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KIRCHHOFF–CARRIER ELASTIC STRINGS IN NONCYLINDRICAL DOMAINS

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Abstract: The existence and uniqueness of local and global solutions for the Kirchhoff–Carrier nonlinear model for the vibrations of elastic strings in noncylindrical domains are investigated by means of the Galerkin Method. The asymptotic behaviour of the energy is also studied.

1 – Introduction

Let $\alpha\colon [0,T] \to \mathbb{R}$ and $\beta\colon [0,T] \to \mathbb{R}$, be two, twice continuously differentiable functions such that:

$$\alpha(t) < \beta(t) \quad \text{ for all } \ 0 \le t \le T \ .$$

We consider the noncylindrical domain $\widehat{Q}$, contained in $\mathbb{R}^2$, defined by:

$$\widehat{Q} = \Big\{ (x,t) \in \mathbb{R}^2 \colon \alpha(t) < x < \beta(t), \ \text{ for all } 0 < t < T \Big\} \ .$$

The lateral boundary $\widehat{\Sigma}$ of $\widehat{Q}$ is given by

$$\widehat{\Sigma} = \bigcup_{0<t<T} \{\alpha(t), \beta(t)\} \times \{t\} \ .$$

In this work we investigate the existence, uniqueness and the asymptotic behaviour of solutions of the following noncylindrical mixed problem, for Kirchhoff–Carrier elastic strings:

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$$
(1.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{in } \widehat{Q} \\ u = 0 \quad \text{on } \widehat{\Sigma} \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{in } \alpha(0) < x < \beta(0) \,. \end{cases}
$$

In the bibliography at the end of this article a complete list of papers dealing with the Kirchhoff–Carrier operator

$$
Lw = \frac{\partial^2 w}{\partial t^2} - M\left(\int_\Omega |\nabla w|^2 \, dx\right) \Delta w
$$

in a cylinder can be found. Regular global solutions are obtained in Arosio–Spagnolo [1], Lions [13] and Pohozhaev [16]; local weak solutions in Ebihara–Medeiros–Miranda [7] for the degenerate case, i.e. $M(s) \geq 0$, cf. also Yamada [17]. For perturbated Kirchhoff–Carrier operators see Arosio–Garivaldi [2]. We refer to Bainov and Minchev [6] for estimates on the interval of existence of local solutions. Variational inequalities for Lu are discussed in Frota–Larkin [8]. In these references complete information is given about the operator Lu.

For global existence in the cylindrical case we refer to Brito [4], Hosya–Yamada [9] and Kouemou Patcheu [12]. Note that in this case initial data are chosen inside a fixed ball. In the present work we also need to choose the initial data satisfying analogous restrictions (see Section 6, Theorem 6.1, condition (6.10)). In Nakao–Nakazaki [14], existence and decay for solutions of the nonlinear wave equation, in noncylindrical domains for the d'Alembert operator $\Box\, u = u_{tt} - \Delta u$ is investigated. They employed the penalty method as in Lions [14]. In the work of Komornik–Zuazua [11], for d'Alembert operator with dissipative nonlinear conditions on part of the boundary a method to study the asymptotic behaviour of the energy was introduced. We adopt this method here to our case.

Some results for noncylindrical domains in dimensions $n \geq 2$ will be published elsewhere.

The plan of this work is as follows:

1. Introduction
2. Notations, Assumptions and Local Results
3. Approximations and Estimates
4. Proof of the Theorems
5. Applications
6. Global Solutions
7. Asymptotic Behaviour.

   Bibliography

2 – Notations, assumptions and local results

We consider real functions $\alpha(t)$, $\beta(t)$ and $M(\lambda)$ satisfying the following conditions:

(H1) $\alpha, \beta \in C^2([0,+\infty];\mathbb{R})$ with $\alpha(t)<\beta(t)$, $\alpha'(t) \le 0$, $\beta'(t) \ge 0$ and

$$\max\Big\{|\alpha'(t)|,\, |\beta'(t)|\Big\} \;\le\; \left(\frac{m_0}{2}\right)^{1/2}$$

for all $0 \le t < \infty$.

(H2) $M \in C^1([0,\infty[;\mathbb{R})$ such that $M(\lambda) \ge m_0 > 0$ for all $\lambda \ge 0$.

Remark 2.1. Note that the assumption $\alpha'(t) \le 0$ and $\beta'(t) \ge 0$ means that $\widehat{Q}$ is increasing in the sense that $\gamma(t) = \beta(t) - \alpha(t)$ is increasing.

Remark 2.2. The condition

$$\max\Big\{|\alpha'(t)|,\, |\beta'(t)|\Big\} \;\le\; \left(\frac{m_0}{2}\right)^{1/2}$$

is equivalent to

$$\Big|\alpha'(t) + y\,\gamma'(t)\Big| \le \left(\frac{m_0}{2}\right)^{1/2}, \qquad \text{for all } \; 0 \le y \le 1 \;.$$

In fact, if $y=0$ and $y=1$ in this last inequality we recover the hypothesis (H1). Reciprocally, if (H1) is true we have:

$$|\alpha'(t)| \le \left(\frac{m_0}{2}\right)^{1/2} \quad \text{and} \quad |\beta'(t)| \le \left(\frac{m_0}{2}\right)^{1/2} .$$

By (H1) we have $\alpha'(t) \le 0$ and $\beta'(t) \ge 0$. Then

$$\alpha'(t) + y\,\gamma'(t) \le y\,\beta'(t) \le \left(\frac{m_0}{2}\right)^{1/2} \quad \text{for all } \; 0 \le y \le 1$$

and

$$-\Big(\alpha'(t) + y\,\beta'(t)\Big) \le -\alpha'(t) - \beta'(t) + \alpha'(t)\,y \le -\alpha'(t) \le \left(\frac{m_0}{2}\right)^{1/2}$$

or

$$\Big|\alpha'(t) + y\,\gamma'(t)\Big| \le \left(\frac{m_0}{2}\right)^{1/2} \quad \text{for all } \; 0 \le y \le 1 \;.$$

Remark 2.3. In the investigation of global solutions of (1.1) we make a stronger assumption; namely $\max\{|\alpha'(t)|, |\beta'(t)\} \le C\Big(\frac{m_0}{2}\Big)^{1/2}$ with

$$C = \frac{1}{6}\left(\frac{2}{5}\right)^{1/2}\left(\frac{\pi}{\pi+1}\right) .$$

Observe that when (x,t) varies in $\widehat{Q}$ the point (y,t), with $y=\dfrac{x-\alpha}{\gamma}$ varies in the cylinder $Q =]0,1] \times]0,T[$. The mapping $\tau\colon \widehat{Q} \to Q$, given by $\tau\colon (x,t) \to (y,t)$ is a diffeomorphism. We transform system (1.1) by means of the change of variables

$$u(x,t) = v(y,t) \quad \text{ with } \; y = \frac{x-\alpha}{\gamma} ,$$

which transforms the operator

$$\widehat{L}u = \frac{\partial^2 u}{\partial t^2} - M\left(\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2} \quad \text{ in } \; \widehat{Q}$$

into the operator

$$\begin{aligned}\check{L}v = {} & \frac{\partial^2 v}{\partial t^2} - \frac{1}{\gamma^2}\,\check{M}\left(\frac{1}{\gamma}\int_0^1 \left(\frac{\partial v}{\partial y}\right)^2 dy\right) \frac{\partial^2 v}{\partial y^2} - \frac{\partial}{\partial y}\left(a(y,t)\,\frac{\partial v}{\partial y}\right) \\ & + b(y,t)\,\frac{\partial^2 v}{\partial t\,\partial y} + c(y,t)\,\frac{\partial v}{\partial y} \quad \text{ in } \; Q .\end{aligned}$$

Here we have

- $dx = \gamma\, dy$,
- $\check{M}(\lambda) = M(\lambda) - \dfrac{m_0}{2} \ge \dfrac{m_0}{2} > 0$,
- $a(y,t) = \dfrac{m_0}{2\,\gamma^2} - \left(\dfrac{\alpha' + \gamma' y}{\gamma}\right)^2 > 0$,
- $b(y,t) = -2\left(\dfrac{\alpha' + \gamma' y}{\gamma}\right)$,
- $c(y,t) = -\left(\dfrac{\alpha'' + \gamma''\, y}{\gamma}\right)$.

Then, the noncylindrical mixed problem (1.1) is transformed in the following cylindrical mixed problem:

$$(2.1)\qquad \begin{cases} \check{L}v(y,t) = g(y,t) \quad \text{in } \ Q\ , \\ v(0,t) = v(1,t) = 0 \quad \text{on } \ 0<t<T\ , \\ v(y,0) = v_0(y), \quad \dfrac{\partial v}{\partial y}(y,t) = v_1(y) \quad \text{on } \ 0<y<1\ . \end{cases}$$

We represent, as usual, by $((\ ,\ ))$, $\|\cdot\|$ and $(\ ,\ )$, $|\cdot|$, respectively, the scalar product and norm in $H_0^1(0,1)$ and $L^2(0,1)$. By $a(t,v,w)$ we denote the positive, continuous, symmetric bilinear form in $H_0^1(0,1)$:

$$a(t,v,w) \ = \int_0^1 a(y,t)\, \frac{\partial v}{\partial y}\, \frac{\partial w}{\partial y}\ dy\ .$$

We have the following results:

Theorem 2.1. *Let Ω_t be the interval $]\alpha(t),\beta(t)[\,,\ 0<t<T$, and suppose that (H1) and (H2) hold. Then, given*

$$u_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)\,, \qquad u_1 \in H_0^1(\Omega_0)\,, \qquad f \in L^\infty([0,T];H_0^1(\Omega_t))\ ,$$

there exists $0<T_0<T$ and a unique function

$$u\colon\ \widehat{Q}_0 \to \mathbb{R}, \qquad \widehat{Q}_0 = \Omega\times\,]0,T_0[\ ,$$

satisfying the conditions:

$$u\in L^\infty(0,T_0;H_0^1(\Omega_t)\cap H^2(\Omega_t)), \quad u'\in L^\infty(0,T_0;H_0^1(\Omega_t)), \quad u''\in L^\infty(0,T_0;L^2(\Omega_t))\ ,$$

solution of (1.1) in $\widehat{Q}_0$.

Theorem 2.2. *Under the assumptions of Theorem 2.1, given*

$$v_0 \in H_0^1(0,1)\cap H^2(0,1)\,, \qquad v_1\in H_0^1(0,1) \qquad \text{and} \qquad g\in L^\infty([0,T];H_0^1(0,1))\ ,$$

there exists $0<T_0<T$ and a unique function

$$v\colon\ Q_0\to\mathbb{R}$$

satisfying the conditions:

$$v\in L^\infty(0,T_0;H_0^1(0,1)\cap H^2(0,1))\,, \qquad v'\in L^\infty(0,T_0;H_0^1(0,1))\ ,$$

which is solution of (2.1), in $Q_0 = \,]0,1[\,\times\,]0,T_0[\,$.

Remark 2.4. By the change of variables $y = \dfrac{x-\alpha}{\gamma}$ in $\widehat{Q}$ we obtain the term $2\,\dfrac{\alpha+\gamma\,y}{\gamma}\,\dfrac{\gamma'}{\gamma}\,\dfrac{\partial v}{\partial y} + \Big(\dfrac{\alpha'+\gamma' y}{\gamma}\Big)^2\,\dfrac{\partial^2 v}{\partial y^2}$ in (2.1) that gives serious trouble when we multiply both sides of the equation (2.1) by $-\dfrac{\partial^2 v'}{\partial y^2}$ and integrate in Q. However, under the assumptions (H1) and (H2), the bad terms can be absorbed by the positive terms that the nonlinearity $M(\lambda)$ provides. Indeed, we incorporate the term $\Big(-\dfrac{m_0}{2\,\gamma^2}+\dfrac{m_0}{2\,\gamma^2}\Big)\dfrac{\partial^2 v}{\partial y^2}$ in $\check{L}v$, so that $\check{M}(\lambda) = M(\lambda) - \dfrac{m_0}{2} > \dfrac{m_0}{2}$. The remaining terms can be written in divergence form:

$$-\frac{m_0}{2\,\gamma^2}\,\frac{\partial^2 v}{\partial y^2} + 2\,\frac{\alpha'+\gamma' y}{\gamma}\,\frac{\gamma'}{\gamma}\,\frac{\partial v}{\partial y} + \left(\frac{\alpha'+\gamma' y}{\gamma}\right)^2\frac{\partial^2 v}{\partial y^2} = \\ = -\frac{\partial}{\partial y}\left(\left[\frac{m_0}{2\,\gamma^2} - \left(\frac{\alpha'+\gamma' y}{\gamma}\right)^2\right]\frac{\partial v}{\partial y}\right)$$

giving a positive symmetric bilinear continuous form $a(t,v,w)$, which can be handled easily when getting a priori estimates. Thus the operator $\check{L}v$ is well adapted to apply the energy method.

