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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: We study asymptotic properties of solutions for certain classes of second order nonlinear differential equations. The main purpose is to investigate when all continuable solutions or just a part of them with initial data satisfying an additional condition behave at infinity like nontrivial linear functions. Making use of Bihari's inequality and its derivatives due to Dannan, we obtain results which extend and complement those known in the literature. Examples illustrating the relevance of the theorems are discussed.

1 – Introduction

In this paper, we study asymptotic properties of solutions of the second order nonlinear differential equation

$$u'' + f(t,u,u') = 0 \; . \tag{1}$$

More precisely, our aim is to establish conditions under which all continuable solutions of equation (1) approach those of equation $u''=0$. In other words, we are interested in the case when continuable solutions of (1) behave like nontrivial linear functions $at+b$ as $t\to\infty$. The origin of this studies goes back at least to the results of Bellman [1], Fubini [8], and Sansone [13] related to some specific, mainly linear, cases of equation (1). Asymptotic behavior of solutions of the equation

$$u'' + f(t,u) = 0 \tag{2}$$

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was discussed for the nonlinear case by Cohen [3] and Tong [15] (see Corollaries 2 and 3 below), and the linear case was studied by Trench [16]. All the results cited have been obtained by using the Gronwall–Bellman inequality [1] or its generalization due to Bihari [2]. For yet another ideas involving the phase plane analysis used for the study of asymptotic behavior of solutions for a particular case of equation (1), the autonomous differential equation

$$u'' + f(u, u') = 0 \ ,$$

we refer the reader to the paper by Rogovchenko and Villari [12].

Dannan [6] introduced a class of functions $\mathbb{H}$ (see definition below) and obtained some derivatives of the well-known Bihari's inequality [2].

Definition ([6]). A function $w\colon\ [0,\infty) \to [0,\infty)$ is said to belong to the class $\mathbb{H}$ if

($\mathbf{H}_1$) $w(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$.

($\mathbf{H}_2$) There exists a function ϕ, continuous on $[0,\infty)$ with $w(\alpha u) \leq \phi(\alpha)\, w(u)$ for $\alpha > 0,\ u \geq 0$. □

Making use of Bihari's type inequality (see [6, Theorem 1]), Dannan proved the following result on asymptotic behavior of solutions of equation (1).

Theorem A ([6]). *Assume the following hypotheses:*

(i) *The function $f(t,u,v)$ is continuous on $D = \{(t,u,v)\colon\ t \geq 1,\ u, v \in \mathbb{R}\}$.*

(ii) *$|f(t,u,v)| \leq \phi(t)\, g(|u|/t) + \psi(t)\, |v|$ for $(t,u,u') \in D$, where $\phi(t)$ and $\psi(t)$ are nonnegative continuous functions on $[0,\infty)$.*

(iii) *$g(u)$ is a nonnegative, continuous, nondecreasing function on $[0,\infty)$, and satisfies*

$$g(\alpha u) \leq \phi_1(\alpha)\, g(u)$$

for $\alpha \geq 1,\ u \geq 0$, where $\phi_1(\alpha) > 0$ is continuous for $\alpha \geq 1$.

(iv) *$\int_1^\infty \psi(t)\, dt = k_1 < \infty,\ \int_1^\infty \phi(t)\, dt = k_2 < \infty$.*

We also assume that there exists $K \geq 1$ such that

$$E(t) \int_1^\infty \phi(s)\, \frac{\phi_1(KE(s))}{E^2(s)}\, ds \ \leq\ K \int_1^\infty \frac{ds}{g(s)} \ ,$$

where $E(s) \equiv \exp(\int_1^s \psi(r)\, dr)$. Then for any solution $u(t)$ of (1) with initial *conditions* $u(1) = c_1,\ u'(1) = c_2$ *such that* $|c_1| + |c_2| \le K$,

$$\lim_{t\to\infty} \int_1^t f(s, u(s), u'(s))\, ds \;=\; \alpha(c_1, c_2) < \infty$$

always exists, and if we set $a = c_2 - \alpha(c_1, c_2)$, *then* $u(t) = b + at + o(t)$ *as* $t \to \infty$, *for any constant* b. ∎

We note that Theorem A establishes sufficient conditions for the desired asymptotic behavior not for all, but only for a part of solutions with initial data satisfying a certain condition.

Recently, Constantin [4] obtained the following result on asymptotic behavior of solutions of equation (1).

