_2 + \alpha_3N \ , \end{aligned} \tag{5.7}$$

where the functions (see Lemma 5.2) α_1, α_2, β_0, β_1 and β_2 depend on the variables x, θ, φ and α_3, β_3 depend on the variables t, x, θ and φ.

Remark 4. If g is near the euclidean metric then T_1, T_2, N, X, δ and Θ_i are close to their euclidean counterparts.

We note that $T = x_1 T_1 + x_2 T_2$. In fact, since Σ is totally geodesic, it follows that N and T are g-orthonormal. Therefore $T = a_1 T_1 + a_2 T_2$; now using the fact that T_i is a g-Killing field, we obtain that a_i is constant.

We introduce the following notation

$$A := \frac{1}{2\,(\beta_1\,x_2 - \beta_2\,x_1)} \;.$$

Lemma 4. *The following statements hold:*

(5.8) $$\nabla_N N = \nabla_{T_i} T_i = \nabla_{\Theta_i} N = 0\,, \qquad i = 1, 2 \;,$$

(5.9) $$[N, T_1] = [N, T_2] = 0 \;.$$

(5.10) *The coefficients α_1, α_2, β_0, β_1, β_2 in (5.7) are independent of the variable t .*

(5.11) $$\nabla_{T_1} T_2 = A T(\beta_3)\, N\,, \qquad \nabla_{T_2} T_1 = -A T(\beta_3)\, N \;,$$

(5.12) $$\nabla_{T_1} N = \nabla_N T_1 = -A T(\beta_3)\, T_2\,, \qquad \nabla_{T_2} N = \nabla_N T_2 = A T(\beta_3)\, T_1 \;,$$

(5.13) $$\nabla_{\Theta_2} T_1 = -\beta_2\, A T(\beta_3)\, N - \beta_3\, A T(\beta_3)\, T_2 \;,$$

(5.14) $$\nabla_{\Theta_2} T_2 = \beta_1\, A T(\beta_3)\, N + \beta_3\, A T(\beta_3)\, T_1 \;,$$

(5.15) $$\nabla_{\Theta_2} N = -\beta_1\, A T(\beta_3)\, T_2 + \beta_2\, A T(\beta_3)\, T_1 \;,$$

(5.16) $$\nabla_{\Theta_1} T_1 = -\beta_0\, A T(\beta_3)\, T_2\,, \qquad \nabla_{\Theta_1} T_2 = \beta_0\, A T(\beta_3)\, T_1 \;,$$

(5.17) $$N(\beta_1) = N(\beta_2) = N(\alpha_1) = N(\alpha_2) = 0 \;.$$

Proof: To establish (5.8), we write

$$\nabla_N N = a_1 T_1 + \, a_2\, T_2 \;;$$

since T_i is a g-Killing field, we have

$$a_i = g(\nabla_N N, T_i) = -g(N, \nabla_N T_i) = 0 \;.$$

The statement (5.9) is a consequence of $[\Theta_1, T] = 0$. To get (5.10), we observe that

$$\nabla_{T_1} T_2 = g(\nabla_{T_1} T_2, N)\, N\,, \qquad \nabla_{T_2} T_1 = g(\nabla_{T_2} T_1, N)\, N\ . \tag{5.18}$$

Using (5.18) and the fact that $\nabla_{\Theta_2} T = \nabla_T \Theta_2$, we obtain that β_1 and β_2 are independent of the variable t. Futhermore,

$$2\, AT(\beta_3) = g(\nabla_{T_1} T_2, N) - g(\nabla_{T_2} T_1, N)\ . \tag{5.19}$$

Since $[X, T] = 0$, it follows that α_1 and α_2 are also independent of the variable t. We observe that

$$g(\nabla_{T_1} T_2, N) = -g(T_2, \nabla_{T_1} N) = -g(T_2, \nabla_N T_1) = -g(N, \nabla_{T_2} T_1)\ . \tag{5.20}$$