Remark 2.5. When getting estimates for v' in $H_0^1(0,1)$ and v in $H^2(0,1)$ terms of the form

$$\frac{\beta'}{\gamma}\left[\frac{\partial\, v'(1,t)}{\partial y}\right]^2 + \frac{(-\alpha')}{\gamma}\left[\frac{\partial\, v'(0,t)}{\partial y}\right]^2$$

appear. In order to guarantee its positivity we need that $\alpha' \le 0$ and $\beta' \ge 0$ or, in other words, that $\widehat{Q}$ is increasing.

3 – Approximations and estimates

Let $\{w_j\}$, $j = 1,2,...$, be the solutions of the spectral problem:

$$((w_j, v)) = \lambda_j (w_j, v), \qquad \text{for all } \ v \in H_0^1(0,1) \ .$$

They can be chosen to constitute an orthonormal basis of $L^2(0,1)$.

We represent by $V_m = \{w_1, w_2, ..., w_m\}$ the subspace of $H_0^1(0,1)$ generated by the first m eigenfunctions w_j orthonormal in $L^2(0,1)$. Note that this is equivalent to say that $-w_j'' = \lambda_j w_j$, $w_j(0) = w_j(1) = 0$, for $j = 1,2,...$, i.e., they are eigenfunctions of the Laplace operator with zero Dirichlet conditions on the boundary. In the one dimensional case we are working on, we obtain $\lambda_j = (j\pi)^2$ and $w_j = \sqrt{2}\sin j\pi x$, $j = 1,2,\ldots$.

We look for $v_m(t) \in V_m$ solution of the system of ordinary differential equations:

$$\text{(3.1)} \qquad \begin{cases} (\check{L} v_m(t), v) = (g(t), v) \quad \text{for all } \ v \in V_m \ , \\ v_m(0) = v_{0m} \ , \\ v_m'(0) = v_{1m} \ . \end{cases}$$

By v_{0m} and v_{1m} we denote the projections of v_0 and v_1 over V_m. Note that

$$v_{0m} \to v_0 \quad \text{in } \ H_0^1(0,1) \cap H^2(0,1)$$

and

$$v_{1m} \to v_1 \quad \text{in } \ H_0^1(0,1) \ ,$$

as $m \to \infty$.

Estimate I. Let us consider $v = v_m'(t)$ in (3.1). We obtain:

$$\text{(3.2)} \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}|v_m'(t)|^2 + \frac{1}{2\gamma^2}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\frac{d}{dt}\|v_m(t)\|^2 + \\ &\quad + a(t, v_m, v_m') + \Big(b(y,t)\frac{\partial v_m'}{\partial y}, v_m'\Big) + \Big(c(y,t)\frac{\partial v_m}{\partial y}, v_m'\Big) = (g, v_m') \ . \end{aligned}$$

If we set

$$\widehat{M}(\lambda) = \int_0^\lambda \check{M}(s)\, ds \ ,$$

we have

$$\text{(3.3)} \qquad \begin{aligned} &\frac{d}{dt}\Big[\frac{1}{2\gamma}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\Big] + \frac{\gamma'}{2\gamma^3}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\|v_m(t)\|^2 + \\ &\quad + \frac{\gamma'}{2\gamma^2}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big) = \frac{1}{2\gamma^2}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\frac{d}{dt}\|v_m(t)\|^2 \ , \end{aligned}$$

$$\text{(3.4)} \qquad a(t, v_m, v_m') = \frac{1}{2}\frac{d}{dt}a(t, v_m, v_m) - \frac{1}{2}a'(t, v_m, v_m) \ ,$$

where

$$a'(t, v, w) = \int_0^1 a'(y,t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\, dy \ .$$

We also have:

$$\text{(3.5)} \qquad \begin{aligned} \Big(b(y,t)\frac{\partial v_m'}{\partial y}, v_m'\Big) &= \frac{1}{2}b(y,t)\, v_m'(y,t)^2\Big|_{y=0}^{y=1} - \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(v_m'(t))^2\, dy \\ &= -\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(v_m'(t))^2\, dy \ . \end{aligned}$$

Substituting (3.3), (3.4) and (3.5) in (3.2), we obtain:

$$(3.6)\qquad \frac{1}{2}\frac{d}{dt}|v'_m(t)|^2+\frac{1}{2}\frac{d}{dt}\Big[\frac{1}{\gamma}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\Big]+\frac{1}{2}\frac{d}{dt}a(t,v_m,v_m)+ \\ +\frac{\gamma'}{2\gamma^3}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\|v_m(t)\|^2+\frac{\gamma'}{2\gamma^2}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)= \\ =\frac{1}{2}a'(t,v_m,v_m)+\frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(v'_m(t))^2\,dy+\Big(c(y,t)\frac{\partial v_m}{\partial y},\,v'_m\Big)+(g,v'_m)\;.$$

From (3.6), we obtain:

$$(3.7)\qquad \frac{1}{2}\frac{d}{dt}|v'_m(t)|^2+\frac{1}{2}\frac{d}{dt}\Big[\frac{1}{\gamma}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\Big]+\frac{1}{2}\frac{d}{dt}a(t,v_m,v_m)+ \\ +\frac{\gamma'}{2\gamma^3}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\|v_m(t)\|^2+\frac{\gamma'}{2\gamma^2}\widehat{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\le \\ \le\frac{1}{2}|g(t)|^2+C_1|v'_m(t)|^2+C_2\|v_m(t)\|^2\;.$$

We have, by hypothesis, $\gamma'\ge 0$ since $\widehat{Q}$ is increasing, and

$$(3.8)\qquad \widehat{M}\Big(\frac{1}{\gamma}\|v_m\|^2\Big)\ge C\|v_m\|^2$$

$$(3.9)\qquad a(t,v_m,v_m)>0\;.$$

Integrating (3.7) on $[0,t[$ contained in the interval of existence of $v_m(t)$ solution of (3.1), we obtain

$$(3.10)\qquad |v'_m(t)|^2+\|v_m(t)\|^2\le K_0+K_1\int_0^t\Big(|v'_m(s)|^2+\|v_m(s)\|^2\Big)\,ds$$

where K_0, K_1 are constants independent of m.

From (3.10) and Gronwall inequality, we obtain:

$$(3.11)\qquad |v'_m(t)|^2+\|v_m(t)\|^2<C\quad\text{on}\;\;[0,T]\;.$$

Estimate II. In the approximate system (3.1) we take $v=-\dfrac{\partial^2 v'_m}{\partial y^2}$. This gives:

$$(3.12)\qquad \frac{1}{2}\frac{d}{dt}\|v'_m(t)\|^2+\frac{1}{2\gamma^2}\check{M}\Big(\frac{1}{\gamma}\|v_m(t)\|^2\Big)\frac{d}{dt}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2+ \\ +a\Big(t,v_m,-\frac{\partial^2 v'_m}{\partial y^2}\Big)+\Big(b(y,t)\frac{\partial v'_m}{\partial y},-\frac{\partial^2 v'_m}{\partial y^2}\Big)+\Big(c(y,t)\frac{\partial v_m}{\partial y},-\frac{\partial^2 v'_m}{\partial y^2}\Big)= \\ =\Big(g,-\frac{\partial^2 v_m}{\partial y^2}\Big)\;.$$

We have:

$$
\begin{aligned}
a\Big(t, v_m, -\frac{\partial^2 v_m'}{\partial y^2}\Big) &= \frac{1}{2}\,\frac{d}{dt}\int_0^1 a(y,t)\,\Big(\frac{\partial^2 v_m}{\partial y^2}\Big)^2\,dy \\
&\quad -\frac{1}{2}\int_0^1 a'(y,t)\,\Big(\frac{\partial^2 v_m}{\partial y^2}\Big)^2\,dy \\
&\quad -\int_0^1 \frac{\partial}{\partial y}\Big[\frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\Big]\,\frac{\partial v_m'}{\partial y}\,dy \;+\; \frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1}\,.
\end{aligned}
\tag{3.13}
$$

Note that:

$$
\begin{aligned}
\Big(b(y,t)\,\frac{\partial v_m'}{\partial y}, -\frac{\partial^2 v_m'}{\partial y^2}\Big) &= -\int_0^1 b(y,t)\,\frac{\partial v_m'}{\partial y}\,\frac{\partial^2 v_m'}{\partial y^2}\,dy \\
&= -\int_0^1 b(y,t)\,\frac{1}{2}\,\frac{\partial}{\partial y}\Big(\frac{\partial v_m'}{\partial y}\Big)^2\,dy\;.
\end{aligned}
$$

Integrating by parts we get:

$$
\Big(b(y,t)\,\frac{\partial v_m'}{\partial y}, -\frac{\partial^2 v_m'}{\partial y^2}\Big) = \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\Big(\frac{\partial v_m'}{\partial y}\Big)^2\,dy - \frac{1}{2}\,b(y,t)\,\Big(\frac{\partial v_m'}{\partial y}\Big)^2\Big|_{y=0}^{y=1}\,.
\tag{3.14}
$$

Remark 3.1. Since $b(y,t) = -2\,\dfrac{\alpha' + \gamma' y}{\gamma}$,

$$
-\frac{1}{2}\,b(y,t)\,\Big(\frac{\partial v_m'}{\partial y}\Big)^2\Big|_{y=0}^{y=1} = \frac{\beta'}{\gamma}\Big(\frac{\partial\, v_m'(1,t)}{\partial y}\Big)^2 - \frac{\alpha'}{\gamma}\Big(\frac{\partial\, v_m'(0,t)}{\partial y}\Big)^2
$$

which is non negative, by the hypothesis $\alpha' \le 0,\;\; \beta' \ge 0$ on the non-decreasing boundary. On the other hand,

$$
\begin{aligned}
\Big(c(y,t)\,\frac{\partial v_m}{\partial y}, -\frac{\partial^2 v_m'}{\partial y^2}\Big) &= \int_0^1 \frac{\partial}{\partial y}\Big[c(y,t)\,\frac{\partial v_m}{\partial y}\Big]\,\frac{\partial v_m'}{\partial y}\,dy \\
&\quad -\, c(y,t)\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1}\,.
\end{aligned}
\tag{3.15}
$$

Substituting (3.13), (3.14) and (3.15) in (3.12), we obtain:

$$
\frac{1}{2}\,\frac{d}{dt}\|v_m'(t)\|^2 \;+\; \frac{1}{2\,\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v_m(t)\|^2\Big)\,\frac{d}{dt}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2 \;+
$$

$$
\begin{aligned}
&+ \frac{1}{2}\frac{d}{dt}\int_0^1 a(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2 dy \;-\; \frac{1}{2}\int_0^1 a'(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2 dy \\
&- \int_0^1 \frac{\partial}{\partial y}\left[\frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}\,dy \;+\; \frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1} \\
&+ \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial v_m'}{\partial y}\right)^2 dy \;-\; \frac{1}{2}\,b(y,t)\left(\frac{\partial v_m'}{\partial y}\right)^2\Big|_{y=0}^{y=1} \qquad (3.16)\\
&+ \int_0^1 \frac{\partial}{\partial y}\left[c(y,t)\,\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}\,dy \;-\; c(y,t)\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1} = \\
&\qquad = \left(\frac{\partial g}{\partial y}, \frac{\partial v_m'}{\partial y}\right).
\end{aligned}
$$