Theorem B ([4]). *Suppose that the function* $f(t, u, v)$ *satisfies the following conditions:*

(i) $f(t, u, v)$ *is continuous in* $D = \{(t, u, v)\colon\ t \in [1, +\infty),\ u, v \in \mathbb{R}\}$;

(ii) *there exist continuous functions* $h_1, h_2, h_3, g\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(t, u, v)| \;\le\; h_1(t)\, g\!\left(\frac{|u|}{t}\right) + h_2(t)\, |v| + h_3(t)\ ,$$

or

$$|f(t, u, v)| \;\le\; h_1(t)\, \frac{|u|}{t} + h_2(t)\, g(|v|) + h_3(t)\ ,$$

where for $s > 0$ *the function* $g(s)$ *is nondecreasing,*

$$\int_1^{+\infty} h_i(s)\, ds = H_i < +\infty\,, \qquad i = 1, 2, 3\ ,$$

and if we denote

$$G(x) = \int_1^x \frac{ds}{g(s)}\ ,$$

then $G(+\infty) = +\infty$.

Then for every solution $u(t)$ *of (1) we have that* $u(t) = at + b + o(t)$ *as* $t \to \infty$, *where* a, b *are real constants.* ∎

We point out that both Theorem A and Theorem B assume linear growth of the function $f(t, u, u')$ either with respect to u or with respect to u', and this assumption has been essential for the technique used in the proofs of the main results both in [4] and [6]. Furthermore, this condition guarantees that all solutions of equation (1) exist for all $t \ge 1$. However, it will be demonstrated below that this hypothesis can be relaxed while speaking only about continuable solutions as it is usual for most part of oscillatory criteria known in the literature (see, for example, [10], [14], and [17], as well as the references cited therein). We obtain results which extend and complement those known in the literature and apply to new classes of equations. Examples are inserted in the text to illustrate the relevance of the theorems, and we point out that the recent results due to Constantin [4] and Dannan [6] fail to apply to equations (10), (17), (20) and (25).

Finally, we note that some of results presented in this paper (namely, Theorems 5 and 6) have been reported at the International Conference "Topological Methods in Differential Equations and Dynamical Systems" (Krakòw, 17–20 July 1996) and have been announced in [9]. For the detailed discussion of results related to particular cases of Theorem 4, we refer the reader to [11].

2 – Main results

We recall that a function $u\colon [t_0,t_1) \to (-\infty,\infty)$, $t_1 > t_0$ is called a solution of equation (1) if $u(t)$ satisfies equation (1) for all $t \in [t_0,t_1)$. A solution $u(t)$ of equation (1) is called continuable if $u(t)$ exists for all $t \ge t_0$. We say that a solution $u(t)$ of equation (1) possesses the property (L) if $u(t) = at + b + o(t)$ as $t \to \infty$, where a, b are real constants.

In what follows it is assumed that the function $f(t,u,v)$ is continuous in $D = \{(t,u,v)\colon\ t \in [1,\infty),\ u,v \in \mathbb{R}\}$.

Theorem 1. *Suppose that there exist continuous functions $h_1, h_2, h_3, g_1, g_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \;\le\; h_1(t)\, g_1\!\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) + h_3(t) \;,$$

where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing,

$$\int_1^\infty h_i(s)\, ds = H_i < +\infty\,, \qquad i = 1,2,3 \;,$$

and if we denote

$$G(x) = \int_1^x \frac{ds}{g_1(s) + g_2(s)} \;,$$

then $G(+\infty) = +\infty$.

Then any continuable solution of equation (1) possesses the property (L).

Proof: By the standard existence results (see, for example, [5, Existence Theorem 3]), it follows from the continuity of the function f that equation (1) has solution $u(t)$ corresponding to the initial data $u(1) = c_1,\ u'(1) = c_2$. Two times integrating (1) from 1 to t, we obtain for $t \geq 1$

$$u'(t) \;=\; c_2 - \int_1^t f(s, u(s), u'(s))\, ds \;, \tag{3}$$

$$u(t) \;=\; c_2(t-1) + c_1 - \int_1^t (t-s)\, f(s, u(s), u'(s))\, ds \;. \tag{4}$$

It follows from (3) and (4) that for $t \geq 1$

$$|u'(t)| \;\leq\; |c_2| + \int_1^t \Big| f(s, u(s), u'(s)) \Big|\, ds \;,$$

$$|u(t)| \;\leq\; \Big(|c_1| + |c_2|\Big)\, t + t \int_1^t \Big| f(s, u(s), u'(s)) \Big|\, ds \;.$$