Using (5.18)–(5.20) we obtain (5.11). To establish (5.17), we use the facts that $[\Theta_1, \Theta_2] = [X, \Theta_1] = 0$. The remaining formulas are immediate. ∎

We introduce the following notation:

$$m_{ij} := m(T_i, T_j), \quad m_{i0} := m(N, T_i), \quad m_{00} := m(N, N)\,, \qquad i, j = 1, 2\ .$$

We shall need the

Proposition 5. *If m satisfies the hypotheses of Theorem 1.1 with Condition II and $M = R_g(m)$, then the following system of equations holds:*

$$X(M_{T_i T_j}) = \Theta_k(M_{T_i T_j}) = 0\,, \qquad i, j, k = 1, 2\ , \tag{5.21}$$

$$\begin{aligned} X(M_{T_j N}) = \sum_{i=1}^{2} \int \Big(& T_i(\delta\alpha_3)\, m_{ij} - T_i(\delta\alpha_i)\, m_{jo} + (-1)^j\, \delta\, \alpha_3\, AT(\beta_3)\, m_{io} \Big) \\ & + \int \Big((-1)^j\, \delta\, \alpha_3\, AT(\beta_3)\, m_{k(j)o} + X(\delta)\, m_{jo} \Big)\ , \end{aligned} \tag{5.22}$$

$$\begin{aligned} \Theta_1(M_{T_j N}) = \sum_{i=1}^{2} \int \Big(& T_i(\delta\beta_0)\, m_{ij} + (-1)^j\, \delta\, \beta_0\, AT(\beta_3)\, m_{io} \Big) \\ & + \int \Big((-1)^j\, \delta\, \beta_0\, AT(\beta_3)\, m_{k(j)o} + \Theta_1(\delta)\, m_{jo} \Big)\ , \end{aligned} \tag{5.23}$$

$$\begin{aligned} \Theta_2(M_{T_j N}) = \sum_{i=1}^{2} \int \Big(& T_i(\delta\beta_3)\, m_{ij} - T_i(\delta\beta_i)\, m_{jo} + (-1)^j\, \delta\, \beta_3\, AT(\beta_3)\, m_{io} \Big) \\ & + \int \Big((-1)^j\, \delta\, \beta_3\, AT(\beta_3)\, m_{k(j)o} + \Theta_2(\delta)\, m_{jo} \Big)\ , \end{aligned} \tag{5.24}$$

$$X(M_{NN}) = \sum_{i=1}^{2} \int \Big(T_i(\delta\alpha_3)\, m_{io} - T_i(\delta\alpha_i)\, m_{oo} + (-1)^{i+1}\, \delta\, \alpha_3\, AT(\beta_3)\, m_{ik(i)}\Big) + \int X(\delta)\, m_{oo} \;, \tag{5.25}$$

$$\Theta_1(M_{NN}) = \sum_{i=1}^{2} \int \Big(-T_i(\delta\beta_0)\, m_{io} + (-1)^{i+1}\, AT(\beta_3)\, m_{ik(i)}\Big) + \int \Theta_1(\delta)\, m_{oo} \;, \tag{5.26}$$

$$\Theta_2(M_{NN}) = \sum_{i=1}^{2} \int \Big(-T_i(\delta\beta_3)\, m_{oo} + T_i(\delta\beta_3)\, m_{io} + (-1)^{i}\, \delta\, \beta_3\, AT(\beta_3)\, m_{ik(i)}\Big) + \int \Theta_2(\delta)\, m_{oo} \;. \tag{5.27}$$

Proof: The meaning of (3.8) in Corollary 3.1 is that $M_{T_iT_j} = 0$ and hence, (5.21) holds. The Transversality Condition satisfied by m means that

$$\nabla_{T_1} m(T_1, T_j) + \nabla_{T_2} m(T_2, T_j) + \nabla_N m(N, T_j) = 0 \;, \tag{5.28}$$

$$\nabla_{T_1} m(T_1, N) + \nabla_{T_2} m(T_2, N) + \nabla_N m(N, N) = 0 \;. \tag{5.29}$$