Remark 3.2. Denoting by $\mu(t) = \dfrac{1}{\gamma^2}\,\check{M}\Big(\dfrac{1}{\gamma}\,\|v\|^2\Big)$, we have:

$$
\mu'(t) \;=\; -\frac{2\,\gamma'}{\gamma^3}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big) + \frac{2}{\gamma^3}\,\check{M}'\Big(\frac{1}{\gamma}\,\|v\|^2\Big)((v,v')) - \frac{\gamma'}{\gamma^4}\,\check{M}'\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\|v\|^2
$$

and, by Estimate I and the fact that $M \in C^1([0,\infty[,\mathbb{R})$ we obtain, by the hypothesis (H1) on α, β:

$$
|\mu'(t)| \le C + \|v_m'\| .
$$

From (3.16) and Remark 3.2, we have:

$$
\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\|v_m'(t)\|^2 + \frac{1}{2}\,\mu(t)\,\frac{d}{dt}\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 + \frac{1}{2}\,\frac{d}{dt}\left(\int_0^1 a(y,t)\left(\frac{\partial v_m}{\partial y}\right)^2 dy\right) + \\
&\qquad\qquad + \frac{\beta'}{\gamma}\left(\frac{\partial\, v_m'(1,t)}{\partial y}\right)^2 + \frac{(-\alpha')}{\gamma}\left(\frac{\partial\, v_m'(0,t)}{\partial y}\right)^2 = \\
&= \frac{1}{2}\int_0^1 a'(y,t)\left(\frac{\partial^2 v_m}{\partial y^2}\right)^2 dy + \int_0^1 \frac{\partial}{\partial y}\left[\frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}\,dy \qquad (3.17)\\
&\quad - \frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1} - \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial v_m'}{\partial y}\right)^2 dy \\
&\quad - \int_0^1 \frac{\partial}{\partial y}\left[c(y,t)\,\frac{\partial v_m}{\partial y}\right]\frac{\partial v_m'}{\partial y}\,dy + c(y,t)\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\Big|_{y=0}^{y=1} + \left(\frac{\partial g}{\partial y}, \frac{\partial v_m'}{\partial y}\right).
\end{aligned}
$$

Remark 3.3. We have the identity:

$$
\frac{\partial v_m}{\partial y}(0,t) \;=\; \int_0^1 \frac{\partial}{\partial y}\left[(1-y)\,\frac{\partial v_m}{\partial y}(y,t)\right] dy \;=\; \int_0^1 (1-y)\,\frac{\partial^2 v_m}{\partial y^2}\,dy - \int_0^1 \frac{\partial v_m}{\partial y}\,dy
$$

or

$$\left|\frac{\partial v_m}{\partial y}(0,t)\right|_{\mathbb{R}} \le \left|\frac{\partial^2 v_m}{\partial y^2}\right|_{L^2(0,1)} + \|v_m\| \le C + \left|\frac{\partial^2 v_m}{\partial y^2}\right|_{L^2(0,1)},$$

by the Estimate II.

A similar estimate is true for $\dfrac{\partial v_m}{\partial y}(1,t)$.

Remark 3.4. We have:

$$a(y,t) = \frac{m_0}{2\gamma^2} - \left(\frac{\alpha' + \gamma' y}{\gamma}\right)^2 > 0 \quad \text{and} \quad \frac{\partial a}{\partial y} = -2\,\frac{\alpha' + \gamma' y}{\gamma}\,\frac{\gamma'}{\gamma},$$

$$c(y,t) = -\frac{\alpha'' + \gamma'' y}{\gamma} - \frac{\alpha' + \gamma' y}{\gamma}\,\frac{\gamma'}{\gamma}.$$

Then, since $\alpha' + \gamma' = \beta'$, we obtain:

$$-\frac{\partial a}{\partial y}\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\bigg|_{y=0}^{y=1} = +\frac{2\,\beta'\gamma'}{\gamma^2}\,\frac{\partial v_m}{\partial y}(1,t)\,\frac{\partial v_m'}{\partial y}(1,t) - \frac{2\,\alpha'\gamma'}{\gamma^2}\,\frac{\partial v_m}{\partial y}(0,t)\,\frac{\partial v_m'}{\partial y}(0,t),$$

$$\begin{aligned} c(y,t)\,\frac{\partial v_m}{\partial y}\,\frac{\partial v_m'}{\partial y}\bigg|_{y=0}^{y=1} &= -\left(\frac{\beta'\gamma' + \gamma\beta''}{\gamma^2}\right)\frac{\partial v_m}{\partial y}(1,t)\,\frac{\partial v_m'}{\partial y}(1,t) \\ &\quad + \left(\frac{\alpha'\gamma' + \alpha''\gamma}{\gamma^2}\right)\frac{\partial v_m}{\partial y}(0,t)\,\frac{\partial v_m'}{\partial y}(0,t). \end{aligned}$$

Let us consider, for example,

$$\left|\frac{\beta'\gamma'}{\gamma^2}\right|\left|\frac{\partial v_m}{\partial y}(1,t)\right|\left|\frac{\partial v_m'}{\partial y}(1,t)\right| \le \lambda\left|\frac{\beta'\gamma'}{\gamma^2}\right|^2\left|\frac{\partial v_m}{\partial y}(1,t)\right|^2 + \frac{1}{4\,\lambda}\left|\frac{\partial v_m'}{\partial y}(1,t)\right|^2.$$

If $\dfrac{1}{\lambda} = \dfrac{\beta'}{\gamma}$, then $\dfrac{1}{4}\,\dfrac{\beta'}{\gamma}\Big(\dfrac{\partial v_m'}{\partial y}(1,t)\Big)^2$ goes to the left side of (3.17) and it is compensated with $\dfrac{\beta'}{\gamma}\Big(\dfrac{\partial v'}{\partial y}(1,t)\Big)^2$ which gives a positive contribution in the first member. The term $\left|\dfrac{\beta'\gamma'}{\gamma^2}\right|\left|\dfrac{\partial v_m}{\partial y}(1,t)\right|^2$ can be estimated as in Remark 3.3. We get

$$\left|\frac{\beta'\gamma'}{\gamma^2}\right|^2\left|\frac{\partial v_m}{\partial y}(1,t)\right|^2 \le C + \left|\frac{\partial^2 v_m}{\partial y}\right|^2,$$

with a possibly different constant C.

The same argument is true for all the other terms above.

Remark 3.5. From the hypothesis on α and β, we can estimate all the other terms in the left side of (3.17) by $C\Big|\dfrac{\partial^2 v_m}{\partial y^2}\Big|^2$ and $C\|v'_m(t)\|^2$.

Then by the Remarks above, we modify (3.17) obtaining:

$$\text{(3.18)}\qquad \frac{d}{dt}\|v'_m\|^2 + \mu(t)\,\frac{d}{dt}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2 + \frac{d}{dt}\int_0^1 a(y,t)\left(\frac{\partial^2 v_m}{\partial y}\right)^2 dy \le$$
$$\le C_0 + C_1\,\|v'_m(t)\|^2 + \left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 .$$

Substituting $\mu\,\dfrac{d}{dt}\Big|\dfrac{\partial^2 v_m}{\partial y^2}\Big|^2$ in (3.18) by $\dfrac{d}{dt}\Big[\mu(t)\Big|\dfrac{\partial^2 v_m}{\partial y^2}\Big|\Big] - \mu'(t)\Big|\dfrac{\partial^2 v_m}{\partial y^2}\Big|^2$ and by Remark 3.2, we obtain:

$$\text{(3.19)}\qquad \frac{d}{dt}\left[\|v'_m(t)\|^2 + \mu(t)\left|\frac{\partial^2 v_m}{\partial y^2}(t)\right|^2 + \int_0^1 a(y,t)\left(\frac{\partial v_m}{\partial y}\right)^2 dy\right] \le$$
$$\le C_0 + C_1\,\|v'_m(t)\|^2 + \|v'_m(t)\|\left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 \quad \text{in } [0,T] .$$

If

$$h_m(t) = \|v'_m(t)\|^2 + \mu(t)\left|\frac{\partial^2 v_m}{\partial y^2}(t)\right|^2 + \int_0^1 a(y,t)\left(\frac{\partial v_m}{\partial y}\right)^2 dy ,$$

we have from (3.19):

$$\frac{dh_m}{dt} \le C_0 + C_1\,h_m + h_m^{3/2} \quad \text{in } [0,T] .$$

From this inequality, we get a number $0 < T_0 < T$, such that $h_m(t)$ is bounded in $[0,T_0]$ independently of m. This gives the second estimate

$$\text{(3.20)}\qquad \|v'_m(t)\|^2 + \left|\frac{\partial^2 v_m}{\partial y^2}\right|^2 < C \quad \text{on } [0,T_0] .$$

Note that $\mu(t)$ is strictly positive on $[0,T]$.

Estimate III. Taking $v = v''_m(t)$ in the approximate system (3.1) we obtain

$$|v''_m(t)|^2 - \mu(t)\left(\frac{\partial^2 v_m}{\partial y^2}, v''_m\right) - \int_0^1 \frac{\partial}{\partial y}\left[a(y,t)\,\frac{\partial v_m}{\partial y}\right] v''_m\,dy \;+$$
$$+\int_0^1 b(y,t)\,\frac{\partial v'_m}{\partial y}\,v''\,dy + \int_0^1 c(y,t)\,\frac{\partial v_m}{\partial y}\,v''_m\,dy \;=\; (g, v''_m) .$$

From the first and second estimates and the hypothesis on α, β, we get:

$$\text{(3.21)}\qquad |v''_m(t)|^2 < C \quad \text{on } [0,T_0] .$$

4 – Proof of the theorems

Proof of Theorem 2.2: In this step we prove that the estimates obtained above are sufficient to take limits in the approximate equation (3.1). In view of (3.11), (3.20) and (3.21) a subsequence represented by (v_k) can be extracted from (v_m) such that:

$$v_k \rightharpoonup v \quad \text{weak star in} \quad L^\infty(0,T_0;H_0^1(0,1)\cap H^2(0,1)) \ , \tag{4.1}$$

$$v_k' \rightharpoonup v' \quad \text{weak star in} \quad L^\infty(0,T_0;H_0^1(\Omega)) \ . \tag{4.2}$$

By the classical compactness argument of Aubin–Lions, cf. Lions [13], it follows that:

$$v_k \to v \quad \text{strongly} \quad L^2(0,T_0;H_0^1(0,1)) \ . \tag{4.3}$$

Because of the estimate (3.21) the subsequence satisfies also:

$$v_k'' \rightharpoonup v'' \quad \text{weak star in} \quad L^\infty(0,T_0;L^2(\Omega)) \ . \tag{4.4}$$

From hypothesis (H1), (H2) and the estimates (3.11) and (3.20), we obtain:

$$\left|\frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v_m(t)\|^2\Big)^2\,\frac{\partial^2 v}{\partial y^2}\right| < C \quad \text{on} \quad [0,T_0[\ . \tag{4.5}$$

Then, the sequence (v_k) is such that

$$\frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v_k(t)\|^2\Big)^2\,\frac{\partial^2 v_k}{\partial y^2} \ \rightharpoonup\ \chi \tag{4.6}$$

weak star in $L^\infty(0,T_0;L^2(\Omega))$. ∎

Lemma 4.1. $\chi = \dfrac{1}{\gamma}\,\check{M}\Big(\dfrac{1}{\gamma}\,\|v(t)\|^2\Big)\,\dfrac{\partial^2 v}{\partial y^2}$ *where v is the limit in (4.1).*

Proof of Lemma 4.1: Let us introduce the notations $\mu_k(t) = \frac{1}{\gamma^2}\check{M}\Big(\frac{1}{\gamma}\|v_k(t)\|^2\Big)$ and $\mu(t) = \frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v(t)\|^2\Big)$. Because of (4.6), for every $w\in L^2(0,T_0;L^2(0,1))$, we have:

$$\begin{aligned}\int_0^{T_0}\Big(\chi-\mu(t)\,\frac{\partial^2 v}{\partial y^2},\,w\Big)\,dt &= \int_0^{T_0}\Big(\chi-\mu_k(t)\,\frac{\partial^2 v_k}{\partial y^2},\,w\Big)\,dt\\ &\quad+\int_0^{T_0}\mu(t)\,\Big(\frac{\partial^2 v_k}{\partial y^2}-\frac{\partial^2 v}{\partial y^2},\,w\Big)\,dt\\ &\quad+\int_0^{T_0}\Big[\mu_k(t)-\mu(t)\Big]\,\Big(\frac{\partial^2 v_k}{\partial y^2},w\Big)\,dt \ .\end{aligned} \tag{4.7}$$

By (4.6) the first right side integral of (4.7) goes to zero when $k \to \infty$ and the second one, by (3.20), also goes to zero. To analyse the third member of the right side of (4.7) we employ hypothesis (H2) on $M(\lambda)$. Then, we have:

$$|\mu_k(t) - \mu(t)| \le c \Big| \|v_k(t)\|^2 - \|v(t)\|^2 \Big| \le c \, \|v_k(t) - v(t)\| \Big(\|v_k(t)\| + \|v(t)\| \Big) \, .$$

It follows from (4.3), estimates (3.11), (3.20) and Lebesgue convergence theorem, that the last term of (4.7) goes to zero when $k \to \infty$. ∎

Integrating by parts we obtain that $a(t, v_k, w) \rightharpoonup a(t, v, w)$ weakly in $L^2(0,T; H^1(0,1))$ and

$$\frac{\partial}{\partial y}\Big(a(y,t)\,\frac{\partial v_k}{\partial y}\Big) \rightharpoonup \frac{\partial}{\partial y}\Big(a(y,t)\,\frac{\partial v}{\partial y}\Big) \quad \text{weakly in} \quad L^2(0,T; L^2(0,1)) \, . \tag{4.8}$$