Making use of the assumptions of the theorem, we have for $t \geq 1$

$$\begin{aligned} |u'(t)| \;\leq\; |c_2| &+ \int_1^t h_1(s)\, g_1\Big(\frac{|u(s)|}{s}\Big)\, ds \\ &+ \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds + \int_1^t h_3(s)\, ds \;, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{|u(t)|}{t} \;\leq\; |c_1| + |c_2| &+ \int_1^t h_1(s)\, g_1\Big(\frac{|u(s)|}{s}\Big)\, ds \\ &+ \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds + \int_1^t h_3(s)\, ds \;. \end{aligned} \tag{6}$$

Denote by $z(t)$ the right-hand side of inequality (6),

$$\begin{aligned} z(t) \;=\; |c_1| + |c_2| &+ \int_1^t h_1(s)\, g_1\Big(\frac{|u(s)|}{s}\Big)\, ds \\ &+ \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds + \int_1^t h_3(s)\, ds \;, \end{aligned}$$

then (5) and (6) yield

$$|u'(t)| \leq z(t)\,, \qquad \frac{|u(t)|}{t} \leq z(t) \;. \tag{7}$$

Since the functions $g_1(s),\ g_2(s)$ are nondecreasing for $s > 0$, we obtain by (7)

$$g_1\Big(\frac{|u(t)|}{t}\Big) \leq g_1(z(t))\,, \qquad g_2\Big(|u'(t)|\Big) \leq g_2(z(t)) \;.$$

Thus, for $t \geq 1$

$$z(t) \;\leq\; 1 + |c_1| + |c_2| + H_3 + \int_1^t h_1(s)\, g_1(z(s))\, ds + \int_1^t h_2(s)\, g_2(z(s))\, ds \;. \tag{8}$$

Furthermore, due to evident inequality

$$h_1(s)\, g_1(z(s)) + h_2(s)\, g_2(z(s)) \;\leq\; \Big(h_1(s) + h_2(s)\Big)\, \Big(g_1(z(s)) + g_2(z(s))\Big) \;,$$

we have by (8)

$$z(t) \;\leq\; 1 + |c_1| + |c_2| + H_3 + \int_1^t \Big(h_1(s) + h_2(s)\Big)\, \Big(g_1(z(s)) + g_2(z(s))\Big)\, ds \;. \tag{9}$$

Applying Bihari's inequality [2] to (9), we obtain for $t \geq 1$

$$z(t) \;\leq\; G^{-1}\bigg(G\Big(1 + |c_1| + |c_2| + H_3\Big) + \int_1^t \Big(h_1(s) + h_2(s)\Big)\, ds\bigg) \;,$$

where

$$G(w) = \int_1^w \frac{ds}{g_1(s) + g_2(s)} \;,$$

and $G^{-1}(w)$ is the inverse function for $G(w)$ defined for $w \in (G(+0), +\infty)$. Note that $G(+0) < 0$, and $G^{-1}(w)$ is increasing. Now, let

$$K \;=\; G\Big(1 + |c_1| + |c_2| + H_3\Big) + H_1 + H_2 \;<\; +\infty \;.$$

Since $G^{-1}(w)$ is increasing, we have

$$z(t) \leq G^{-1}(K) < +\infty \;,$$

so (7) yields

$$\frac{|u(t)|}{t} \leq G^{-1}(K) \qquad \text{and} \qquad |u'(t)| \leq G^{-1}(K) \;.$$

Using assumptions of the theorem, we have

$$\int_1^t \Big|f(s, u(s), u'(s))\Big|\, ds \;\leq\; \leq\; |c_1| + |c_2| + \int_1^t h_1(s)\, g_1\bigg(\frac{|u(s)|}{s}\bigg)\, ds + \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds + \int_1^t h_3(s)\, ds \;= \;=\; z(t) \;\leq\; G^{-1}(K) \;.$$

Therefore, the integral

$$\int_1^{+\infty} \Big| f(s, u(s), u'(s)) \Big| \, ds$$

converges, and there exists an $a \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} u'(t) = a \ .$$

In the same way as it has been done in [3, 13], we can ensure that there exists a solution $u(t)$ of equation (1) such that

$$\lim_{t \to +\infty} u'(t) \neq 0 \ .$$

Further, by the l'Hospital's rule, we conclude that

$$\lim_{t \to +\infty} \frac{|u(t)|}{t} = \lim_{t \to +\infty} u'(t) = a \ ,$$

and the proof is now complete. ∎

Corollary 1 ([4])**.** *Suppose that the function $f(t,u,u')$ satisfies the following conditions:*