We begin by computing (5.22); we have

$$X(M_{T_jN}) \;= \int \delta\, X\, m(T_j, N) \;+ \int X(\delta)\, m(T_j, N) \;.$$

Now using integration by parts (taking into account that m is compactly supported in $\overline{\Omega}$), Lemma 5.2, (5.28) and taking $k(j) \neq j \in \{1,2\}$, we obtain

$$\begin{aligned}
\int \delta\, X\, m(T_j, N) &= \sum_{i=1}^{2} \int \delta\, \alpha_i\, T_i\, m_{jo} + \int \delta\, \alpha_3\, N\, m_{jo} \\
&= \sum_{i=1}^{2} \int -T_i(\delta\alpha_i)\, m_{jo} + \int \delta\, \alpha_3\, \nabla_N m(T_j, N) + \delta\, \alpha_3\, m(\nabla_N T_j, N) \\
&= \sum_{i=1}^{2} \int \Big(-T_i(\delta\alpha_i)\, m_{jo} - \delta\, \alpha_3\, T_i\, m_{ij} + \delta\, \alpha_3\, m(T_i, \nabla_{T_i} T_j)\Big) \\
&\quad + \int \delta\, \alpha_3\, m(\nabla_N T_j, N) \\
&= \sum_{i=1}^{2} \int \Big(-T_i(\delta\alpha_i)\, m_{jo} + T_i(\delta\alpha_3)\, m_{ij} + (-1)^j\, \delta\, \alpha_3\, AT(\beta_3)\, m_{io}\Big) \\
&\quad + \int (-1)^j \delta\, \alpha_3\, AT(\beta_3)\, m_{k(j)o} \;.
\end{aligned}$$

To establish (5.23), we note that

$$\Theta_1(M_{T_j N}) = \int \delta\, \Theta_1\, m(T_j, N) + \int \Theta_1(\delta)\, m(T_j, N) \ .$$

It follows from similar arguments that

$$\begin{aligned}
&\int \delta\, \beta_0\, \nabla_N m(T_j, N) + \delta\, \beta_0\, m(\nabla_N T_j, N) = \\
&= \sum_{i=1}^{2} \int -\delta\, \beta_0\, \nabla_{T_i}\, m(T_i, T_j) + \int \delta\, \beta_0\, m(\nabla_N T_j, N) \\
&= \sum_{i=1}^{2} \int \Big(-\delta\, \beta_0\, T_i\, m_{ij} + \delta\, \beta_0\, m(T_i, \nabla_{T_i} T_j)\Big) + \int (-1)^j\, \delta\, \beta_0\, AT(\beta_3)\, m_{k(j)o} \\
&= \sum_{i=1}^{2} \int \Big(T_i(\delta\beta_0)\, m_{ij} + (-1)^j\, \delta\, \beta_0\, AT(\beta_3)\, m_{io}\Big) + \int (-1)^j\, \delta\, \beta_0\, AT(\beta_3)\, m_{k(j)o} \ .
\end{aligned}$$

The remaining formulas follow from similar computations. ∎

Proof of Theorem 1.1 with Condition II: It follows from Proposition 5.1 that there is a constant $C_1 > 0$ such that

$$\|dM\|_0 \le C_1\, \varepsilon\, \|m\|_0 \ , \tag{5.30}$$

where ε can be made arbitrarily small by requiring that g be sufficiently close to the euclidean metric. On the other hand, by the Poincaré inequality, we obtain that there is a constant $C_2 > 0$ such that

$$\|H\|_1 \le C_2\, \|dH\|_0 \ , \tag{5.31}$$

and using Lemma 5.1 with $n = 3$, it follows that there is a constant $C_3 > 0$ such that

$$\|m\|_0 \le C_3\, \|M\|_1 \ . \tag{5.32}$$

Using (5.30)–(5.32) we get that $M = 0$ and, consequently, $m = 0$. ∎

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