We also have, by the same argument,

$$b(y,t)\,\frac{\partial v_k'}{\partial y} \rightharpoonup b(y,t)\,\frac{\partial v'}{\partial y} \quad \text{weakly in } L^2(0,T; L^2(0,1)) \, , \tag{4.9}$$

$$c(y,t)\,\frac{\partial v_k}{\partial y} \rightharpoonup c(y,t)\,\frac{\partial v}{\partial y} \quad \text{weakly in } L^2(0,T; L^2(0,1)) \, . \tag{4.10}$$

Because of (4.4), (4.6), Lemma 1, (4.8), (4.9) and (4.10) we take $m = k$ in the approximate equation (3.11) and we let k go to $+\infty$ obtaining:

$$(\check{L}v, w) = (g, w) \quad \text{for all } w \in L^2(0, T_0; L^2(0,1))$$

or, equivalently,

$$\check{L}v = g \quad \text{in } L^2(0, T_0; L^2(0,1)) \, . \tag{4.11}$$

From (4.1), (4.2) and (4.4), we obtain

$$v(0) = v_0 \, , \qquad v'(0) = v_1 \ \text{ on } \Omega \, . \tag{4.12}$$

Uniqueness

If v and $\hat{v}$ are two solutions in the conditions of Theorem 2.2, then $w = v - \hat{v}$ satisfies:

$$\begin{aligned} w'' - \frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\frac{\partial^2 v}{\partial y^2} + \frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|\hat{v}\|^2\Big)\,\frac{\partial^2 \hat{v}}{\partial y^2} - \frac{\partial}{\partial y}\Big(a(y,t)\,\frac{\partial w}{\partial y}\Big) + \\ + b(y,t)\,\frac{\partial w'}{\partial y} + c(y,t)\,\frac{\partial w}{\partial y} = 0 \quad \text{in } L^2(0,T; L^2(0,1)) \, , \end{aligned} \tag{4.13}$$

$w(0) = w'(0) = 0$ in Ω and $w = 0$ on $]0,1[\times]0, T_0[$.

Multiplying (4.13) by w' and integrating we obtain:

$$\begin{aligned}(4.14)\qquad &\frac{1}{2}\,\frac{d}{dt}|w'(t)|^2 + \frac{1}{2\,\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v(t)\|^2\Big)\,\frac{d}{dt}\|w(t)\|^2 \;+\\ &+\frac{1}{2}\,\frac{d}{dt}\int_0^1 a(y,t)\,\Big(\frac{\partial w}{\partial y}\Big)^2\,dy + \int_0^1 b(y,t)\,\frac{\partial w'}{\partial y}\,w'\,dy + \int_0^1 c(y,t)\,\frac{\partial w}{\partial y}\,w'\,dy \;=\\ &=\Big[\frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big) - \frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|\hat{v}\|^2\Big)\Big]\,\Big(\frac{\partial^2\hat{v}}{\partial y^2}, w'\Big)\;.\end{aligned}$$

We have:

$$\begin{aligned}(4.15)\qquad \frac{1}{2\,\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\frac{d}{dt}\|w\|^2 \;=\;& \frac{1}{2}\,\frac{d}{dt}\Big(\frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\|w\|^2\Big)\\ &+\frac{\gamma'}{\gamma^3}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\\ &-\frac{1}{2\gamma^2}\,\check{M}'\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\frac{d}{dt}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\|w\|^2\end{aligned}$$

$$(4.16)\qquad \int_0^1 b(y,t)\,\frac{\partial w'}{\partial y}\,w'\,dy \;=\; -\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(w')^2\,dy\;.$$

Substituting (4.4) and (4.5) in (4.3) we obtain:

$$\begin{aligned}(4.17)\qquad &\frac{d}{dt}\Big(|w'(t)|^2 + \frac{1}{\gamma^2}\,\check{M}\Big(\frac{1}{\gamma}\,\|v\|^2\Big)\,\|w\|^2 + \int_0^1 a(y,t)\,\Big(\frac{\partial w}{\partial y}\Big)^2\,dy\Big) \;\le\\ &\qquad\le\; c\Big(|w'(t)|^2 + \|w(t)\|^2\Big)\;.\end{aligned}$$

Integrating (4.17) over $0 \le t < T_0$, we have:

$$|w'(t)|^2 + \|w(t)\|^2 \;\le\; C_0\int_0^t\Big(|w'(s)|^2 + \|w(s)\|^2\Big)\,ds\;.$$

This implies $w = 0$ by Gronwall's inequality. ∎

Proof of Theorem 2.1: If v is the solution of Theorem 2.2 we consider the function

$$(4.18)\qquad u(x,t) = v(y,t)\,,\qquad x = \alpha + \gamma\,y\;.$$

We also set

$$(4.19)\qquad g(y,t) = f(\alpha + \gamma y,\, t)\;,$$

$$v_0(y) = u(x,0) = u_0\Big(\alpha(0) + \gamma(0)\,y\Big) \;, \tag{4.20}$$

$$\begin{aligned} v_1(y) = u'(x,0) &= u_1\Big(\alpha(0) + \gamma(0)\,y\Big) \\ &\quad + \Big(\alpha'(0) + \gamma'(0)\,y\Big)\,u_0'\Big(\alpha(0) + \gamma(0)\,y\Big) \;. \end{aligned} \tag{4.21}$$

The function $u(x,t)$ defined by (4.18) is the solution of Theorem 2.1. To see this it is sufficient to observe that the application

$$(x,t) \;\to\; \Big(\frac{x-\alpha}{\gamma},\, t\Big)$$

from $\widehat{Q}$ into $]0,1[\,\times\,]0,T_0[$ is of class C^2 and we have:

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{\gamma^2}\,\frac{\partial^2 v}{\partial y}(y,t) \;, \tag{4.22}$$

$$u''(x,t) \;=\; v''(y,t) - \frac{\partial}{\partial y}\Big(a(y,t)\,\frac{\partial v}{\partial y}\Big) + b(y,t)\,\frac{\partial v'}{\partial y}(y,t) + c(y,t)\,\frac{\partial v}{\partial y}(y,t) \;, \tag{4.23}$$

with $y = \dfrac{x-\alpha}{\gamma}$,

$$\int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx \;=\; \frac{1}{\gamma}\int_0^1 \Big(\frac{\partial v}{\partial y}\Big)^2 dy \;. \tag{4.24}$$

From (4.22)–(4.24) we obtain:

$$u''(x,t) - M\Big(\int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\,\frac{\partial^2 u}{\partial x^2} \;=\; \check{L}\,v(y,t) \;, \tag{4.25}$$

with $y = \dfrac{x-\alpha}{\gamma}$. Then u solves (1.1), with initial conditions u_0 and u_1.

The regularity of v given by Theorem 2.2, implies the regularity of u claimed in Theorem 2.1.

To prove the uniqueness observe that from (4.22)–(4.25) we have the equivalence between the mixed problems (1.1) and (2.1). Then, if u and $\hat{u}$ are two solutions of (1.1) given by Theorem 2.1, then v and $\hat{v}$ obtained by (4.18) are solutions in the conditions of Theorem 2.2. Since we have uniqueness for v, i.e., $v = \hat{v}$ it implies $u = \hat{u}$. ∎

5 – Applications

We will give examples of functions $\alpha(t)$ and $\beta(t)$ in $C^1([0,\infty[\,,\mathbb{R})$, such that $\alpha(t)<\beta(t)$, $\alpha'(t)\le 0$ and $\beta'(t)\ge 0$. If $\gamma(t)=\beta(t)-\alpha(t)$, we have $\gamma'(t)\ge 0$ and, by hypothesis, (H1), Remark 2.2, we must have:

$$\Big|\alpha'(t)+\gamma'(t)\,y\Big| \;\le\; \left(\frac{m_0}{2}\right)^{1/2} ,$$

for all $0\le t\le T$ and $0\le y\le 1$.

Let us consider the family of straight lines:

$$x=\alpha'(t)+\gamma'(t)\,y$$

depending of the parameter $t\ge 0$.

We rewrite (H1) as:

$$-\alpha'(t)\le \left(\frac{m_0}{2}\right)^{1/2} \quad \text{and} \quad \beta'(t)\le \left(\frac{m_0}{2}\right)^{1/2}$$

Case 1. Let us consider in the (x,t) plane the lines:

$$\alpha(t)=\alpha_0-\alpha_1 t \quad \text{with } \ 0\le\alpha_1\le \left(\frac{m_0}{2}\right)^{1/2} ,$$

$$\beta(t)=\beta_0+\beta_1 t \quad \text{with } \ 0\le\beta_1\le \left(\frac{m_0}{2}\right)^{1/2} ,$$

with $\alpha_0<\beta_0$ positive constants.

The noncylindrical domains are cones with basis the intervals $[\alpha_0,\beta_0]$.

Case 2. Let us consider the curves

$$\alpha(t)=\frac{\alpha_0+\beta_0}{2}-(t+t_0)^{\frac{1}{2k}} ,$$

$$\beta(t)=\frac{\alpha_0+\beta_0}{2}+(t+t_0)^{\frac{1}{2k}} , \qquad k=1,2,\dots .$$

By the conditions of Lemma 1, we can choose

$$t_0=(2k)^{\frac{2k}{1-2k}}\left(\frac{m_0}{2}\right)^{\frac{k}{1-2k}} .$$

These curves could be written as:

$$(x-x_0)^{2k}=t+t_0 ,$$

$t\ge 0$, with $x_0=\dfrac{\alpha_0+\beta_0}{2}$.

6 – Global solutions

In this section we prove that if we add some damping to the noncylindrical Kirchhoff–Carrier model, with certain restrictions on the initial data, we obtain a global solution in time. Note, however, that we also impose certain restrictions on the boundary of the noncylindrical domain.

In fact, with the notation and hypothesis fixed in Section 2, we consider now, for the sake of simplicity, the domains of the form

$$\alpha(t) = -\beta(t) \quad \text{for all} \ \ t \geq 0 \ .$$

Then

$$\gamma(t) = 2\beta(t) \quad \text{for all} \ \ t \geq 0$$

and by Remark 2.2, we must have:

$$0 < \beta'(t) < C\left(\frac{m_0}{2}\right)^{1/2} . \tag{6.1}$$

with

$$C = \frac{1}{6}\left(\frac{2}{5}\right)^{1/2}\left(\frac{\pi}{\pi+1}\right) .$$

We suppose, in the present section, that

$$M(\lambda) = m_0 + m_1\lambda\,, \qquad m_0, m_1 > 0, \quad \lambda \geq 0 \ . \tag{6.2}$$

This is a particular case in which (H2) holds.

The mapping from $\widehat{Q}$ into the cylinder Q is given by:

$$y = \frac{x+\beta}{2\beta} \qquad \text{or} \qquad x = (2\,y-1)\,\beta \ , \tag{6.3}$$

with $0 \leq y \leq 1$.

We consider the perturbated system (6.4). The modified Kirchhoff–Carrier model with damping is, for $\delta > 0$ fixed, of the type:

$$\begin{cases} u'' - M\left(\displaystyle\int_{-\beta(t)}^{\beta(t)}\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\dfrac{\partial^2 u}{\partial x^2} + \delta\left(\dfrac{\beta'}{\beta}\,x\,\dfrac{\partial u}{\partial x} + u'\right) = 0 \quad \text{in} \ \ \widehat{Q} \ , \\ u = 0 \quad \text{on} \ \ \widehat{\Sigma} \ , \\ u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x) \quad \text{on} \ \ -\beta_0 < x < \beta_0 \ , \end{cases} \tag{6.4}$$

where $\beta_0 = \beta(0)$.

In addition to (H1), (H2) we assume that

(**H3**) For all $t \geq 0$, $|\beta''(t)| \leq \dfrac{(\beta'(t))^2}{\beta(t)}$.

If we consider the mapping (6.3) from $\widehat{Q}$ into Q, the mixed problem (6.4) in the noncylindrical domain $\widehat{Q}$ has the following form in Q:

$$(6.5)\qquad \begin{cases} v'' - \dfrac{1}{4\beta^2}\,\check{M}\Big(\dfrac{1}{2\beta}\,\|v\|^2\Big)\,\dfrac{\partial^2 v}{\partial y^2} - \dfrac{\partial}{\partial y}\Big(a(y,t)\,\dfrac{\partial v}{\partial v}\Big) + \\ \qquad\qquad + \; b(y,t)\,\dfrac{\partial v'}{\partial y} + c(y,t)\,\dfrac{\partial v}{\partial y} + \delta\, v' \;=\; 0 \quad \text{in } \; Q \;, \\ v = 0 \quad \text{ on } \; \Sigma \;, \\ v(y,0) = v_0(y), \;\; v'(y,0) = v_1(y)\,, \quad 0 < y < 1 \;. \end{cases}$$

Note that $v(y,t) = u((2y-1)\beta, t)$ and $Q = (0,1) \times (0,T)$.