(i) *$f(t,u,v)$ is continuous in $D = \{(t,u,v)\colon\ t \in [1,+\infty),\ u,v \in \mathbb{R}\}$;*

(ii) *there are exist continuous functions $h_1, h_2, h_3, g\colon\ \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \ \le\ h_1(t)\, g\Big(\frac{|u|}{t}\Big) + h_2(t)\, |v| + h_3(t) \ ,$$

or

$$|f(t,u,v)| \ \le\ h_1(t)\, \frac{|u|}{t} + h_2(t)\, g(|v|) + h_3(t) \ ,$$

where for $s > 0$ the function $g(s)$ is nondecreasing,

$$\int_1^{+\infty} h_i(s)\, ds = H_i < +\infty \,, \qquad i = 1,2,3 \ ,$$

and if we denote

$$G(x) = \int_1^x \frac{ds}{s + g(s)} \ ,$$

then $G(+\infty) = +\infty$. Then any solution of equation (1) possesses the property (L).

Proof: We note first that by (ii) and by standard extension theorems (see, for example, [5, Extension Theorem 3]), all solutions of equation (1) are continuable.

In order to show that the conclusion of the corollary follows from Theorem 1, we need to prove that if for any nondecreasing function $g(s)\colon \mathbb{R}_+ \to \mathbb{R}_+$ the integral

$$\int_1^{\infty} \frac{ds}{g(s)}$$

diverges, so does the integral

$$\int_1^{\infty} \frac{ds}{s+g(s)} \ ,$$

or, equivalently, to prove that the divergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{g(k)}$$

implies the divergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{k+g(k)} \ .$$

By the Cauchy theorem, it suffices to show that

$$\sum_{k=1}^{\infty} \frac{2^k}{g(2^k)} = \infty \quad \Longrightarrow \quad \sum_{k=1}^{\infty} \frac{2^k}{2^k+g(2^k)} = \infty \ ,$$

or

$$\sum_{k=1}^{\infty} \frac{1}{\dfrac{g(2^k)}{2^k}} = \infty \quad \Longrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{1+\dfrac{g(2^k)}{2^k}} = \infty \ ,$$

but the latter implication is clear. Now the conclusion of the corollary follows from Theorem 1. ∎

Remark 1. We point out that actually it has been proved that Theorem B is a consequence of our Theorem 1. □

Corollary 2 ([3]). *Suppose that $f(t,u)$ satisfies the following conditions:*

(i) *$f(t,u)$ is continuous in $D\colon\ t \geq 1,\ u \in \mathbb{R}$;*

(ii) *the derivative f_u exists on D and satisfies $f_u(t,u) > 0$ on D;*

(iii) *$|f(t,u(t))| \leq f_u(t,0)\,|u(t)|$ on D;*

(iv) *$\int_1^{+\infty} t\, f_u(t,0)\, dt < +\infty$.*

Then equation (2) has solutions which are asymptotic to $a+bt$ as $t \to +\infty$.

Proof: Let

$$h_1(t) = t\, f_u(t,0), \quad h_2(t) \equiv 0, \quad h_3(t) \equiv 0, \quad g_1(s) = s, \quad g_2(s) \equiv 0 \ .$$

Then the conclusion of corollary follows from Theorem 1. ∎

Corollary 3 ([15]). *Let $f(t,u)$ be continuous in D: $t \geq 1$, $u \in \mathbb{R}$. Assume that there are nonnegative continuous functions $v(t)$ and $\phi(t)$ defined for $t \geq 0$, and a continuous function $g(u)$ defined for $u \geq 0$ such that*

- **(i)** $\int_1^{+\infty} v(t)\, \phi(t)\, dt < +\infty$;
- **(ii)** *$g(u)$ is positive and nondecreasing for $u > 0$;*
- **(iii)** $\int_1^{+\infty} \frac{dt}{g(t)} = +\infty$;
- **(iv)** $|f(t,u(t))| \leq v(t)\, \phi(t)\, g(\frac{|u|}{t})$ *in D.*

Then equation (2) has solutions which are asymptotic to $a + bt$, where a, b are constants.

Proof: Let

$$h_1(t) = v(t)\, \phi(t), \quad h_2(t) \equiv 0, \quad h_3(t) \equiv 0, \quad g_1(s) = s, \quad g_2(s) \equiv 0 \ .$$

The conclusion of corollary follows from Theorem 1. ∎

Remark 2. We note that Corollary 2 without assumption (iii) becomes false as it has been pointed out by Fan Wei Meng [7]. This assumption, crucial for the application of Bihari's inequality [2], has been added later by Constantin [4]. □

Example 1. Consider the nonlinear differential equation

$$u'' + t^{-\frac{3}{2}}\, u' \ln(u') + t^{-\frac{5}{2}}\, u \ln(u) \ = \ 0 \ . \tag{10}$$

By Theorem 1, all continuable solutions of equation (10) are asymptotic to $at + b$ as $t \to +\infty$. □

An important feature of Theorem 1 is that all continuable solutions of equation (1) are asymptotic to $at + b$ as $t \to +\infty$, and this type of behavior requires corresponding restrictions on the growth of the function $f(t,u,u')$ with respect to u and u'. The following result (cf. Theorem A) relaxes them for a certain class of functions, but one has desired asymptotic behavior only for a part of continuable solutions with initial data satisfying an additional condition.

Theorem 2. *Suppose that the following assumptions hold:*

(i) *there exist nonnegative continuous functions* $h_1, h_2, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(t,u,v)| \le h_1(t)\, g_1\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) \ ;$$

(ii) *for* $s > 0$ *the functions* $g_1(s)$, $g_2(s)$ *are nondecreasing, and*

$$g_1(\alpha u) \le \psi_1(\alpha)\, g_1(u)\,, \qquad g_2(\alpha u) \le \psi_2(\alpha)\, g_2(u)$$

for $\alpha \ge 1$, $u \ge 0$, *where the functions* $\psi_1(\alpha)$, $\psi_2(\alpha)$ *are continuous for* $\alpha \ge 1$;

(iii) $\int_1^{+\infty} h_i(s)\, ds = H_i < +\infty$, $i = 1, 2$.

Assume that there exists a constant $K \ge 1$ *such that*

$$K^{-1}\Big(\psi_1(K) + \psi_2(K)\Big)\,(H_1 + H_2) \le \int_1^{+\infty} \frac{ds}{g_1(s) + g_2(s)} \ . \tag{11}$$

Then any continuable solution $u(t)$ *of equation (1) with initial data* $u(1) = c_1$, $u'(1) = c_2$ *such that* $|c_1| + |c_2| \le K$ *possesses the property (L).*

Proof: Arguing in the same way as in Theorem 1, we obtain by (i)

$$|u'(t)| \le |c_2| + \int_1^t h_1(s)\, g_1\left(\frac{|u(s)|}{s}\right) ds + \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds\ , \tag{12}$$

$$\frac{|u(t)|}{t} \le K + \int_1^t h_1(s)\, g_1\left(\frac{|u(s)|}{s}\right) ds + \int_1^t h_2(s)\, g_2\Big(|u'(s)|\Big)\, ds\ , \tag{13}$$

where $t \ge 1$. Denoting by $z(t)$ the right-hand side of inequality (13), we have by (12) and (13)

$$|u'(t)| \le z(t)\,, \qquad \frac{|u(t)|}{t} \le z(t)\ . \tag{14}$$

Since the functions $g_1(s)$, $g_2(s)$ are nondecreasing for $s > 0$, (14) yields for $t \ge 1$

$$z(t) \le K + \int_1^t \Big(h_1(s) + h_2(s)\Big)\Big(g_1(z(s)) + g_2(z(s))\Big)\, ds\ . \tag{15}$$

By assumption (ii), the functions $g_1(u)$, $g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, it follows from [6, Lemma 1] that if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with the corresponding multiplier functions $\psi_1(\alpha)$ and $\psi_2(\alpha)$ respectively, then the sum $g_1(u)+g_2(u)$ also belongs to $\mathbb{H}$, and the corresponding multiplier function is $\psi_1(\alpha)+\psi_2(\alpha)$. Applying [6, Theorem 1] to (15), we have for $t \geq 1$

$$z(t) \leq K\,W^{-1}\Bigg(K^{-1}\Big(\psi_1(K)+\psi_2(K)\Big)\int_1^t \Big(h_1(s)+h_2(s)\Big)\,ds\Bigg)\,, \tag{16}$$

where

$$W(u) = \int_1^u \frac{ds}{g_1(s)+g_2(s)}\,,$$

and $W^{-1}(u)$ is the inverse function for $W(u)$. Inequality (16) holds for all $t \geq 1$ because