The coefficients of the operator $\check{L}v(y,t)$ in (6.5) are given by:

- $a(y,t) = \dfrac{1}{4\beta^2}\left[\dfrac{m_0}{2} - \Big(\beta'(1-2\,y)\Big)^2\right]$,
- $b(y,t) = \dfrac{1}{\beta}\,\Big[\beta'(1-2\,y)\Big]$,
- $c(y,t) = \dfrac{1}{2\,\beta}\,\Big[\beta''(1-2\,y)\Big]$.

We have:

$$-\frac{1}{2}\,a'(y,t) \;=\; \frac{\beta'}{4\,\beta^3}\left[\beta''\beta\,(1-2\,y)^2 + \frac{m_0}{2} - \Big(\beta'(1-2\,y)\Big)^2\right] .$$

Then, by (4.20), we obtain:

$$\Big|\beta''\beta\,(1-2\,y)^2\Big| \leq |\beta''\beta| \leq (\beta')^2 < \frac{m_0}{8}$$

and

$$\Big|\Big(\beta'(1-2\,y)\Big)^2\Big| \leq (\beta')^2 < \frac{m_0}{8} \;.$$

We have, consequently

$$(6.6)\qquad -\frac{1}{2}\,a'(y,t) \geq \frac{m_0\,\beta'}{16\,\beta^3} \;.$$

Note that

$$v' = \frac{\beta'}{\beta}\, x\, \frac{\partial u}{\partial x} + u'$$

which gives in (6.5) the damping $\delta v'$.

Now we determine the condition on the initial data v_0, v_1 which implies the global existence of solution for the mixed problem (6.5).

Let us fix the number δ such that

$$\frac{\delta}{2} > \frac{5}{2\,\beta_0} \left(\frac{m_0}{2}\right)^{1/2} \left(\frac{4+\pi^2}{\pi^2}\right) . \tag{6.7}$$

Let us define the function $H(t)$ by:

$$\begin{aligned} H(t) &= \|v'(t)\|^2 + \frac{\delta}{4}\,(\nabla v'(t), \nabla v(t)) + \frac{\delta^2}{8}\,\|v(t)\|^2 \\ &\quad + \frac{1}{8\,\beta^2}\,\check{M}\left(\frac{1}{2\,\beta}\,\|v(t)\|^2\right) |\Delta v(t)|^2 + \int_0^1 a(y,t)\,(\Delta v)^2\, dy\ , \end{aligned} \tag{6.8}$$

where $\Delta v = \dfrac{\partial^2 v}{\partial y^2}$, $\nabla v = \dfrac{\partial v}{\partial y}$ and v is a solution of the approximate problem for (6.5).

Lemma 6.1. *We have $H(t) > 0$, for all $t \geq 0$.*

Proof: In fact,

$$\left|\frac{\delta}{4}(\nabla v, \nabla v')\right| \leq \frac{\delta}{4}\left|(\nabla v, \nabla v')\right| \leq \frac{\delta^2}{32}\,\|v\|^2 + |v'|^2\ .$$

We know that $\check{M}(\lambda) = \dfrac{m_0}{2} + m_1\lambda \geq \dfrac{m_0}{2}$, then

$$\frac{1}{8\,\beta^2}\,\check{M}(\lambda) \geq \frac{m_0}{16\,\beta^2}$$

and $a(y,t) \geq 0$. It implies:

$$H(t)\ \geq\ \frac{1}{2}\,\|v'\|^2 + \frac{3\,\delta^2}{32}\,\|v\|^2 + \frac{m_0}{16\,\beta^2}\,|\Delta v|^2\ . \ \blacksquare$$

We represent by C_1 the constant:

$$C_1\ =\ \frac{1}{2}\,|v_1|^2 + \frac{1}{4\,\beta_0}\,\widehat{M}\left(\frac{1}{2\,\beta_0}\,\|v_0\|^2\right) + a(0, v_0, v_0)\ . \tag{6.9}$$

Theorem 6.1. *Suppose (H1) and (H3) are true and* $\alpha(t) = -\beta(t)$. *Suppose also that* $M = M(\lambda)$ *is of the form (6.2). Then given*

$$v_0 \in H_0^1(0,1) \cap H^2(0,1)\,, \qquad v_1 \in H_0^1(0,1)$$

and $\delta > 0$ *satisfying (6.7), if* v_0, v_1 *are such that*

$$\sqrt{C_1\, H(0)} < \frac{2\, m_0 \sqrt{m_0}}{16\, m_1}\, \frac{5}{2\,\beta_0} \left(\frac{m_0}{2}\right)^{1/2} \left(\frac{4+\pi^2}{\pi^2}\right) \tag{6.10}$$

with C_1 *and* $H(0)$ *as in (6.9) and (6.8), then the mixed problem (6.5) has a unique weak solution* $v(y,t)$, *defined for all* $t \geq 0$, $y \in (0,1)$.

Remark 6.1. Before going into the proof of Theorem 6.1, we analyse the restriction (6.10) on the initial data u_0 , u_1 of (6.4).

We have

(i) $v_0(y) = u_0\Big((2\,y-1)\,\beta_0\Big)$, with $\beta_0 = \beta(0)$,

(ii) $v_1(y) = u_1\Big((2\,y-1)\,\beta_0\Big) + (2\,y-1)\,\beta'(0)\,\dfrac{d}{dx} u_0\Big((2\,y-1)\,\beta_0\Big)$.

Let us represent $(-\beta_0, \beta_0)$ by Ω_0. Then we have:

(iii) $\|v_0\|^2 = 2\,\beta_0\, \|u_0\|^2_{H^1(\Omega_0)}$,

(iv) $a(t, v_0, v_0) \leq \dfrac{m_0}{8\,\beta_0^2}\, \|v_0\|^2 = \dfrac{m_0}{4\,\beta_0}\, \|u_0\|^2_{H^1(\Omega_0)}$,

(v) $|4\, v_0|^2 = 8\,\beta_0^2\, |4\, u_0|^2_{L^2(\Omega_0)}$,

(vi) $|v_1|^2 \leq \dfrac{1}{\beta_0}\, |u_1|^2_{L^2(\Omega_0)} + \dfrac{(\beta'(0))^2}{\beta_0}\, \|u_0\|^2_{H_0^1(\Omega_0)}$,

(vii) $\|v_1\|^2 \leq 4\,\beta_0\, \|u_1\|^2_{H_0^1(\Omega_0)} + 8\,\beta_0\, \beta'^2(0)\, |\Delta u_0|^2_{L^2(\Omega_0)} + 8\,\beta_0\, \|u_0\|^2_{H_0^1(\Omega_0)}$,

(viii) $\displaystyle\int_0^1 a(y, 0, (\Delta v_0)^2)\, dy \leq \beta_0\, m_0\, |\Delta u_0|^2_{L^2(\Omega_0)}$.

Because of (iii)–(viii) and Remark 2.2, we obtain:

(ix) $C_1 \leq \check{C}_0 = \dfrac{1}{2\,\beta_0}\, |u_1|^2_{L^2(\Omega_0)} + \dfrac{m_0}{2\,\beta_0}\, \|u\|^2_{H_0^1(\Omega_0)} + \dfrac{1}{4\,\beta_0}\, \widehat{M}\Big(\|u_0\|^2_{H_0^1(\Omega_0)}\Big)$

and

(x)
$$\begin{aligned} H(0) \,\leq\, G_0 \,=\, & 12\,\beta_0\, \|u_1\|^2_{H_0^1(\Omega_0)} + 13\,\beta_0\, m_0\, |\Delta u_0|^2_{L^2(\Omega_0)} \\ & + \left(24\,\beta_0 + \frac{58^2}{16}\,\beta_0\right) \|u_0\|^2_{H_0^1(\Omega_0)} \\ & + \beta_0\, \check{M}\Big(\|u_0\|^2_{H_0^1(\Omega_0)}\Big)\, |\Delta u_0|^2_{L^2(\Omega_0)} \,. \end{aligned}$$

Therefore, if we take

$$u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0) \quad \text{and} \quad u_1 \in H_0^1(\Omega_0)$$

and fix δ satisfying (6.4), and u_0 and u_1 are restricted to

$$\sqrt{\check{C}_0\, G_0} < \frac{2\,m_0\,\sqrt{m_0}}{32\,m_1}\,\delta\;,$$

then the mixed problem (6.5) has a unique solution u defined for all $t > 0$ and $x \in \Omega_t$.

Proof of Theorem 6.1: We will employ the Faedo–Galerkin method choosing a Hilbertian basis in $H_0^1(0,1) \cap H^2(0,1)$ (cf. Brezis [3]). We do only the a priori estimates that imply the convergence of approximate solutions as we have done in Section 3, employing classical compactness arguments as in Lions [13]. In fact, we use the basis of eigenvectors of the spectral problem

$$((v,w)) = \lambda(v,w) \quad \text{for all } \; w \in H_0^1(0,1)\;.$$

Note that this basis can be obtained explicitly as in Section 3. Let us obtain the a priori estimates.

Estimate I. Multiply both sides of $(6.4)_1$ by v' and integrate on $(0,1)$. We obtain:

$$\begin{aligned}(6.11)\qquad & \frac{1}{2}\,\frac{d}{dt}|v'(t)|^2 + \frac{d}{dt}\left(\frac{1}{4\,\beta}\,\widehat{M}\left(\frac{1}{2\,\beta}\,\|v(t)\|^2\right)\right) + \\ & + \frac{\beta'}{8\,\beta^2}\,\check{M}\left(\frac{1}{2\beta}\,\|v(t)\|^2\right)\|v(t)\|^2 + \frac{\beta'}{4\,\beta^2}\,\widehat{M}\left(\frac{1}{2\beta}\,\|v(t)\|^2\right) \\ & + \frac{1}{2}\,\frac{d}{dt}\,a(t,v,v) - \frac{1}{2}\,a'(t,v,v) - \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(v'(t))^2\,dy \\ & + \int_0^1 c\,\frac{\partial v}{\partial y}\,v'\,dy + \delta\,|v'|^2 \;=\; 0\end{aligned}$$

with

$$-\int_0^1 \frac{\partial b}{\partial y}\,(v')^2\,dy > 0\;.$$

By (6.6) we obtain:

$$(6.12)\qquad -\frac{1}{2}\,a'(t,v,v) = -\int_0^1 \frac{1}{2}\,a'(y,t)\left(\frac{\partial v}{\partial y}\right)^2 dy \geq \frac{m_0\,\beta'}{16\,\beta^3}\,\|v\|^2\;.$$

Because of (H3), i.e., $|\beta''| < \dfrac{(\beta')^2}{\beta}$, we obtain:

$$\left|\int_0^1 c(y,t)\,\frac{\partial v}{\partial y}\,v'\,dy\right| \le \int_0^1 \left|\frac{\beta''(1-2y)}{2\,\beta}\,\frac{\partial v}{\partial y}\,v'\right| dy$$
$$\le \int_0^1 \frac{(\beta')^2}{2\,\beta^2}\left|\frac{\partial v}{\partial y}\right|\cdot|v'|\,dy$$
$$\le \int_0^1 \frac{\mu(\beta')^4}{4\,\beta^4}\left|\frac{\partial v}{\partial y}\right|^2 dy + \int_0^1 \frac{1}{4\,\mu}\,|v'|^2\,dy\ .$$

If we consider μ such that $\dfrac{\mu(\beta')^4}{4\,\beta^4} = \dfrac{m_0\,\beta'}{16\,\beta^3}$, we obtain $\dfrac{1}{4\,\mu} = \dfrac{(\beta')^3}{m_0\,\beta}$. Then:

$$\left|\int_0^1 c(y,t)\,\frac{\partial v}{\partial y}\,v'\,dy\right| \;\le\; \frac{m_0\,\beta'}{16\,\beta^3}\,\|v\|^2 + \frac{1}{16\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}|v'|^2\ .$$

Whence,

$$\int_0^1 c(y,1)\,\frac{\partial v}{\partial y}\,v'\,dy \;\ge\; -\frac{m_0\,\beta'}{16\,\beta^3}\,\|v\|^2 - \frac{\sqrt{m_0}}{16\,\sqrt{2}\,\beta_0}\,|v'|^2\ . \tag{6.13}$$

From (6.12) and (6.13), we have:

$$-\,\frac{1}{2}\,a'(t,v,v) + \int_0^1 c(y,t)\,\frac{\partial v}{\partial y}\,v'\,dy + \delta\,|v'|^2 \;\ge\; \ge \left(\delta - \frac{1}{16\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\right)|v'|^2\ . \tag{6.14}$$

By (6.14) we modify (6.11), obtaining:

$$\frac{1}{2}\,\frac{d}{dt}|v'(t)|^2 + \frac{d}{dt}\left(\frac{1}{4\,\beta}\,\widehat{M}\left(\frac{1}{2\,\beta}\,\|v(t)\|^2\right)\right) + {}+\,\frac{1}{2}\,\frac{d}{dt}a(t,v,v) + \left(\delta - \frac{1}{8\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\right)|v'|^2 \;\le\; 0\ . \tag{6.15}$$

The parameter δ was fixed satisfying the condition (6.7), what implies:

$$\frac{\delta}{2} > \frac{1}{8\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\ .$$

Then, from (6.15) we get:

$$\frac{1}{2}\,\frac{d}{dt}|v'(t)|^2 + \frac{d}{dt}\left[\frac{1}{4\,\beta}\,\widehat{M}\left(\frac{1}{2\,\beta}\,\|v(t)\|^2\right)\right] + \frac{1}{2}\,\frac{d}{dt}a(t,v,v) + \frac{\delta}{2}\,|v'(t)|^2 \;\le\; 0\ .$$

Integrating from 0 to t, we get:

$$\frac{1}{2}|v'(t)|^2 + \frac{1}{4\beta}\widehat{M}\Big(\frac{1}{2\beta}\|v(t)\|^2\Big) + \frac{1}{2}a(t,v,v) + \frac{\delta}{2}\int_0^t |v'(s)|^2\,ds \le$$
$$\le \frac{1}{2}|v_1|^2 + \frac{1}{4\beta_0}\widehat{M}\Big(\frac{1}{2\beta_0}\|v_0\|^2\Big) + \frac{1}{2}a(0,v_0,v_0)$$

for all $t \ge 0$.

If we set

$$C_1 = \frac{1}{2}|v_1|^2 + \frac{1}{4\beta_0}\widehat{M}\Big(\frac{1}{2\beta_0}\|v_0\|^2\Big) + a(0,v_0,v_0)$$

we obtain the first estimate

$$\frac{1}{2}|v'(t)|^2 + \frac{m_0}{16\beta^2}\|v(t)\|^2 < C_1 \tag{6.16}$$

for all $t > 0$. Note that C_1 does not depend on t, but depends on v_0, v_1, β_0, $a(y,0)$.

Estimate II. We multiply both sides of the equation $(6.5)_1$ by $-\Delta v' = -\dfrac{\partial^2 v'}{\partial y^2}$ and integrate on $(0,1)$.

We obtain:

$$\begin{aligned}
(6.17)\qquad & \frac{1}{2}\frac{d}{dt}\|v'(t)\|^2 + \frac{1}{8\beta^2}\check{M}\Big(\frac{1}{2\beta}\|v(t)\|^2\Big)\frac{d}{dt}|\Delta v(t)|^2 + \\
&+ \frac{1}{2}\frac{d}{dt}\int_0^1 a(y,t)\,(\Delta v)^2\,dy - \frac{1}{2}\int_0^1 a'(y,t)\,(\Delta v)^2\,dy \\
&+ \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\Big(\frac{\partial v'}{\partial y}\Big)^2 dy - \frac{1}{2}b(y,t)\Big(\frac{\partial v'}{\partial y}\Big)\Big|_{y=0}^{y=1} \\
&- \int_0^1 \frac{\partial}{\partial y}\Big(\frac{\partial a}{\partial y}\frac{\partial v}{\partial y}\Big)\frac{\partial v'}{\partial y}\,dy + \frac{\partial a}{\partial y}\frac{\partial v}{\partial y}\frac{\partial v'}{\partial y}\Big|_{y=0}^{y=1} \\
&+ \int_0^1 \frac{\partial}{\partial y}\Big[c(y,t)\frac{\partial v}{\partial y}\Big]\frac{\partial v'}{\partial y}\,dy - c(y,t)\frac{\partial v}{\partial y}\frac{\partial v'}{\partial y}\Big|_{y=0}^{y=1} + \delta\,\|v'(t)\|^2 = 0\,.
\end{aligned}$$

The inequalities below for the various terms of this identity are true. In fact, we have:

$$-\frac{1}{2}\int_0^1 a'(y,t)\,(\Delta v)^2\,dy \ge \frac{m_0\,\beta'}{32\beta^3}|\Delta v|^2\,, \tag{6.18}$$

$$\left|\int_0^1 \frac{\partial}{\partial y}\left[\frac{\partial a}{\partial y}\,\frac{\partial v}{\partial y}\right]\frac{\partial v'}{\partial y}\,dy\right| \le \frac{m_0\,\beta'}{64\,\beta^3}\,|\Delta v|^2 + \frac{2}{\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\left(\frac{4+\pi^2}{\pi^2}\right)\|v'\|^2\ , \tag{6.19}$$

$$\left|\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}\left(\frac{\partial v'}{\partial y}\right)^2 dy\right| \le \frac{1}{2\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\|v'\|^2\ , \tag{6.20}$$

$$\left|\int_0^1 \frac{\partial}{\partial y}\left[c(y,t)\,\frac{\partial v}{\partial y}\right]\frac{\partial v'}{\partial y}\,dy\right| \le \frac{m_0\,\beta'}{64\,\beta^2}\,|\Delta v|^2 + \frac{1}{2\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\left(\frac{4+\pi^2}{\pi^2}\right)\|v'\|^2\ , \tag{6.21}$$

$$-b(y,t)\left(\frac{\partial v'}{\partial y}\right)^2\Bigg|_{y=0}^{y=1} = \frac{\beta'}{2\,\beta}\left(\frac{\partial\, v'(1,t)}{\partial y}\right)^2 + \frac{\beta'}{2\,\beta}\left(\frac{\partial\, v'(0,t)}{\partial y}\right)^2\ , \tag{6.22}$$

$$\begin{aligned}\left|\,\frac{\partial a}{\partial y}\,\frac{\partial v}{\partial y}\,\frac{\partial v'}{\partial y}\Bigg|_{y=0}^{y=1}\right| \le\ & \frac{\beta'}{8\,\beta}\left(\frac{\partial v'}{\partial y}(1,t)\right)^2 + \frac{\beta'}{8\,\beta}\left(\frac{\partial v'}{\partial y}(0,t)\right)^2 \\ & + 4\left(\frac{\beta'}{\beta}\right)^3\left(\frac{\pi+1}{\pi}\right)^2|\Delta v|^2\ ,\end{aligned} \tag{6.23}$$

$$\begin{aligned}\left|\,c(y,t)\,\frac{\partial v}{\partial y}\,\frac{\partial v'}{\partial y}\Bigg|_0^1\right| \le\ & \frac{\beta'}{8\,\beta}\left(\frac{\partial v}{\partial y}(1,t)\right)^2 + \frac{\beta'}{8\,\beta}\left(\frac{\partial v'}{\partial y}(0,t)\right)^2 \\ & + \left(\frac{\beta'}{\beta}\right)^3\left(\frac{\pi+1}{\pi}\right)^2|\Delta v|^2\ .\end{aligned} \tag{6.24}$$

Using (6.18)–(6.24) the following inequality can be obtained from (6.17):

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\|v'(t)\|^2 + \frac{1}{8\,\beta^2}\,\check{M}\left(\frac{1}{2\,\beta}\,\|v(t)\|^2\right)\frac{d}{dt}|\Delta v(t)|^2\ + \\ &\quad + \frac{1}{2}\,\frac{d}{dt}\int_0^1 a(y,t)\,(\Delta v)^2\,dy - \left(\frac{\beta'}{\beta}\right)^3\left(\frac{\pi+1}{\pi}\right)^2|\Delta v|^2 + \frac{m_0\,\beta'}{32\,\beta^2}\,|\Delta v|^2 \\ &\quad + \frac{\beta'}{4\,\beta}\left(\frac{\partial v'}{\partial y}(1,t)\right)^2 + \frac{\beta'}{4\,\beta}\left(\frac{\partial v'}{\partial y}(0,t)\right)^2 \\ &\quad + \left[\delta - \frac{5}{2\,\beta_0}\left(\frac{m_0}{2}\right)^{1/2}\left(\frac{4+\pi^2}{\pi^2}\right)\right]\|v'(t)\|^2\ \le\ 0\ .\end{aligned} \tag{6.25}$$

The coefficient of $|\Delta v|^2$ is assumed to be positive, i.e.,

$$\frac{m_0}{32}\,\frac{\beta'}{\beta^3} - \frac{\beta'^3}{\beta^3}\left(\frac{\pi+1}{\pi}\right)^2 > 0\ .$$

From the condition (6.7) on δ, it follows from (6.25):

$$\frac{1}{2}\frac{d}{dt}\|v'(t)\|^2 + \frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v(t)\|^2\right)\frac{d}{dt}|\Delta v(t)|^2 + \\ + \frac{1}{2}\frac{d}{dt}\int_0^1 a(y,t)\,(\Delta v(t))^2\,dy + \frac{\delta}{2}\|v'(t)\|^2 \le 0\,. \tag{6.26}$$

Estimate III. Multiply both sides of $(6.7)_1$ by $w = -\Delta v$ and integrate on $(0,1)$. We have:

$$-(v'',\Delta v) + \frac{1}{4\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)|\Delta v|^2 - a(t,v,\Delta v) - \\ - \delta(v',\Delta v) + \left(b\,\frac{\partial v'}{\partial y}, -\Delta v\right) + \left(c\,\frac{\partial v}{\partial y}, -\Delta v\right) = 0\,.$$

Whence,

$$\frac{d}{dt}(\nabla v',\nabla v) - \|v'\|^2 + \frac{1}{4\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)|\Delta v|^2 - \\ - a(t,v,\Delta v) - \frac{\delta}{2}\frac{d}{dt}\|v\|^2 + \left(b\,\frac{\partial v'}{\partial y}, -\Delta v\right) + \left(c\,\frac{\partial v}{\partial y}, -\Delta v\right) = 0\,. \tag{6.27}$$

We obtain the inequalities:

$$a(t,v,-\Delta v) \ge \frac{m_0}{8\beta^2}\left(\frac{3}{4} - \frac{1}{\pi}\right)|\Delta v|^2\,, \tag{6.28}$$

$$\left|\left(b(y,1)\,\frac{\partial v'}{\partial y},\,-\Delta v\right)\right| \le \frac{m_0}{32\beta^2}|\Delta v|^2 + \|v'\|^2\,, \tag{6.29}$$

$$\left|\left(c(y,t)\,\frac{\partial v}{\partial y},\,-\Delta v\right)\right| \le \frac{m_0}{16\pi\beta^2}|\Delta v|^2\,. \tag{6.30}$$

By (6.28), (6.29), (6.30), we modify (6.27) obtaining:

$$\frac{d}{dt}(\nabla v',\nabla v) + \frac{\delta}{2}\frac{d}{dt}\|v\|^2 + \frac{m_0}{8\beta^2}|\Delta v|^2 - 2\|v'\|^2 \le 0\,, \tag{6.31}$$

since $\check{M}(\lambda) > \dfrac{m_0}{2} > 0$ and the coefficient of $|\Delta v|^2$ is also positive. Note that δ is a fixed parameter. Multiply (6.31) by $\dfrac{\delta}{4}$ and adding to (6.26) we cancel $-2\|v'\|^2$, obtaining:

$$\frac{1}{2}\frac{d}{dt}\left[\|v'\|^2 + \frac{\delta}{4}(\nabla v', \nabla v) + \frac{\delta^2}{8}\|v'\|^2 + \frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)|\Delta v|^2 + \right. \\ \left. + \frac{1}{2}\int_0^1 a(y,t)\,(\Delta v)^2\,dy\right] + \frac{m_0\,\delta}{32\,\beta^2}|\Delta v|^2 \le \\ \le \left[\frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v(t)\|^2\right)\right]'|\Delta v|^2 \,. \tag{6.32}$$