$$K^{-1}\Big(\psi_1(K)+\psi_2(K)\Big)\int_1^t \Big(h_1(s)+h_2(s)\Big)\,ds \in \operatorname{Dom}(W^{-1})$$

for all $t \geq 1$ due to assumption (11). Let

$$K^{-1}\Big(\psi_1(K)+\psi_2(K)\Big)\,(H_1+H_2) = L < +\infty\,.$$

Since $W^{-1}(u)$ is increasing, we get

$$z(t) \leq K\,W^{-1}(L) < +\infty\,,$$

so it follows from (14) that

$$\frac{|u(t)|}{t} \leq KW^{-1}(L) \quad \text{and} \quad |u'(t)| \leq KW^{-1}(L)\,.$$

The rest of the proof is similar to that of Theorem 1 and thus is omitted. ∎

Example 2. Consider the nonlinear differential equation

$$u'' + (2\,t)^{-4}\,u^2\cos u + (4\,t)^{-2}\,(u')^2\sin^3 u = 0\,. \tag{17}$$

For equation (17), we have

$$g_1(u) = g_2(u) = u^2, \quad h_1(t) = h_2(t) = (4t)^{-2}, \quad \psi_1(\alpha) = \psi_2(\alpha) = \alpha^2\,.$$

After a straightforward computation, we conclude by Theorem 2 that all continuable solutions of equation (17) with initial data satisfying $|c_1|+|c_2| \leq 2$ are asymptotic to $at+b$ as $t \to +\infty$. □

Theorem 3. *Suppose that assumptions (i) and (iii) of Theorem 2 hold, while (ii) is replaced by*

(**ii′**) *for $s>0$ the functions $g_1(s)$, $g_2(s)$ are nonnegative, continuous and nondecreasing, $g_1(0)=g_2(0)=0$, and satisfy a Lipschitz condition*

$$\Big|g_1(u+v)-g_1(u)\Big| \le \lambda_1\, v\,, \qquad \Big|g_2(u+v)-g_2(u)\Big| \le \lambda_2\, v\ ,$$

where λ_1, λ_2 are positive constants.

Then any continuable solution $u(t)$ of equation (1) with initial data $u(1)=c_1$, $u'(1)=c_2$ such that $|c_1|+|c_2|\le K$ possesses the property (L).

Proof: Applying [6, Corollary 2] to (15), we have for $t\ge 1$

$$\begin{aligned} z(t) \;&\le\; K+\int_1^t\Big(h_1(s)+h_2(s)\Big)\Big(g_1(K)+g_2(K)\Big)\cdot \\ &\qquad\cdot\exp\biggl(\int_1^t(\lambda_1+\lambda_2)\Big(h_1(\tau)+h_2(\tau)\Big)\,d\tau\biggr)\,ds \\ &\le\; K+(H_1+H_2)\Big(g_1(K)+g_2(K)\Big)\exp\Big((\lambda_1+\lambda_2)\,(H_1+H_2)\Big) \;<\; +\infty\ . \end{aligned}$$

The proof can be completed with the same argument as in Theorem 1. ∎

In what follows, we present results analogous to Theorems 1–3 for another class of equations (cf. [11]).

Theorem 4. *Suppose that there exist continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \;\le\; h(t)\, g_1\Big(\frac{|u|}{t}\Big)\, g_2(|v|)\ ,$$

where for $s>0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing,

$$\int_1^\infty h(s)\,ds<\infty\ ,$$

and if we denote

$$G(x)=\int_1^x\frac{ds}{g_1(s)\,g_2(s)}\ ,$$

then $G(+\infty)=+\infty$.

Then any continuable solution of equation (1) possesses the property (L).

Proof: Arguing as in the proof of Theorem 1, we obtain for $t \geq 1$

$$
\begin{aligned}
|u'(t)| \;&\leq\; |c_2| + \int_1^t h(s)\, g_1\Bigl(\frac{|u(s)|}{s}\Bigr)\, g_2\Bigl(|u'(s)|\Bigr)\, ds \;, \\
\frac{|u(t)|}{t} \;&\leq\; |c_1| + |c_2| + \int_1^t h(s)\, g_1\Bigl(\frac{|u(s)|}{s}\Bigr)\, g_2\Bigl(|u'(s)|\Bigr)\, ds \;. \qquad (18)
\end{aligned}
$$

Denoting by $z(t)$ the right-hand side of inequality (18) and using the assumptions of the theorem, we have for $t \geq 1$

$$
z(t) \;\leq\; 1 + |c_1| + |c_2| + \int_1^t h(s)\, g_1(z(s))\, g_2(z(s))\, ds \;. \tag{19}
$$

Applying Bihari's inequality [2] to (19), we obtain for $t \geq 1$

$$
z(t) \;\leq\; G^{-1}\Bigl(G\Bigl(1+|c_1|+|c_2|\Bigr) + \int_1^t h(s)\, ds\Bigr) \;\leq\; G^{-1}(K) \;,
$$

where

$$
G(w) = \int_1^w \frac{ds}{g_1(s)\, g_2(s)} \;,
$$

and $G^{-1}(w)$ is the inverse function for $G(w)$. The function $G^{-1}(w)$ is defined for $w \in (G(+0), +\infty)$, where $G(+0) < 0$, it is increasing, and

$$
K \;=\; G\Bigl(1+|c_1|+|c_2|\Bigr) + \int_1^\infty h(s)\, ds \;<\; \infty \;.
$$

The rest of the proof is similar to that of Theorem 1 and thus is omitted. ∎

Example 3. Consider the nonlinear differential equation

$$
u'' + h(t)\left(\frac{u^2}{u^2+t^2}\right)^{3/4} \left(\frac{(u')^2}{(u')^2+1}\right)^{1/4} \;=\; 0\,, \qquad t > 1 \;, \tag{20}
$$

where

$$
h(t) \;=\; \frac{2}{t^3}\left(\frac{2\,t^4 - 2\,t^2 + 1}{(t^2-1)^2}\right)^{3/4} \left(\frac{2\,t^4 + 2\,t^2 + 1}{(t^2+1)^2}\right)^{1/4} \;.
$$

The functions

$$
g_1(t) = \left(\frac{t^2}{t^2+1}\right)^{3/4}\,, \qquad g_2(t) = \left(\frac{t^2}{t^2+1}\right)^{1/4}
$$

are continuous and nondecreasing for $t > 1$,

$$
\int_{t_0}^{+\infty} h(s)\, ds < +\infty \;,
$$

and

$$G(+\infty) \;=\; \int_{t_0}^{+\infty} \frac{ds}{\left(\dfrac{s^2}{s^2+1}\right)^{3/4} \left(\dfrac{s^2}{s^2+1}\right)^{1/4}} \;=\; \int_{t_0}^{+\infty} \frac{s^2+1}{s^2}\, ds \;=\; +\infty \;,$$

for any $t_0 > 0$. Thus, by Theorem 4, for any continuable solution $u(t)$ of equation (20) there exist real numbers a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Observe that $u(t) = t - 1/t$ is the solution of equation (20) satisfying the initial data $u(2) = 3/2$, $u'(2) = 5/4$, which is asymptotic to t as $t \to \infty$. □

Theorem 5. *Suppose that the following conditions hold:*

(i) *there exist nonnegative continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \;\le\; h(t)\, g_1\!\left(\frac{|u|}{t}\right) g_2(|v|) \;;$$

(ii) *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing, and*

$$g_1(\alpha u) \le \psi_1(\alpha)\, g_1(u)\,, \qquad g_2(\alpha u) \le \psi_2(\alpha)\, g_2(u)$$

for $\alpha \ge 1$, $u \ge 0$, where the functions $\psi_1(\alpha)$, $\psi_2(\alpha)$ are continuous for $\alpha \ge 1$;

(iii) $\int_1^{+\infty} h(s)\, ds = H < +\infty$.

Assume also that there exists a constant $K \ge 1$ such that

$$K^{-1} H\, \psi_1(K)\, \psi_2(K) \;\le\; \int_1^{+\infty} \frac{ds}{g_1(s)\, g_2(s)} \;. \tag{21}$$

Then any continuable solution $u(t)$ of equation (1) with initial data $u(1) = c_1$, $u'(1) = c_2$ such that $|c_1| + |c_2| \le K$ possesses the property (L).