Note that $\check{M}(\lambda) = \dfrac{m_0}{2} + m_1\lambda$ with $m_0, m_1 > 0$. Then $\check{M}'(\lambda) = m_1$ is constant. We obtain:

$$\left[\frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v(t)\|^2\right)\right]' = -\frac{\beta'}{4\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right) + \frac{m_1}{8\beta^3}(\nabla v, \nabla v') \\ - \frac{\beta'}{16\beta^4}\|v\|^2, \qquad \text{with} \quad -\beta' = \alpha' < 0 \,,$$

or

$$\left[\frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)\right]' \le \frac{m_1}{8\beta^2}\,\frac{\|v\|}{\beta}\,\|v'\| \,. \tag{6.33}$$

From Estimate I, we have:

$$\frac{\|v\|}{\beta^2} \le 4\left(\frac{C_1}{m_0}\right)^{1/2} \tag{6.34}$$

with

$$C_1 = \frac{1}{2}|v_1|^2 + \frac{1}{4\beta_0}\widehat{M}\left(\frac{1}{2\beta_0}\|v_0\|^2\right)a(0, v_0, v_0) \,.$$

Then, from (6.31) and (6.32) it follows:

$$\left[\frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)\right]'|\Delta v|^2 \le \frac{m_1}{2\beta}\left(\frac{C_1}{m_0}\right)^{1/2}\|v'\|\,|\Delta v|^2 \,. \tag{6.35}$$

Then, from (6.31) and (6.34), we have:

$$\frac{1}{2}\frac{d}{dt}\left[\|v'\|^2 + \frac{\delta}{2}(\nabla v', \nabla v) + \frac{\delta^2}{8}\|v'\|^2 + \frac{1}{8\beta^2}\check{M}\left(\frac{1}{2\beta}\|v\|^2\right)|\Delta v|^2 + \right. \\ \left. + \int_0^1 a(y,t)\,(\Delta v)^2\,dy\right] + \left(\frac{m_0}{32}\,\delta - \frac{m_1}{2}\left(\frac{C_1}{m_0}\right)^{1/2}\|v'\|\right)\frac{|\Delta v|^2}{\beta^2} \le 0 \,. \tag{6.36}$$

With the notations of Theorem 6.1, we obtain from (6.35):

$$\frac{1}{2}\,H'(t) \le \left(-\frac{m_0}{32}\,\delta + \gamma(t)\right)\frac{|\Delta v|^2}{\beta^2} \tag{6.37}$$

where

$$\gamma(t) = \left(\frac{C_1}{m_0}\right)^{1/2}\frac{m_1}{2}\,\|v'(t)\|$$

and $\|v'(t)\| < \sqrt{H(t)}$.

Suppose that $\gamma(t) \ge \dfrac{m_0\,\delta}{32}$ and let us see that this leads us to a contradiction.

In fact, we have:

$$\gamma(t) < \frac{1}{\sqrt{m_0}}\,\frac{m_1}{2}\,\sqrt{C_1\,H(t)}\ ,$$

then

$$\gamma(0) \le \frac{1}{\sqrt{m_0}}\,\frac{m_1}{2}\,\sqrt{C_1\,H(0)}\ . \tag{6.38}$$

By assumption (6.10) of Theorem 6.1 about v_0, v_1, we obtain, from (6.37):

$$\gamma(0) < \frac{m_0\,\delta}{32}\ .$$

Let us consider $t^* = \min\left\{t > 0,\ \gamma(t) = \dfrac{m\,\delta}{32}\right\}$. This minimum exists and it is greater than zero, by continuity of $\gamma(t)$. We have:

$$\begin{cases} \gamma(t) < \dfrac{m_0}{32}\,\delta & \text{on}\ \ 0 \le t < t^*\ , \\[2mm] \gamma(t^*) = \dfrac{m_0}{32}\,\delta\ . \end{cases} \tag{6.39}$$

We obtain:

$$H(t^*) - H(0) = \int_0^{t^*} H'(s)\,ds\ . \tag{6.40}$$

By (6.34) we have $H'(s) < 0$ on $(0, t^*)$, then $H(t^*) < H(0)$ or $H(t^*)^{1/2} < H(0)^{1/2}$. Consequently,

$$\gamma(t^*) < \frac{1}{\sqrt{m_0}}\left(\frac{m_1}{2}\right)\sqrt{C_1\,H(t^*)} < \frac{1}{\sqrt{m_0}}\left(\frac{m_1}{2}\right)\sqrt{C_1\,H(0)} < \frac{m_0}{32}\,\delta$$

which is in contradiction with $(6.39)_2$.

Then, from (6.37) we have $H'(t) \le 0$ for all $t \ge 0$ and, therefore, the global estimate

$$\|v'(t)\|^2 + |\Delta v(t)|^2 < c, \qquad \text{for all} \ \ t \ge 0 \ . \tag{6.41}$$

The estimate for $v''(t)$ and the uniqueness can be proved as in the local case. ∎

7 – Asymptotic behaviour

In this section we investigate the behaviour, when t goes to infinity, of a perturbed energy associated to the global solution v obtained in Theorem 6.1. In fact we consider

$$E(v,t) \;=\; \frac{1}{2}\,|v'(t)|^2 + \frac{1}{4\,\beta}\,\widehat{M}\Big(\frac{1}{2\,\beta}\,\|v(t)\|^2\Big) + \frac{1}{2}\,a(t,v(t),v(t)) \ , \tag{7.1}$$

which we still call the energy associated to the solution v of the mixed problem (6.5).

Remember that $\check{M}(\lambda) = \dfrac{m_0}{2} + m_1\lambda$, then $\widehat{M}(\lambda) = \dfrac{m_0}{2}\lambda + \dfrac{m_1}{2}\lambda^2$ and (7.1) takes the form

$$E(v,t) = \frac{1}{2}\,|v'(t)|^2 + \frac{m_0}{16\,\beta^2}\,\|v(t)\|^2 + \frac{m_1}{32\,\beta^3}\,\|v(t)\|^4 + \frac{1}{2}\,a(t,v(t),v(t)) \ . \tag{7.2}$$

Observe that $E(v,t)$ depends on the boundary $\beta = \beta(t)$ of the noncylindrical domain $\widehat{Q}$, and that $-\alpha(t) = \beta(t)$.

Theorem 7.1. *If v is the global solution of the mixed problem (6.5), then we have*

$$E(v,t) \;\le\; C_1\, e^{-C_2 \int_0^t \frac{ds}{\beta(s)^2}} \ ,$$

for all $t > 0$.

Remark 7.1. Note that $C_1 = C_0\,\beta_0^2\,E(v_0,0)$, $C_2 = \dfrac{\delta}{60\,C_0}$ and

$$C_0 = \frac{1+2\,k}{\beta_0^2} + \frac{16\,(k+\delta)}{m_0}$$

with k such that $20\,k = \dfrac{\delta}{4}$.

Proof of Theorem 7.1: Let us consider the perturbation of $E(v,t)$ given by:

$$F(v,t) \;=\; E(v,t) + k\left[2\,(v'(t),v(t)) + \delta\,|v'(t)|^2\right] \tag{7.3}$$

where k is a constant to be fixed; see Remark 7.1. This method of working with a perturbed energy $F(v,t)$ was employed by Komornik–Zuazua [11]. We also refer to Nakao–Narazaki [15] and Hosya–Yamada [9].

Since

$$\Big|2\,k\,(v'(t),v(t))\Big| \;\le\; \frac{k}{\delta}\,|v'(t)|^2 + k\,\delta\,|v(t)|^2 \;, \tag{7.4}$$

we obtain:

$$2\,k\,(v'(t),v(t)) + k\,\delta\,|v(t)|^2 \;\ge\; -\frac{k}{\delta}\,|v'(t)|^2 \;. \tag{7.5}$$

Then, from (7.4) and (7.5) we modify (7.3), obtaining:

$$F(v,t) \;\ge\; \left(\frac{1}{2}-\frac{k}{\delta}\right)|v'(t)|^2 + \frac{m_0}{16\,\beta^2}\,\|v(t)\|^2 + \frac{m_1}{32\,\beta^3}\,\|v(t)\|^4 + \frac{1}{2}\,a(t,v,v) \;.$$

If we take $k>0$ such that $\dfrac{k}{\delta}<\dfrac{1}{4}$ we obtain:

$$F(v,t) \;\ge\; \frac{1}{2}\,E(v,t) \;. \tag{7.6}$$

From (7.3) we also have

$$F(v,t) \;\le\; E(v,t) + k\,|v'(t)|^2 + (k+\delta)\,|v(t)|^2 \;, \tag{7.7}$$

since

$$|v'(t)|^2 \le 2\,E(v,t) \qquad\text{and}\qquad |v(t)|^2 \le \frac{16}{m_0}\,\beta^2(t)\,E(v,t) \;.$$

From (7.7) we obtain:

$$\begin{aligned} F(v,t) \;&\le\; (1+2\,k)\,E(v,t) + \left(\frac{k+\delta}{m_0}\right)16\,\beta^2\,E(v,t) \\ &\le\; \frac{1+2\,k}{\beta_0}\,\beta^2\,E(v,t) + \left(\frac{k+\delta}{m_0}\right)16\,\beta^2\,E(v,t) \;, \end{aligned}$$

since $\beta(t)\ge\beta(0)=\beta_0$. Then:

$$F(v,t) \;\le\; C_0\,\beta^2\,E(v,t) \;, \tag{7.8}$$

with

$$C_0 = \frac{1+2\,k}{\beta_0^2} + \frac{16\,(k+\delta)}{m_0} \ .$$

Taking the time derivative of $F(v,t)$ we have:

$$\frac{dF}{dt} \;=\; \frac{dE}{dt} + k\left[2\,(v''(t),v(t)) + 2\,|v'(t)|^2 + 2\,\delta\,(v'(t),v(t))\right] \ . \tag{7.9}$$

From the first estimate in Section 7, we have:

$$\frac{dE}{dt} \le -\frac{\delta}{2}\,|v'(t)|^2 \ . \tag{7.10}$$

Multiplying both sides of $(6.5)_1$ by v and integrating on $(0,1)$, we obtain:

$$\begin{aligned}(v''(t),v(t)) + \frac{1}{4\,\beta^2}\,\check{M}\Big(\frac{1}{2\,\beta}\,\|v(t)\|^2\Big)\,\|v(t)\|^2 + a(t,v(t),v(t)) \;+ \\ + \Big(b(y,t)\,\frac{\partial v'}{\partial y},\,v(t)\Big) + \Big(c(y,t)\,\frac{\partial v}{\partial y},\,v(t)\Big) + \delta(v'(t),v(t)) \;=\; 0 \ .\end{aligned} \tag{7.11}$$

We verify that:

$$\Big|\Big(b(y,t)\,\frac{\partial v'}{\partial y},\,v(t)\Big)\Big| \;\le\; \frac{1}{4}\Big(\frac{\beta'}{\beta}\Big)^2\,\|v(t)\|^2 + 9\,|v'(t)|^2 \ , \tag{7.12}$$

$$\Big|\Big(c(y,t)\,\frac{\partial v}{\partial y},\,v(t)\Big)\Big| \;\le\; \frac{1}{2}\Big(\frac{\beta'}{\beta}\Big)^2\,\|v(t)\|^2 \ , \tag{7.13}$$

$$a(t,v(t),v(t)) \;\ge\; \frac{1}{4\,\beta^2}\,\Big(\frac{m_0}{2} - \beta'^2\Big)\,\|v(t)\|^2 \ . \tag{7.14}$$

Then, from (7.12), (7.13) and (7.14) we obtain:

$$\begin{aligned}a(t,v(t),v(t)) + \Big(b(y,t)\,\frac{\partial v'}{\partial y},\,v(t)\Big) + \Big(c(y,t)\,\frac{\partial v}{\partial y},\,v(t)\Big) \;\ge \\ \ge\; \frac{1}{4\,\beta^2}\Big(\frac{m_0}{2} - 4\,\beta'^2\Big)\,\|v(t)\|^2 - 9\,|v'(t)|^2 \\ \ge\; -9\,|v'(t)|^2 \ ,\end{aligned} \tag{7.15}$$

since $\dfrac{m_0}{2} - 4\beta'^2 > 0$.

Then, from (7.11), (7.15) and by definition of $\check{M}(\lambda)$, we have:

$$(v''(t),v(t)) + \delta(v'(t),v(t)) \;\le\; -\frac{m_0}{8\,\beta^2}\,\|v(t)\|^2 - \frac{m_1}{8\,\beta^3}\,\|v(t)\|^4 + 9\,|v'(t)|^2 \ . \tag{7.16}$$

Then, substituting (7.10) and (7.16) in (7.9), we obtain:

$$\frac{dF}{dt} \leq -\frac{\delta}{2}\,|v'(t)|^2 + k\left[2\,|v'(t)|^2 - \frac{m_0}{4\,\beta^2}\,\|v(t)\|^2 - \frac{m_1}{4\,\beta^3}\,\|v(t)\|^4 + 18\,|v'(t)|^2\right]$$

or

$$\frac{dF}{dt} + \left(\frac{\delta}{2} - 20\,k\right)|v'(t)|^2 + \frac{k\,m_0}{4\,\beta^2}\,\|v(t)\|^2 + \frac{k\,m_1}{4\,\beta^2}\,\|v(t)\|^2 \leq 0\ .$$

Taking k such that $20\,k = \dfrac{\delta}{4}$, we get

$$\frac{dF}{dt} + \frac{\delta}{4}\,|v'(t)|^2 + \frac{\delta}{20}\left[\frac{m_0}{16\,\beta^2}\,\|v(t)\|^2 + \frac{m_1}{16\,\beta^3}\,\|v(t)\|^2\right] \leq 0\ . \tag{7.17}$$

We have:

$$0 < a(t, v(t), v(t)) \leq \frac{m_0}{8\,\beta^2}\,\|v(t)\|^2$$

which substituted in (7.2) gives:

$$E(v,t) \leq \frac{1}{2}\,|v'(t)|^2 + \frac{3\,m_0}{16\,\beta^2}\,\|v(t)\|^2 + \frac{m_1}{32\,\beta^3}\,\|v(t)\|^4\ ,$$

$$\frac{\delta}{20}\,E(v,t) \leq \frac{\delta}{4}\,|v'(t)|^2 + \frac{\delta}{20}\left(\frac{m_0}{16\,\beta^2}\,\|v(t)\|^2 + \frac{m_1}{16\,\beta^3}\,\|v(t)\|^4\right)\ . \tag{7.18}$$

Substituting (7.18) in (7.17) we obtain:

$$\frac{dF}{dt} + \frac{\delta}{60}\,E(v,t) \leq 0 \quad \text{for all} \ \ t \geq 0\ . \tag{7.19}$$

From (7.8) and (7.19) we obtain:

$$F(v,t) \leq F(v_0, 0)\,e^{-\frac{\delta}{60\,C_0}\int_0^t \frac{ds}{\beta(s)^2}}\ .$$

From (7.6), this last inequality implies:

$$E(v,t) \leq 2\,F(v_0, 0)\,e^{-\frac{\delta}{60\,C_0}\int_0^t \frac{ds}{\beta(s)^2}}\ . \blacksquare$$

Applications

Case 1. Suppose that $m > 2$, is an integer and

$$\beta(t) = (t + t_0)^{\frac{1}{m}}\ , \qquad t, t_0 > 0\ .$$

We have:

$$E(v,t) \;\leq\; C_m\, e^{-\frac{m\,C_2}{m-2}(t+t_0)^{\frac{m-2}{m}}}$$

which gives an exponential decay.

Case 2. Suppose that $m = 2$. Then

$$\beta(t) = (t+t_0)^{1/2}\ .$$

We obtain:

$$E(v,t) \leq C_1 (t+t_0)^{-C_2}\,, \qquad C_2 > 0\ ,$$

which gives an algebraic decay.

Remark 7.2. We have:

$$\begin{aligned} E(v,t) \;&=\; \frac{1}{2}\int_0^1 (v')^2\,dy + \frac{m_0}{16\,\beta^2}\int_0^1\Big(\frac{\partial v}{\partial y}\Big)^2 dy \\ &\quad + \frac{m_1}{32\,\beta^3}\Big(\int_0^1\Big(\frac{\partial v}{\partial y}\Big)^2 dy\Big)^2 + \frac{1}{2}\int_0^1 a(y,t)\,\Big(\frac{\partial v}{\partial y}\Big)^2 dy\ . \end{aligned}$$

If

$$2\,\beta\, y = x+\beta\,, \qquad 2\,\beta\, dy = dx\,, \qquad 2\,\beta\,\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ,$$

then the above quantity, in the cylinder Q, is transformed, into the noncylindrical domain $\widehat{Q}$, in the following one

$$\begin{aligned} E(u,t) \;&=\; \frac{1}{2\,\beta}\Bigg[\frac{1}{2}\int_{-\beta}^{+\beta}\Big(\frac{\beta' x}{\beta}\,\frac{\partial u}{\partial x} + u'\Big)^2 dx + \frac{m_0}{4}\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2 dx \\ &\quad + \frac{m_1}{4}\Big(\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)^2 + \int_{-\beta}^{+\beta}\Big(\frac{m_0}{2} - \Big(\frac{\beta' x}{\beta}\Big)^2\Big)\,\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Bigg]\ . \end{aligned}$$

We have $\widehat{M}(\lambda) = \dfrac{m_0}{2}\lambda + \dfrac{m_1}{2}\lambda^2$. Then

$$\begin{aligned} E(u,t) \;&=\; \frac{1}{2\,\beta}\Bigg[\frac{1}{2}\int_{-\beta}^{+\beta}\Big(\frac{\beta' x}{\beta}\,\frac{\partial u}{\partial x} + u'\Big)^2 dx + \frac{1}{2}\,\widehat{M}\Big(\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big) \\ &\quad + \int_{-\beta}^{+\beta}\Big(\frac{m_0}{2} - \Big(\frac{\beta' x}{\beta}\Big)^2\Big)\,\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Bigg]\ . \end{aligned} \tag{7.20}$$

The last integral is positive, then:

$$E(u,t) \;\geq\; \frac{1}{2\,\beta}\,\widehat{E}(u,t)$$

where

$$\widehat{E}(u,t) \;=\; \frac{1}{2}\int_{-\beta}^{+\beta}\Big(\frac{\beta' x}{\beta}\,\frac{\partial u}{\partial x}+u'\Big)^2\,dx \;+\; \frac{1}{2}\,\widehat{M}\Big(\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2\,dx\Big)\;.$$

The behaviour of $\widehat{E}(u,t)$ depends essentially on β because

$$\frac{1}{2}\,\widehat{E}(u,t) \;\le\; \beta\,C_1\,e^{C_2\int_0^t \frac{ds}{\beta(s)^2}}\;.$$

For example, if $\beta(t)=(1+t)^{1/2}$, then

$$\frac{1}{2}\,\widehat{E}(u,t) \;\le\; C_1(1+t)^{2-C_2}\;,$$

and we need $C_2>2$ to have a decay of algebraic type. Note that C_2 depends on m_0, δ and β_0.

Remark 7.3. We found in (7.20)

$$\begin{aligned} E(u,t) \;=\; &\frac{1}{2\beta}\Big[\frac{1}{2}\int_{-\beta}^{+\beta}\Big(\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}+u'\Big)^2\Big]\,dx \;+\; \frac{1}{2}\int_{-\beta}^{+\beta}\Big(\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}\Big)^2dx\\ &+\frac{1}{2}\,\widehat{M}\Big(\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2dx\Big)+\int_{-\beta}^{+\beta}\Big[\frac{m_0}{2}-\frac{3}{2}\Big(\frac{\beta'}{\beta}x\Big)^2\Big]\Big(\frac{\partial u}{\partial x}\Big)^2dx\;.\end{aligned}$$

Observe that

$$(u')^2 \;=\; \Big(u'+\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}-\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}\Big)^2 \le \frac{1}{2}\Big(u'+\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}\Big)^2+\frac{1}{2}\Big(\frac{\beta'}{\beta}x\,\frac{\partial u}{\partial x}\Big)^2\;.$$

Then,

$$E(u,t) \;\ge\; \frac{1}{2\beta}\Big(\int_{-\beta}^{+\beta}(u')^2\,dx \;+\; \frac{1}{2}\,\widehat{M}\Big(\int_{-\beta}^{+\beta}\Big(\frac{\partial u}{\partial x}\Big)^2dx\Big)\Big)\;.$$

If we represent by

$$\widehat{\widehat{E}}(u,t) \;=\; \int_{-\beta}^{\beta}(u')^2\,dx+\frac{1}{2}\,\widehat{M}\Big(\int_{-\beta}^{\beta}\Big(\frac{\partial u}{\partial x}\Big)^2dx\Big)$$

we have

$$E(u,t)\le\frac{1}{2\beta}\,\widehat{\widehat{E}}(u,t),\qquad \text{for all}\;\; t\ge 0\;.$$

Now suppose that β' is strictly positive and

$$0<\beta'(t)<\frac{\delta}{120\,C_0}\,\frac{1}{\beta}\qquad \text{for all}\;\; t\ge 0$$

with

$$\frac{\delta}{120\, C_0\, \beta_0} < C \Big(\frac{m_0}{2}\Big)^{1/2} .$$

Note that δ, m_0, C_0, C are the constants fixed above.

We obtain

$$-\frac{\delta}{60\, C_0} \int_0^t \frac{ds}{\beta^2(s)} \;\leq\; -2 \int_0^t \frac{\beta'}{\beta}\, ds$$

or

$$\exp\Big(-\frac{\delta}{60\, C} \int_0^t \frac{ds}{\beta^2}\Big) \;<\; \frac{\beta_0^2}{\beta(t)^2} .$$

Substituting in the inequality of Theorem 7.1, we obtain:

$$E(v,t) \;\leq\; \frac{C_0\, \beta_0}{\beta^2(t)}\, E(v,0) .$$

But $\hat{\hat{E}}(u,t) < 2\, \beta\, E(v,t)$. This gives

$$\hat{\hat{E}}(u,t) \leq \frac{2\, C_0\, \beta_0^4\, E(v_0,0)}{\beta(t)} .$$

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BIBLIOGRAPHY

[**1**] Arosio, A. and Spagnolo, S. – *Global solutions to the Cauchy problem for a nonlinear hyperbolic equation*, in "Nonlinear Partial Differential Equations and Their Applications", Collège de France Seminar (H. Brezis and J.L. Lions, Eds.), Pitman Research Notes in Mathematics Series, Vol. 16, London, 1984, pp. 1–24.

[**2**] Arosio, A. and Garivaldi, S. – On degenerate Kirchhoff strings, *Math. Appl. Science,* 14 (1991), 177–195.

[**3**] Brezis, H. – *Analyse Fonctionnelle (Théorie et Applications)*, Masson, Paris, 1983.

[**4**] Brito, E. – The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Appl. Anal.,* 13 (1982), 219–233.

[**5**] Carrier, C.F. – On the vibrations problem of elastic string, *Q. J. Appl. Math.,* 3 (1945), 151–165.

[**6**] Bainov, D. and Minchev, E. – An estimate of the interval of existence of solutions of the nonlinear Kirchhoff equation, *Computers Math. Appl.,* 31(2) (1996), 63–65.

[7] Ebihara, Y., Medeiros, L.A. and Milla Miranda, M. – Local solutions for a nonlinear degenerate hyperbolic equation, *Nonlinear Analysis,* 10 (1986), 27–40.

[8] Frota, C. and Larkin, N.A. – On global existence and uniqueness for unilateral problem associated to the degenerate Kirchhoff equation, *Nonlinear Analysis*, 3 (1997), 443–452.

[9] Hosya, M. and Yamada, Y. – On some nonlinear wave equations. II – Global existence and energy decay of solutions, *Journal of the Faculty of Science, The University of Tokyo, Sec IA,* 38(2) (1991), 239–250.

[10] Kirchhoff, G. – *Vorlesungen über Mechanik*, ch. 9, §7, p. 144, Leipzig, Tauber, 1883.

[11] Komornik, V. and Zuazua, E. – A direct method for the boundary stabilization of the wave equation, *J. Math. Pure Appl.,* 69 (1980), 33–54.

[12] Kouemou Patcheu, S. – Existence globale et decroissance exponentielle de l'énergie d'une équation quasilinéaire, *C. R. Acad. Sci. Paris,* t. 322, Série I (1996), 631–634.

[13] Lions, J.L. – *On some questions in boundary value problems of Mathematical Physics*, in "Contemporary Development in Continuum Mechanics and Partial Differential Equations", (G.M. de La Penha and L.A. Medeiros, Eds.), North Holland, Amsterdam, (1978), pp. 284–329.

[14] Lions, J.L. – *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[15] Nakao, M. and Narazaki, T. – Existence and decay of solutions of some nonlinear wave equations in noncylindrical domains, *Math. Rep.,* XI (1978), 117–125.

[16] Pohozhaev, S. – On a class of quasilinear hyperbolic equations, *Math. Sbornik,* 95 (1975), 152–166.

[17] Sidelnik, Ya.I. – Existence and uniqueness of a generalized solution of the mixed problem for an equation of plate oscillation type in noncylindrical domain, *Math. Fisiko Mekhanicheskic, Polya,* 28 (1988), 87–89.

[18] Yamada, Y. – Some nonlinear degenerate equations, *Nonlinear Analysis,* 11(10) (1987), 1155–1168.

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