Proof: With the same argument as in Theorem 2, we have for $t \ge 1$

$$|u'(t)| \;\le\; |c_2| + \int_1^t h(s)\, g_1\!\left(\frac{|u(s)|}{s}\right) g_2\Big(|u'(s)|\Big)\, ds \;,$$

$$\frac{|u(t)|}{t} \;\le\; |c_1| + |c_2| + \int_1^t h(s)\, g_1\!\left(\frac{|u(s)|}{s}\right) g_2\Big(|u'(s)|\Big)\, ds \;. \tag{22}$$

Denoting by $z(t)$ the right-hand side of inequality (22), we obtain for $t \ge 1$

$$z(t) \;\le\; K + \int_1^t h(s)\, g_1(z(s))\, g_2(z(s))\, ds \;. \tag{23}$$

Assumption (ii) implies that the functions $g_1(u), g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, it follows from [6, Lemma 1] that if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with the corresponding multiplier functions $\psi_1(\alpha)$ and $\psi_2(\alpha)$ respectively, then the product $g_1(u)\, g_2(u)$ also belongs to $\mathbb{H}$ and the corresponding multiplier function is $\psi_1(\alpha)\,\psi_2(\alpha)$. Thus, applying [6, Theorem 1] to (23), we have for $t \ge 1$

$$z(t) \;\le\; K\, W^{-1}\left(K^{-1}\,\psi_1(K)\,\psi_2(K)\,\int_1^t h(s)\,ds\right)\,, \tag{24}$$

where

$$W(u) \;=\; \int_1^u \frac{ds}{g_1(s)\,g_2(s)}\,,$$

and $W^{-1}(u)$ is the inverse function for $W(u)$. Evidently, inequality (24) holds for all $t \ge 1$ since by (21)

$$K^{-1}\,\psi_1(K)\,\psi_2(K)\int_1^t h(s)\,ds \;\in\; \mathrm{Dom}(W^{-1})$$

for all $t \ge 1$. The rest of the proof is analogous to that of Theorem 2 and is omitted. ∎

Example 4. Consider the nonlinear differential equation

$$u'' + (3\,t)^{-4}(u\,u')^2 \sin^3 u \;=\; 0\,. \tag{25}$$

For equation (25), we have

$$g_1(u) = g_2(u) = u^2\,, \qquad h(t) = (9\,t)^{-2}\,, \qquad \psi_1(\alpha) = \psi_2(\alpha) = \alpha^2\,.$$

After a straightforward computation, we conclude by Theorem 5 that all continuable solutions of equation (25) with initial data

$$|c_1| + |c_2| \le 3$$

are asymptotic to $at + b$ as $t \to +\infty$.

Note that we may also apply to equation (25) Theorem 2. Indeed, making use of the elementary inequality, we obtain the following estimate

$$|f(t,u,u')| \;\le\; 2^{-1}\,3^{-4}\,t^{-2}\Big((t^{-1}\,u)^4 + (u')^4\Big)\,.$$

Keeping the same notation as in Theorem 2, we have

$$g_1(u) = g_2(u) = u^4\,, \qquad h_1(t) = h_2(t) = 2^{-1}(9\,t)^{-2}\,, \qquad \psi_1(\alpha) = \psi_2(\alpha) = \alpha^4\,.$$

After a simple computation, we conclude by Theorem 2 that all continuable solutions of equation (25) with initial data

$$|c_1| + |c_2| \le \frac{3}{4^{1/3}}$$

are asymptotic to $at + b$ as $t \to +\infty$, but we point out that the domain of the initial data for the solutions with desired asymptotic behavior is reduced in comparison with that obtained by Theorem 5. □

Theorem 6. *Suppose that assumptions (i) and (iii) of Theorem 5 hold, while (ii) is replaced by*

(**ii′**) *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are continuous and nondecreasing, $g_1(0) = g_2(0) = 0$, and satisfy a Lipschitz condition*

$$\Big|g_1(u+v) - g_1(u)\Big| \le \lambda_1 v\,, \qquad \Big|g_2(u+v) - g_2(u)\Big| \le \lambda_2 v\,,$$

where λ_1, λ_2 are positive constants.

Then any continuable solution $u(t)$ of equation (1) with initial data $u(1) = c_1$, $u'(1) = c_2$ such that $|c_1| + |c_2| \le K$ possesses the property (L).

Proof: Applying [6, Corollary 2] to (23), we have for $t \ge 1$

$$\begin{aligned} z(t) \;&\le\; K + g_1(K)\, g_2(K) \int_1^t h(s)\, \exp\Big(\lambda_1\, \lambda_2 \int_1^t h(\tau)\, d\tau\Big)\, ds \\ &\le\; K + H\, g_1(K)\, g_2(K)\, \exp(\lambda_1\, \lambda_2\, H) \;<\; +\infty\,. \end{aligned}$$

The proof can be completed with the same argument as in Theorem 1. ∎

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